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EXPONENTIAL SMOOTHING FOR IRREGULAR DATA*

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Abstract. Various types of exponential smoothing for data observed at irregular time intervals are surveyed. Double exponential smoothing and some modifications of Holt's method for this type of data are suggested. A real data example compares double exponential smoothing and Wright's modification of Holt's method for data observed at irregular time intervals.

Keywords: double exponential smoothing, exponential smoothing, Holt's method, Holt-Winters method, irregular time intervals, missing observations, time series

MSC 2000: 62M10, 62M20, 90A20, 60G35

1. Introduction

Various types of exponential smoothing seem to be a simple and practical instrument for smoothing and forecasting time series. Moreover, modifications of exponential smoothing have been developed which can be used in special situations, e.g., robust exponential smoothing for time series with outliers (e.g. [1], [3], [4]), exponential smoothing for exponential or damped trends (e.g. [6]), exponential smoothing for multivariate time series (e.g. [2], [6]) and others.

Another special time series typical in practice are data observed (or published) at irregular time intervals. One may denote them briefly as irregular data

$$y_{t_1}, y_{t_2}, y_{t_3}, \ldots \tag{1}$$

Such a situation can include as a special case time series with missing observations (see also [2], [5], [8]). Simple exponential smoothing and Holt's method for irregular

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data have been suggested by Wright (see [9]). This approach is recalled in Section 2. Double exponential smoothing for irregular data is suggested in Section 3. Some modifications of Holt's method including the seasonal additive and multiplicative Holt-Winters methods are shown in Section 4. Finally a real data example in Section 5 demonstrates Holt's method from Section 2 and double exponential smoothing from Section 3 numerically for irregular data.

2. Simple exponential smoothing and Holt's method for irregular data

Wright's simple exponential smoothing for irregular data (i.e. the exponential smoothing method with one smoothing coefficient for local constant trends) has the form which generalizes in a natural way the regular case

$$S_{t_n} = \alpha_{t_n} y_{t_n} + (1-\alpha_{t_n}) S_{t_{n-1}}, \tag{2}$$

$$\alpha_{t_n} = \frac{\alpha_{t_{n-1}}}{\beta^{t_n - t_{n-1}} + \alpha_{t_{n-1}}}, \tag{3}$$

$$\hat{y}_{t_n+k}(t_n) = S_{t_n}, \quad k \geqslant 0. \tag{4}$$

It is easy to show that (2) and (3) are recursive versions of relations

$$S_{t_n} = \alpha_{t_n} \sum_{j=1}^{n} \beta^{t_n - t_j} y_{t_j}, \tag{5}$$

$$\alpha_{t_n} = \frac{1}{\sum\limits_{j=1}^{n} \beta^{t_n - t_j}} \tag{6}$$

which generalize the well-known relations of the regular simple exponential smoothing. The discount coefficient β $(0 < \beta < 1)$ or the smoothing coefficient $\alpha = 1 - \beta$ would be used in the regular case but in the irregular case one must recalculate a nonconstant smoothing coefficient recursively according to (3). The initial values of the smoothing statistics S_0 and the smoothing coefficient α_0 are recommended as

$$S_0 = S_{t_0} = \frac{1}{n_0} \sum_{j=1}^{n_0} y_{t_j}, \tag{7}$$

$$\alpha_0 = \alpha_{t_0} = \frac{1}{\sum\limits_{j=0}^{\infty} \beta^{q \cdot j}} = 1 - \beta^q = 1 - (1-\alpha)^q \tag{8}$$

where n_0 is the length of an initial block of observations (e.g. $n_0 = 6$ or $n_0 = n/4$) and q is the average time spacing of the data.

Similarly, Wright's modification of Holt's method for irregular data (i.e. the ad hoc exponential smoothing method with two smoothing coefficients for local linear trends) has the form

$$S_{t_n} = \alpha_{t_n} y_{t_n} + (1-\alpha_{t_n})[S_{t_{n-1}} + (t_n - t_{n-1})T_{t_{n-1}}], \tag{9}$$

$$T_{t_n} = \gamma_{t_n} \frac{S_{t_n} - S_{t_{n-1}}}{t_n - t_{n-1}} + (1-\gamma_{t_n})T_{t_{n-1}}, \tag{10}$$

$$\alpha_{t_n} = \frac{\alpha_{t_{n-1}}}{(1-\alpha)^{t_n - t_{n-1}} + \alpha_{t_{n-1}}}, \quad \gamma_{t_n} = \frac{\gamma_{t_n - 1}}{(1-\gamma)^{t_n - t_{n-1}} + \gamma_{t_{n-1}}}, \tag{11}$$

$$\hat{y}_{t_n+k}(t_n) = S_{t_n} + kT_{t_n}, \quad k \geqslant 0. \tag{12}$$

The initial values of the level S_0, the slope T_0 and the smoothing coefficients α_0 and γ_0 are recommended similarly as for the irregular simple exponential smoothing, i.e.

$$S_0 = S_{t_0} = \hat{b}_0(0), \tag{13}$$

$$T_0 = T_{t_0} = \hat{b}_1(0), \tag{14}$$

$$\alpha_0 = \alpha_{t_0} = 1 - (1-\alpha)^q, \quad \gamma_0 = \gamma_{t_0} = 1 - (1-\gamma)^q \tag{15}$$

where $\hat{b}_0(0)$ and $\hat{b}_1(0)$ are regression estimates of the linear trend $b_0 + b_1 t$ in an initial block of observations with length n_0, the smoothing coefficients α and γ ($0 < \alpha < 1$, $0 < \gamma < 1$) would be used in the regular case and q is again the average time spacing of the data.

3. Double exponential smoothing for irregular data

Double exponential smoothing for irregular data (i.e. the exponential smoothing method with one smoothing coefficient for local linear trends) is not an ad hoc procedure as Holt's method in the previous section and has the form

$$S_{t_n} = \alpha_{t_n} y_{t_n} + (1-\alpha_{t_n})S_{t_{n-1}}, \tag{16}$$

$$S_{t_n}^{[2]} = \alpha_{t_n} S_{t_n} + (1-\alpha_{t_n})S_{t_{n-1}}^{[2]}, \tag{17}$$

$$\alpha_{t_n} = \frac{\alpha_{t_{n-1}}}{\beta^{t_n - t_{n-1}} + \alpha_{t_{n-1}}}, \quad w_{t_n} = \frac{w_{t_{n-1}}}{\beta^{t_n - t_{n-1}} + (t_n - t_{n-1})\beta^{t_n - t_{n-1}} \frac{w_{t_{n-1}}}{\alpha_{t_{n-1}}}},$$
$$z_{t_n} = \frac{z_{t_{n-1}}}{\beta^{t_n - t_{n-1}} + \frac{\alpha_{t_n} z_{t_{n-1}}}{w_{t_n}}}, \tag{18}$$

$$\hat{y}_{t_n+k}(t_n) = S_{t_n} + \Big(\frac{z_{t_n}}{w_{t_n}} + \frac{z_{t_n}}{\alpha_{t_n}} k\Big)(S_{t_n} - S_{t_n}^{[2]}), \quad k \geqslant 0. \tag{19}$$

One can see that in addition to the smoothing coefficient α one must recalculate recursively two auxiliary values w and z (see (18)). It is not difficult to show that in the regular case one obtains from (16)–(19) the regular double exponential smoothing formulas (the so called Brown's method). The initial values of the smoothing statistics S_0 and $S_0^{[2]}$ and the values α_0, w_0 and z_0 can be recommended as

$$S_0 = S_{t_0} = \hat{b}_0(0) - \frac{q\beta^q}{1-\beta^q}\hat{b}_1(0), \tag{20}$$

$$S_0^{[2]} = S_{t_0}^{[2]} = \hat{b}_0(0) - 2\frac{q\beta^q}{1-\beta^q}\hat{b}_1(0), \tag{21}$$

$$\alpha_0 = \alpha_{t_0} = 1-\beta^q,\ w_0 = w_{t_0} = \frac{(1-\beta^q)^2}{q\beta^q}, \quad z_0 = z_{t_0} = \frac{(1-\beta^q)^2}{q\beta^q} \tag{22}$$

where $\hat{b}_0(0)$, $\hat{b}_1(0)$, $\beta = 1-\alpha$ and q are as in the irregular Holt's method in Section 2.

Derivation of double exponential smoothing for irregular data: One can proceed similarly as for the regular double exponential smoothing (see e.g. [7]) in the model

$$y_{t_j} = b_0 + b_1 t_j + \varepsilon_{t_j}, \quad j = 1, 2, \dots \tag{23}$$

where the residuals ε form a white noise. Let us denote

$$S_{t_n} = \alpha_{t_n} \sum_{j=1}^{n} \beta^{t_n - t_j} y_{t_j}, \tag{24}$$

$$S_{t_n}^{[2]} = \alpha_{t_n} \sum_{j=1}^{n} \beta^{t_n - t_j} S_{t_j}, \tag{25}$$

$$\alpha_{t_n} = \frac{1}{\sum\limits_{j=1}^{n} \beta^{t_n - t_j}}, \quad w_{t_n} = \frac{1}{\sum\limits_{j=1}^{n} (t_n - t_j)\beta^{t_n - t_j}}, \quad z_{t_n} = \frac{1}{\sum\limits_{j=1}^{n} \frac{\alpha_{t_j}}{w_{t_j}} \beta^{t_n - t_j}} \tag{26}$$

for $n > 1$. Then it is easy to verify that the recursive relations (16)–(18) hold. Moreover, the expectation of the smoothing statistics S satisfies

$$\begin{aligned} E(S_{t_n}) &= \alpha_{t_n} \sum_{j=1}^{n} (b_0 + b_1 t_j)\beta^{t_n - t_j} \\ &= \alpha_{t_n}(b_0 + b_1 t_n) \sum_{j=1}^{n} \beta^{t_n - t_j} - \alpha_{t_n} b_1 \sum_{j=1}^{n} (t_n - t_j)\beta^{t_n - t_j} \\ &= b_0 + b_1 t_j - b_1 \frac{\alpha_{t_n}}{w_{t_n}} = E(y_{t_n}) - b_1 \frac{\alpha_{t_n}}{w_{t_n}} \end{aligned} \tag{27}$$

and similarly for the expectation of the double smoothing statistics $S^{[2]}$

$$E(S_{t_n}^{[2]}) = E(S_{t_n}) - b_1 \frac{\alpha_{t_n}}{z_{t_n}}. \tag{28}$$

Now one can construct the unbiased estimator of b_1 according to (28) as

$$\hat{b}_1(t_n) = \frac{z_{t_n}}{\alpha_{t_n}}(S_{t_n} - S_{t_n}^{[2]}) \tag{29}$$

and the unbiased estimator of $E(y_{t_n})$ according to (27) and (29) as

$$\hat{y}_{t_n} = S_{t_n} + \hat{b}_1(t_n)\frac{\alpha_{t_n}}{w_{t_n}} = S_{t_n} + \frac{z_{t_n}}{w_{t_n}}(S_{t_n} - S_{t_n}^{[2]}). \tag{30}$$

Hence one obtains finally the unbiased predictor (19) as

$$\hat{y}_{t_n+k}(t_n) = \hat{y}_{t_n} + k\hat{b}_1(t_n) = S_{t_n} + \left(\frac{z_{t_n}}{w_{t_n}} + \frac{z_{t_n}}{\alpha_{t_n}}k\right)(S_{t_n} - S_{t_n}^{[2]}), \quad k \geqslant 0. \tag{31}$$

As the initial values (20)–(21) are concerned one can use relations

$$\hat{b}_1(t_n) = \frac{z_{t_n}}{\alpha_{t_n}}(S_{t_n} - S_{t_n}^{[2]}), \tag{32}$$

$$\hat{b}_0(t_n) = \hat{y}_{t_n}(t_n) - \hat{b}_1(t_n)t_n = S_{t_n} + \left(\frac{z_{t_n}}{w_{t_n}} - \frac{z_{t_n}}{a_{t_n}}t_n\right)(S_{t_n} - S_{t_n}^{[2]}) \tag{33}$$

for $t_n = 0$, and e.g. the initial value w_0 in (22) follows as

$$w_0 = w_{t_0} = \frac{1}{\sum\limits_{j=0}^{\infty} qj\beta^{q\cdot j}} = \frac{(1-\beta^q)^2}{q\beta^q}. \tag{34}$$

4. Modifications of Holt's method for irregular data

Further modifications of Holt's method for irregular data (9)–(12) may be useful in practical situations. In the case of exponential trends one can use

$$S_{t_n} = \alpha_{t_n} y_{t_n} + (1-\alpha_{t_n})S_{t_{n-1}} T_{t_{n-1}}^{t_n - t_{n-1}}, \tag{35}$$

$$T_{t_n} = \gamma_{t_n}\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right)^{1/(t_n - t_{n-1})} + (1-\gamma_{t_n})T_{t_{n-1}}, \tag{36}$$

$$\hat{y}_{t_n+k}(t_n) = S_{t_n} T_{t_n}^k, \quad k \geqslant 0 \tag{37}$$

where S is the level and T is the base of the power.

In the case of damped linear trends one can use

$$S_{t_n} = \alpha_{t_n} y_{t_n} + (1-\alpha_{t_n})\Big(S_{t_{n-1}} + \sum_{i=1}^{t_n - t_{n-1}} \varphi^i T_{t_{n-1}}\Big), \tag{38}$$

$$T_{t_n} = \gamma_{t_n} \frac{\varphi^{t_n - t_{n-1}}}{\sum\limits_{i=1}^{t_n - t_{n-1}} \varphi^i}(S_{t_n} - S_{t_{n-1}}) + (1-\gamma_{t_n})\varphi^{t_n - t_{n-1}} T_{t_{n-1}}, \tag{39}$$

$$\hat{y}_{t_n+k}(t_n) = S_{t_n} + \sum_{i=1}^{k} \varphi^i T_{t_n}, \quad k \geqslant 0 \tag{40}$$

where S is the level, T is the slope and φ is the damping coefficient ($0 < \varphi < 1$). The smoothing coefficients both for (35)–(37) and for (38)–(40) are calculated according to (11).

In the case of seasonality one can use the modification of Holt-Winters method (i.e. the ad hoc exponential smoothing method with three smoothing coefficients for local linear trends combined with seasonality) for irregular data suggested by Cipra (see [5]). The additive method is

$$S_{t_n} = \alpha_{t_n}(y_{t_n} - I_{t_n^*}) + (1-\alpha_{t_n})[S_{t_{n-1}} + (t_n - t_{n-1})T_{t_{n-1}}], \tag{41}$$

$$T_{t_n} = \gamma_{t_n} \frac{S_{t_n} - S_{t_{n-1}}}{t_n - t_{n-1}} + (1-\gamma_{t_n})T_{t_{n-1}}, \tag{42}$$

$$I_{t_n} = \delta_{t_n}(y_{t_n} - S_{t_n}) + (1-\delta_{t_n})I_{t_n^*}, \tag{43}$$

$$\alpha_{t_n} = \frac{\alpha_{t_{n-1}}}{(1-\alpha)^{t_n - t_{n-1}} + \alpha_{t_{n-1}}}, \quad \gamma_{t_n} = \frac{\gamma_{t_{n-1}}}{(1-\gamma)^{t_n - t_{n-1}} + \gamma_{t_{n-1}}}, \tag{44}$$

$$\delta_{t_n} = \frac{\delta_{t_{n-1}}}{(1-\delta)^{t_n - t_{n-1}} + \delta_{t_{n-1}}},$$

$$\hat{y}_{t_n} = S_{t_n} + I_{t_n}, \tag{45}$$

$$\hat{y}_{t_n+k}(t_n) = S_{t_n} + kT_{t_n} + I_{(t_n+k)^*}, \quad k > 0 \tag{46}$$

where S is the level, T is the slope, I is the seasonal index, p is the length of season, t_n^* is the largest value among $t_{n-1}, t_{n-2}, \ldots$ such that the time t_n^* corresponds to the same seasonal period as the time t_n and $(t_n + k)^*$ is the largest value among $t_n, t_{n-1}, \ldots$ such that the time $(t_n + k)^*$ corresponds to the same seasonal period as the time $t_n + k$.

Analogously, the multiplicative method includes the relations (42), (44) and

$$S_{t_n} = \alpha_{t_n} \frac{y_{t_n}}{I_{t_n^*}} + (1 - \alpha_{t_n})[S_{t_{n-1}} + (t_n - t_{n-1})T_{t_{n-1}}], \tag{47}$$

$$I_{t_n} = \delta_{t_n} \frac{y_{t_n}}{S_{t_n}} + (1 - \delta_{t_n}) I_{t_n^*}, \tag{48}$$

$$\hat{y}_{t_n} = S_{t_n} I_{t_n}, \tag{49}$$

$$\hat{y}_{t_n+k}(t_n) = (S_{t_n} + kT_{t_n}) I_{(t_n+k)^*}, \quad k > 0. \tag{50}$$

5. An example

In Figs. 1 and 2 one considers relative price changes of investment units (in basis points 0.01 %) on 60 irregular dates for a Czech investment fund: 2 on January 3, 11 on January 6, 26 on January 11, …, 257 on June 30. In addition to smoothing predicted values for July 1–18 should be constructed.

In Fig. 1 one has used Wright's modification of Holt's method for irregular data according to (9)–(12) with $\alpha = 0.1$, $\gamma = 0.1$, $n_0 = 6$, $q = 3$. The mean square error for 60 historical observations is 163.9 (i.e. $\sqrt{\text{MSE}}$ is 12.8 basis points). The predicted values decrease linearly (since $T_{t_{60}}$ in (12) is -0.190).

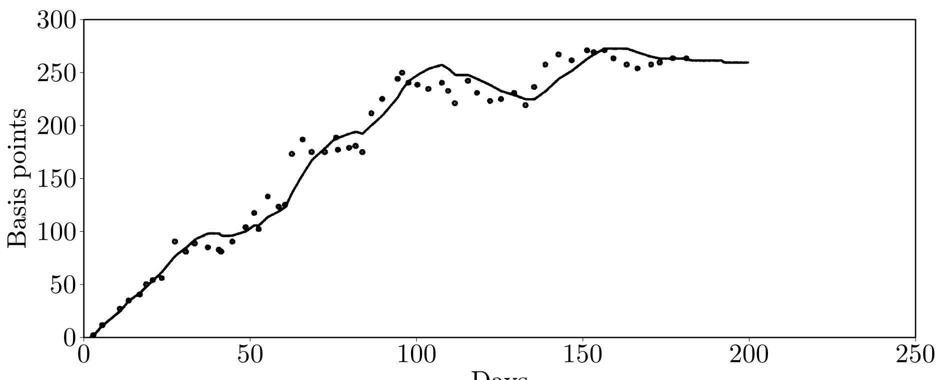

Figure 1. Relative price changes of investment units (in basis points) on irregular dates for a Czech investment fund (dotted line) and its smoothed and predicted values by means of Wright's modification of Holt's method for irregular data (full line).

In Fig. 2 one has used double exponential smoothing on irregular data according to (16)–(19) with $\alpha = 0.1$, $n_0 = 6$, $q = 3$. The mean square error for 60 historical observations is 58.6 (i.e. $\sqrt{\text{MSE}}$ is only 7.7 basis points). In contrast to Fig. 1 the predicted values increase linearly according to the substance of the method suggested

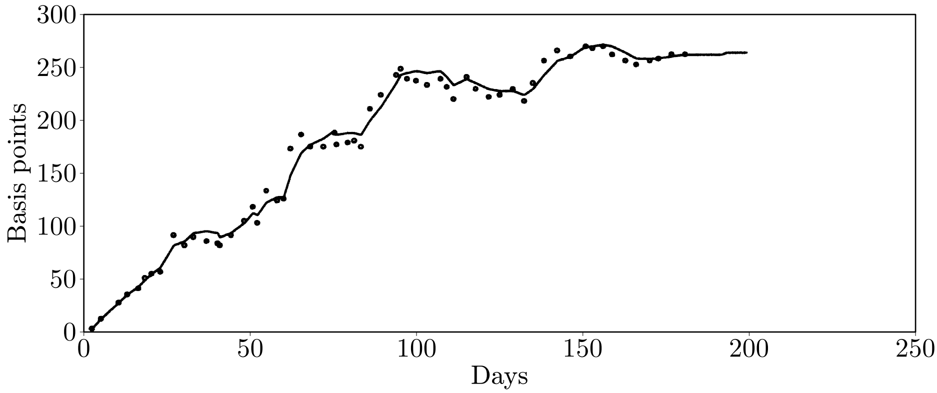

Figure 2. Relative price changes of investment units (in basis points) on irregular dates for a Czech investment fund (dotted line) and its smoothed and predicted values by means of double exponential smoothing for irregular data (full line).

and match better the real data. Similar conclusions hold also for other real data applications.

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