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EXPONENTIAL STABILITY AND GLOBAL EXISTENCE IN THERMOELASTICITY WITH RADIAL SYMMETRY

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Abstract. In this paper we consider equations of linear and nonlinear thermoelasticity with various boundary conditions. We assume radial symmetry of the initial data to prove exponential decay and to show the global existence of solutions of the nonlinear problem for small initial data.

1. Introduction. In this article we consider the equations of thermoelasticity. In the linearized case they take the following form (where $u : G \subset \mathbb{R}^n \to \mathbb{R}^n$ denotes the displacement vector and $\theta : G \to \mathbb{R}$ denotes the temperature):

$$u_{tt} - \mu\Delta u - (\mu + \lambda)\nabla\mathrm{div}\, u + \beta\nabla\theta = 0, \tag{1}$$

$$c\theta_t - \kappa\Delta\theta + \beta\mathrm{div}\, u_t = 0, \tag{2}$$

together with initial data and various boundary conditions as will be specified later.

An existence theory for this problem is well established and there are also local existence results for the nonlinear case. For an overview on this, see [4].

The time asymptotic behaviour is, in general, quite complicated. We expect very different behaviour for the hyperbolic (elasticity) and the parabolic (heat equation) part of the system. In two and three dimensions, the hyperbolic part tends to dominate the behaviour of the hole systems as there is in general no decay rate, see e.g. [5]. However, in special cases one can prove exponential decay for the linear system and apply this to prove global existence for small initial data in the nonlinear case. This was done for *rotation free* solutions in the case of Dirichlet boundary conditions by Jiang, Muñoz Rivera, and Racke in [3]. Here we extend their ideas to different boundary conditions of Neumann- and Robin-type. In Sec. 2 we consider the linear case and prove exponential decay for the solution. In Sec. 3 we study the nonlinear case and prove global existence for small initial data. The new difficulties we have to solve for the nonlinear problem arise in particular from the nonlinearity of the Neumann boundary conditions (whereas the Dirichlet boundary condition studied in [3] is the same as in the linear case). This

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leads to interesting technical problems, especially we need an elliptic regularity property that is proved in Sec. 4.

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2. Linear Thermoelasticity. In this section we study the system of linear thermoelasticity (1)–(2) for a homogenous, isotropic medium (with appropriate initial conditions). For a physical derivation of these equations, see [1] and the references therein.

Throughout this paper we always consider initial boundary value problems in bounded domains $G \subset \mathbb{R}^n$, where $n = 2$ or $n = 3$ in most cases. (The results of this section may also be extended to other dimensions.) We denote the space of functions on G with k weak derivatives in $\mathcal{L}^2(G)$ by $H^k(G)$ (often abbreviating this by H^k); i.e., $H^k(G) := W^{k,2}(G)$. The standard norms in these Sobolev spaces are denoted by $\|\cdot\|_{H^k}$. We denote the $\mathcal{L}^2$-norm on G by $\|\cdot\|_{\mathcal{L}^2}$ or simply $\|\cdot\|$. By $|\cdot|$ we denote the absolute value of a number or the length of a vector in $\mathbb{R}^n$. Finally, by $\langle\cdot,\cdot\rangle$ we denote the scalar product in $\mathcal{L}^2(G)$, i.e., $\langle u, v\rangle := \int_G u(x)v(x)\,dx$.

Our goal will be to describe the asymptotic behaviour of special solutions for various boundary conditions. For u and θ we consider Dirichlet and Neumann conditions and for θ we consider also a mixed boundary condition, the so-called Robin condition.

The physical meaning of these boundary conditions is shown in the following table:

	Dirichlet	Neumann	Robin boundary condition
u	body fixed on the boundary	free boundary	—
θ	temperature fixed on the boundary	perfect isolated boundary	heat flow on the boundary

To prove the existence of solutions to these problems we can use semigroup theory. We only want to state the result; a proof can be found, for example, in [6].

REMARK 2.1. There exists a unique solution (u, θ) of the initial boundary value problems with $u \in \mathcal{C}^2([0,\infty), \mathcal{L}^2) \cap \mathcal{C}^1([0,\infty), H^1), \theta \in \mathcal{C}^1([0,\infty), \mathcal{L}^2), \Delta\theta \in \mathcal{C}([0,\infty), \mathcal{L}^2)$, $\mathcal{D}'\mathcal{S}\mathcal{D}u \in \mathcal{C}([0,\infty), \mathcal{L}^2)$. (For a definition of $\mathcal{S}$ and $\mathcal{D}$ see the next subsection.)

We will assume radially symmetric initial data and therefore explicit radially symmetric solutions (u, θ).

The boundary condition $u\,|_{\partial G} = 0$, $\theta\,|_{\partial G} = 0$ was considered by Jiang, Muñoz Rivera, and Racke. They proved exponential decay in the case of rotation-free solutions [3]. In the next two sections we try to find similar results for the Neumann boundary condition in u resp. the Robin boundary condition in θ. The case $u\,|_{\partial G} = 0$, $\frac{\partial\theta}{\partial\vec{n}}\,|_{\partial G} = 0$ is omitted because it is the easiest case, where the assumptions of rotation free solutions without explicit radial symmetry is sufficient; for a proof, see [10].

2.1. *The Neumann boundary conditions for u.* To investigate the Neumann boundary conditions for u we assume explicitly radial symmetry, so we only consider discs, balls, and annular discs and spheres as domain G with radially symmetric initial values. The resulting solutions $u(x,t)$ and $\theta(x,t)$ can be written as:

$$u(x,t) = w(|x|,t)x, \quad \theta(x,t) = \Theta(|x|,t). \tag{3}$$

We notice that under these assumptions $\theta(\cdot,t)$ is locally constant on ∂G, i.e., it is constant on all components of ∂G.

We now define some auxiliary operators to formulate the Neumann boundary condition in the cases $n=2$ and $n=3$.

DEFINITION 2.2. Let τ, λ, μ be the Lamé moduli, then define for $\vec{n} = (n_1, \dots, n_n)$:

$$\underline{n=2}: \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \\ n_2 & n_1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \tau & \lambda & 0 \\ \lambda & \tau & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

$$\underline{n=3}: \quad \mathcal{D} = \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \\ 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \tau & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \tau & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}.$$

As a convention we denote in this context the transposed matrix $\mathcal{D}^T$ with $\mathcal{D}'$. The equations of thermoelasticity take the form:

$$u_{tt} - \mathcal{D}'\mathcal{S}\mathcal{D}u + \mathcal{D}'\vec{\beta}\theta = 0, \tag{4}$$

$$\theta_t + c\Delta\theta - \vec{\beta}'\mathcal{D}u_t = 0, \tag{5}$$

where we have defined for $n=2$: $\vec{\beta} := (\beta,\beta,0)^T$ and for $n=3$: $\vec{\beta} := (\beta,\beta,\beta,0,0,0)^T$. We are now able to formulate the boundary conditions we want to study:

$$\mathcal{N}'\mathcal{S}\mathcal{D}u \,|_{\partial G} = 0, \quad \theta \,|_{\partial G} = 0. \tag{6}$$

Before we start our energy estimates we need the following lemmata:

LEMMA 2.3. For $v \in H^1$ we have for a.e. $x \in G$: $|\mathcal{D}v(x)|^2 \geq \frac{1}{2}|\text{div}\, v(x)|^2$.

LEMMA 2.4. For $v \in H^1$ and $\text{rot}\, v = 0$ we have for a.e. $x \in G$: $|\mathcal{D}v(x)|^2 \geq |\nabla v(x)|^2 \geq \frac{1}{2}|\mathcal{D}v(x)|^2$.

LEMMA 2.5. For $v \in H^1$ radially symmetric we have: $||\text{div}\, v||^2 \geq ||\nabla v||^2 \geq \frac{1}{2}||\text{div}\, v||^2$.

Proof. One can easily check the lemmata 2.3 and 2.4 using the Cauchy-Schwarz inequality. The same is true for the first inequality in 2.5. For the proof of the second inequality it is necessary to use explicit radial symmetry to show that the boundary terms appearing by the partial integration fit together, i.e., that one gets:

$$\int_{\partial G} (\nabla v\vec{n} - \text{div}\, v\vec{n})v \;=\; -\int_{\partial G} (n-1)w(|x|)^2|x|.$$

Now we are ready to state the main result of this section and to prove it:

THEOREM 2.6 (Exponential decay). Let (u, θ) be a solution of (4)–(5) with respect to the boundary condition (6) with radially symmetric initial values (u^0, u^1, θ^0), and let

$$E(t) := e^{\gamma t}\left\{\sum_{k=0}^{2} ||\partial_t^k u(t)||^2_{H^{2-k}} + ||\theta_t(t)||^2 + ||\theta(t)||^2_{H^2}\right\}. \tag{7}$$

Then there are constants $\Gamma \geq 1$ and $\gamma > 0$ such that for all $t \geq 0$:

$$E(t) + \int_0^t e^{\gamma s}||\nabla\theta_t(s)||^2\, ds \leq \Gamma E(0). \tag{8}$$

Proof. To apply an energy method we define the "energies":

$$\begin{aligned} F_1(t) &:= \frac{1}{2}\left(||u_t||^2 + \langle \mathcal{D}u, \mathcal{S}\mathcal{D}u\rangle + c||\theta||^2\right)(t), \\ F_2(t) &:= \frac{1}{2}\left(||u_{tt}||^2 + \langle \mathcal{D}u_t, \mathcal{S}\mathcal{D}u_t\rangle + c||\theta_t||^2\right)(t), \\ F_3(t) &:= \frac{1}{2}\left(||\Delta u||^2 + \langle \mathcal{D}u_t, \mathcal{S}\mathcal{D}u_t\rangle + c||\nabla\theta||^2\right)(t). \end{aligned}$$

For F_1 and F_2 we derive by multiplying the differential equations with suitable terms and integrating by parts:

$$\frac{d}{dt}F_1(t) = -\kappa||\nabla\theta||^2, \quad \frac{d}{dt}F_2(t) = -\kappa||\nabla\theta_t||^2. \tag{9}$$

For F_3 we get:

$$\frac{d}{dt}F_3(t) = -\kappa||\Delta\theta||^2 + \beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\operatorname{div} u_t - \frac{1}{\tau}\int_{\partial G} u_{tt}\mathcal{N}'\vec{\beta}\theta_t. \tag{10}$$

As an important tool we need the Poincaré inequality and the Korn inequality for u in the same form as in the Dirichlet case. We have two possible attempts stated in the two following lemmata, where we have $\mathcal{D}_0 := \{v \in (H^1(G))^n | \mathcal{D}v = 0\}$:

LEMMA 2.7 (Korn inequality for u). Let u_0 and $u_1 \in \mathcal{D}_0^\perp$, then:
$u(t) \in \mathcal{D}_0^\perp$ for all $t \geq 0$, and there exists $C > 0$ with: $||u|| \leq C||\mathcal{D}u||$.

REMARK 2.8. This is the "classical" attempt resulting from the nullspace of the differential operator. With lemma 2.4 we get the intended type of the Poincaré inequality for u (see lemma 2.9).

Proof. It follows from the (normal) Korn inequality (see, e.g., [9]):

$$||u||^2 \leq C\left(||\mathcal{D}u||^2 + \sup_{v\in\mathcal{D}_0, ||v||=1} |\langle u, v\rangle|\right)$$

On the other hand we have $u \in \mathcal{D}_0^\perp$, because for $v \in \mathcal{D}_0$ we have:

$$\langle u_{tt}, v\rangle = \langle \mathcal{D}'\mathcal{S}\mathcal{D}u - \mathcal{D}'\vec{\beta}\theta, v\rangle = -\langle \mathcal{S}\mathcal{D}u - \vec{\beta}\theta, \underbrace{\mathcal{D}v}_{=0}\rangle + \int_{\partial G}\underbrace{\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta}_{=0} = 0.$$

LEMMA 2.9 (Poincaré inequality for u). Let $\int_G u_0 = 0$ and $\int_G u_1 = 0$ (which we can assume after appropriate normalization); then there exists a $C \geq 0$, such that:

$$||u|| \leq C||\nabla u||.$$

Proof. We start with the (normal) Poincaré inequality:

$$||u|| \leq C||\nabla u|| + \int_G u.$$

Under the given assumptions we have $\int_G u = 0$ (using the differential equation, the divergence theorem and the boundary condition).

With the help of lemma 2.4 we again arrive at the Korn inequality in the form of lemma 2.7.

Of course this leads to the question whether the assumptions made for the initial data are satisfied by the physics. Furthermore we have to find some *radially symmetric* functions satisfying the assumptions to avoid an "empty" result. In the first case we can show that for *simply connected* domains (i.e., balls and discs), *all* radially symmetric functions are automatically in $\mathcal{D}_0^\perp$ (for the simple proof consider [10]). In the second case we even have a stronger result: For radially symmetric functions we obviously have $u(-x) = -u(x)$, so it follows (independently of the topology of G) that $\int_G u = 0$. (This corresponds to choosing the center of gravity of the body G in the origin.) Using (10) we get:

$$\frac{d}{dt}F_3(t) = -\kappa||\Delta\theta||^2 + \beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\mathrm{div}\, u_t. \tag{11}$$

We decompose the boundary integral with the Young inequality to:

$$\beta\int_{\partial G}\frac{\partial\theta}{\partial\vec{n}}\mathrm{div}\, u_t \leq \underbrace{\beta\frac{C}{\varepsilon}\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2}_{:=I_1} + \underbrace{\beta\varepsilon\int_{\partial G}|\mathrm{div}\, u_t|^2}_{:=I_2}. \tag{12}$$

To estimate I_1 and I_2 we need two theorems. We will also apply these theorems in the next section for the nonlinear case; for this purpose it makes sense to extend the results needed in this section slightly and not to use the boundary condition.

THEOREM 2.10. Assume that θ satisfies the differential equation:

$$c\theta_t + \kappa\Delta\theta = h_2. \tag{13}$$

Furthermore let θ be locally constant on ∂G and $\sigma \in (\mathcal{C}^1(\bar{G}))^n$ with $\sigma = \vec{n}$ on ∂G. (Such a σ can be constructed by gluing appropriate functions with a partition of unity argument; see [8].)
Then we have:

(14)
$$\kappa\int_{\partial G}\left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 = 2c\int_G\theta_t\sigma\nabla\theta + 2\kappa\int_G\nabla\theta\nabla\sigma_k\partial_k\theta - \kappa\int_G\mathrm{div}\,\sigma|\nabla\theta|^2 - \int_G h_2\sigma\nabla\theta.$$

Proof. We start with (13), multiply with $\sigma_k\partial_k v$ and integrate over G. θ is locally constant on ∂G, so we have: $\frac{\partial\theta}{\partial\vec{n}} = \nabla\theta$. If we apply this, an elementary calculation and partial integration leads to the theorem's statement.

THEOREM 2.11. Let $v(x) = (v_1, \dots, v_n)(x) = w(|x|)x$ be a radially symmetric solution of:

$$v_{tt} - \tau \Delta u = h_1. \tag{15}$$

Let $\sigma \in (\mathcal{C}^1(\bar{G}))^n$ with $\sigma = \vec{n}$ on ∂G.

Then we have:

$$\begin{aligned}
&\tau \int_{\partial G} |\text{div } v|^2 + \int_{\partial G} |v_t|^2 - \tau \int_{\partial G} (n-1) w(|x|) \text{div } v \\
&= 2\frac{d}{dt} \int_G v_t \sigma_k \partial_k v + \int_G \text{div } \sigma |v_t|^2 - \tau \int_G \text{div } \sigma |\text{div } v|^2 \\
&\quad + \tau \int_G \text{div } v \nabla \sigma_k \partial_k v - 2 \int_G h_1 \sigma_k \partial_k v.
\end{aligned} \tag{16}$$

Proof. We start with equation (1), multiply with $\sigma_k \partial_k v$ and integrate over G:

$$\int_G v_{tt} \sigma_k \partial_k v - \tau \int_G \nabla \text{div } v \sigma_k \partial_k v. \tag{17}$$

Using integration by parts we get:

$$\begin{aligned}
&\frac{d}{dt} \int_G v_t \sigma_k \partial_k v + \frac{1}{2} \int_G |v_t|^2 \partial_k \sigma_k - \frac{1}{2} \int_{\partial G} |v_t|^2 \\
&+ \tau \int_G \text{div } v \nabla \sigma_k \partial_k v - \frac{1}{2} \tau \int_G \text{div } \sigma |\text{div } v|^2 \\
&+ \frac{1}{2} \tau \int_{\partial G} |\text{div } v|^2 - \tau \int_{\partial G} \partial_i v_i \sigma_j \sigma_k \partial_k v_j \quad = \quad \int_G h_1 \sigma_k \partial_k v.
\end{aligned}$$

Now we explicitly use the radial symmetry of v to "sum up" the boundary integrals. After an elementary calculation where we use that $\sigma |_{\partial G} = \pm \vec{n}$, we arrive at (16).

We can apply this theorem for thermoelasticity. It is useful to extend the equations slightly for reasons we will see later, so we reach the following corollary:

COROLLARY 2.12. Let u and θ be a smooth radially symmetric vector field respective a smooth radially symmetric function. For a smooth vector field f we assume the differential equation:

$$u_{tt} - \tau \Delta u + \beta \nabla \theta = f.$$

Furthermore we assume the Poincaré inequality:

$$||u||^2 \leq C ||\nabla u||^2.$$

Then we have:

$$\int_{\partial G} |\text{div } u|^2 + \int_{\partial G} |u_t|^2 \quad \leq \quad C \frac{d}{dt} \int_G u_t \sigma_k \partial_k u + C ||(u_t, \nabla u, \nabla \theta)||^2 + C \int_G f \sigma_k \partial_k u.$$

Proof. The proof is an immediate consequence of theorem 2.11 that we get using the Young inequality, the Sobolev trace theorem, and the assumed Poincaré inequality.

REMARK 2.13. We get other useful inequalities of the same type by differentiating with respect to t. For example we get:

$$\int_{\partial G} |\mathrm{div}\, u_t|^2 + \int_{\partial G} |u_{tt}|^2 \quad \leq \quad C\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C||(u_{tt}, \nabla u_t, \nabla\theta_t)||^2 + C\int_G f_t\sigma_k\partial_k u_t.$$

Now we use corollary 2.12 and theorem 2.10 to estimate the terms I_1 and I_2 to get

(18)
$$\frac{d}{dt}F_3(t) \leq -\kappa||\Delta\theta||^2 + K\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + C\varepsilon||(u_{tt}, \nabla u_t, \nabla\theta_t)||^2 + \frac{C}{\varepsilon^3}||(\nabla\theta, \nabla\theta_t)||^2.$$

We are now able to prove three auxiliary estimates we will need below:

$$\frac{d}{dt}\int_G u_t u \quad \leq \quad -\frac{1}{2}\tau s_1||\nabla u||^2 + C||(\theta, u_t)||^2, \tag{19}$$

$$\begin{aligned} \frac{d}{dt}\int_G \mathrm{div}\, u_t \mathrm{div}\, u \quad \leq \quad & -\frac{\tau}{2}||\Delta u||^2 + C||\nabla\theta||^2 \\ & +K_1\alpha\left\{\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + ||(u_{tt}, \nabla u_t, \nabla\theta_t)||^2\right\} \\ & +\frac{K_2}{\alpha}\left\{\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + ||(u_t, \nabla u, \nabla\theta)||^2\right\} + ||\nabla u_t||^2, \end{aligned} \tag{20}$$

and

$$-\frac{\tau}{2}||\Delta u||^2 \leq -\frac{\tau}{4}||\Delta u||^2 - \frac{1}{8\tau}||u_{tt}||^2 + \frac{\beta^2}{4\tau}||\nabla\theta||^2. \tag{21}$$

To prove these statements we have to use the differential equation in the form (4) resp. (1), the lemmata 2.4 and 2.5 and the corollary 2.12.

We now define the auxiliary energy

$$\begin{aligned} H(t) \quad := \quad & \eta F_1 + \eta F_2 + F_3 - (K + K_1\alpha)\int_G u_{tt}\sigma_k\partial_k u_t - \frac{K_2}{\alpha}\int_G u_t\sigma_k\partial_k u \\ & +\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \mathrm{div}\, u \mathrm{div}\, u_t. \end{aligned}$$

Now our goal is to show the exponential decay of the auxiliary energy $H(t)$ by using the Gronwall inequality. For a suitably large η we estimate $\frac{d}{dt}H(t)$:

$$\begin{aligned}
\frac{d}{dt}H(t) \;\leq\; & -C\eta||(\nabla\theta,\nabla\theta_t)||^2 - C||\Delta\theta||^2 + \frac{d}{dt}\left\{\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \operatorname{div} u \operatorname{div} u_t\right\} \\
& + C\varepsilon||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 - K_1\alpha\left\{\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t\right\} \\
& - \frac{K_2}{\alpha}\left\{\frac{d}{dt}\int_G u_t\sigma_k\partial_k u\right\} \qquad \text{using (9) and (18)} \\
\leq\; & -C\eta||(\nabla\theta,\theta,\nabla\theta_t,\theta_t)||^2 - C||(\Delta\theta,\nabla u_t,u_t)||^2 \\
& + \frac{d}{dt}\left\{\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \operatorname{div} u \operatorname{div} u_t\right\} + C\varepsilon||u_{tt}||^2 \\
& - K_1\alpha\left\{2\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t\right\} \\
& - \frac{K_2}{\alpha}\left\{2\frac{d}{dt}\int_G u_t\sigma_k\partial_k u\right\} \text{ using Poincaré for } \theta, \text{ (5) and Poincaré for } u_t \\
\leq\; & -C_1(\eta)||(\nabla\theta,\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2 + \varepsilon^{1/4}\left\{-\frac{\tau s_1}{2}||\nabla u||^2 + C||(\theta,u_t)||^2\right\} \\
& + \varepsilon^{1/2}\left\{-\frac{\tau}{2}||\Delta u||^2 + \frac{C}{\alpha}||(u_t,\nabla u,\nabla\theta)||^2 + C\alpha||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + ||\nabla u_t||^2\right\} \\
& + C\varepsilon||u_{tt}||^2 \qquad \text{using (19) and (20)} \\
\leq\; & -C_1(\eta)||(\nabla\theta,\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2 + \varepsilon^{1/4}\left\{-\frac{\tau s_1}{2}||\nabla u||^2 + C||(\theta,u_t)||^2\right\} \\
& + \varepsilon^{1/2}\left\{-\frac{\tau}{4}||\Delta u||^2 - \frac{1}{8\tau}||u_{tt}||^2 + \frac{\beta^2}{4\tau}||\nabla\theta||^2 + \frac{C}{\alpha}||(u_t,\nabla u,\nabla\theta)||^2\right. \\
& \left. + C\alpha||(u_{tt},\nabla u_t,\nabla\theta_t)||^2 + ||\nabla u_t||^2\right\} + C\varepsilon||u_{tt}||^2 \qquad \text{using (21).}
\end{aligned}$$

Now we define: $\alpha := \varepsilon^{1/8}$, and we choose ε small enough. Then using the Poincaré inequality we get:

$$\begin{aligned}
\frac{d}{dt}H(t) \;\leq\; & -C_1||(\nabla\theta,\theta,\nabla\theta_t,\theta_t,\Delta\theta,\nabla u_t,u_t)||^2 - C\varepsilon^{1/4}||\nabla u||^2 - C\varepsilon^{1/4}||u||^2 - C\varepsilon^{1/2}||\Delta u||^2 \\
& - C\varepsilon||u_{tt}||^2.
\end{aligned}$$

Using the elliptic regularity property $||u||^2_{H^2} \leq C||\Delta u||^2_{\mathcal{L}^2}$ (see Sec. 4), we have for large η constants C_1, $C_2 > 0$, such that:

$$C_1 E(t) \leq H(t)e^{\gamma t} \leq C_2 E(t). \tag{22}$$

This is an immediate consequence of the definitions of $E(t)$ and $H(t)$. Hence we have:

$$\frac{d}{dt}H(t) \leq -CE(t) - C||\nabla\theta_t||^2 \leq -CH(t) - C||\nabla\theta_t||^2.$$

Now we can use the Gronwall inequality and (22) to get:

$$E(t) \le e^{-\gamma t}\left\{E(0) - \int_0^t \|\nabla\theta_t\|^2 e^{Cr}\,dr\right\}. \tag{23}$$

This is the statement of theorem 2.6. □

2.2. *The Robin boundary condition for θ.* In this section we want to consider the Robin boundary condition (sometimes called "third kind boundary condition") for θ combined with the Dirichlet boundary condition for u. (It should be possible to modify these arguments slightly for the Neumann boundary condition in u.) The physical interpretation to this problem is a thermoelastic body fixed on its boundary with heat flow through its boundary. We normalize the constant temperature of the environment to zero. This leads to

$$a(x)\frac{\partial\theta}{\partial\vec{n}} = -b(x)\theta \text{ on } \partial G, \tag{24}$$

where a and b describe the heat flow in $x \in \partial G$.

We assume:

$$\begin{aligned}\partial G &= \Gamma_D \cup \Gamma_N \cup \Gamma_R,\\ a(x) = 0,\ b(x) &= 1 \text{ for } x \in \Gamma_D,\\ a(x) = 1,\ b(x) &= 0 \text{ for } x \in \Gamma_N,\\ \frac{b(x)}{a(x)} > 0,\quad \left|\frac{b(x)}{a(x)}\right| &\text{ bounded for } x \in \Gamma_R.\end{aligned} \tag{25}$$

Furthermore, we assume in this section that (u,θ) is a radially symmetric solution. Our goal is to prove the following theorem:

THEOREM 2.14 (Exponential decay for Robin boundary condition). If Γ_D or Γ_R have positive $(n-1)$–measure, we have under the assumptions (25) for a radially symmetric solution (u,θ) of the equations (1), (2) together with appropriate initial conditions and the boundary condition (24):
There exist constants $\Gamma \ge 1$ and $\gamma > 0$ with

$$E(t) + \int_0^t e^{\gamma s}\|\nabla\theta_t(s)\|^2\,ds \le \Gamma E(0), \tag{26}$$

where

$$E(t) := e^{\gamma t}\left\{\sum_{k=0}^{2}\|\partial_t^k u(t)\|_{H^{2-k}}^2 + \|\theta_t(t)\|^2 + \|\theta(t)\|_{H^2}^2\right\}. \tag{27}$$

To prove this we need again a certain form of Poincaré inequality for θ. We therefore quote the following lemma (see [11]):

LEMMA 2.15 (Poincaré inequality for θ). Let $\partial G = \Gamma_D \cup \Gamma_N \cup \Gamma_R$; let Γ_D or Γ_R have positive $(n-1)$–measure. Furthermore let $\theta \in H^2(G)$, and θ satisfy (24) and the assumptions (25).

Then there is a $C > 0$ with: $\|\theta\|^2 \le C\|\nabla\theta\|^2$.

Proof. To prove the exponential decay we use a modification of the proofs for the other boundary conditions. Estimating the integral $\int_{\partial G} \frac{\partial\theta}{\partial\vec{n}}\theta$ we get:

$$\int_{\partial G} \frac{\partial\theta}{\partial\vec{n}}\theta = \int_{\Gamma_D} \frac{\partial\theta}{\partial\vec{n}} \underbrace{\theta}_{=0} + \int_{\Gamma_N} \underbrace{\frac{\partial\theta}{\partial\vec{n}}}_{=0} \theta + \int_{\Gamma_R} \frac{\partial\theta}{\partial\vec{n}}\theta = -\int_{\Gamma_R} \frac{b(x)}{a(x)}|\theta|^2.$$

We put this into the equation for $\frac{d}{dt}F_1(t)$ and arrive at:

$$\frac{d}{dt}F_1(t) = -\kappa||\nabla\theta||^2 - \int_{\Gamma_R} \underbrace{\frac{b(x)}{a(x)}}_{>0} |\theta|^2 \leq -\kappa||\nabla\theta||^2.$$

Similarly we can estimate the other integrals on the boundary, e.g.:

$$\begin{aligned}\int_{\partial G} \frac{\partial\theta}{\partial\vec{n}}\theta_t &= -\int_{\Gamma_R} \frac{b(x)}{a(x)}\theta_t\theta \\ &\leq C\int_{\Gamma_R} \left|\frac{b(x)}{a(x)}\right|^2 |\theta|^2 + C\int_{\Gamma_R} |\theta_t|^2 \\ &\leq C||\nabla\theta||^2 + C||\nabla\theta_t||^2.\end{aligned}$$

(Such a term is small if η (compare with the previous section) is large enough.)

The estimate of

$$\int_{\partial G} \operatorname{div} u_t \frac{\partial\theta}{\partial\vec{n}} \leq \varepsilon\int_{\partial G} |\operatorname{div} u_t|^2 + \frac{C}{\varepsilon}\int_{\partial G} \left|\frac{\partial\theta}{\partial\vec{n}}\right|^2 \tag{28}$$

is the same as in Sec. 2.1. — Remember that the theorems 2.10 and 2.11 did not depend on the boundary condition as long as u and θ are radially symmetric!

3. Nonlinear thermoelasticity.

3.1. *Formulation of the problem.* After studying the linearized equations of thermoelasticity, we want to consider the nonlinear case. In general we have no global existence of solutions. In this section, however, we will show global existence for radially symmetric initial values and radially symmetric boundary conditions for small initial data.

First we formulate the initial boundary value problem. We start with the nonlinear equations in the case $n = 3$. The case $n = 2$ is similar; the condition $n < 4$, however, is necessary as we will see later.

We define (starting with a smooth Helmholtz potential ψ):

$$\begin{aligned}C_{i\alpha j\beta}(\nabla u, \theta) &:= \frac{\partial^2\psi(\nabla u, \theta)}{\partial(\partial u_i/\partial x_\alpha)\partial(\partial u_j/\partial x_\beta)}, \\ \tilde{C}_{i\alpha} &:= \frac{\partial^2\psi(\nabla u, \theta)}{\partial(\partial u_i/\partial x_\alpha)\partial\theta}, \\ a(\nabla u, \theta) &:= -\frac{\partial^2\psi(\nabla u, \theta)}{\partial\theta^2}.\end{aligned}$$

This leads to the following differential equations:

$$
\begin{aligned}
\frac{\partial^2 u_i}{\partial t^2} &= \operatorname{div} S(\nabla u, \theta) \\
&= C_{i\alpha j\beta}(\nabla u, \theta)\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} + \tilde{C}_{i\alpha}(\nabla u, \theta)\frac{\partial \theta}{\partial x_\alpha}, \quad i = 1, 2, 3.
\end{aligned} \tag{29}
$$

$$
a(\nabla u, \theta)\theta_t = \frac{1}{b(\theta)}\operatorname{div} q(\nabla u, \theta, \nabla\theta) + \tilde{C}_{i\alpha}(\nabla u, \theta)\frac{\partial^2 u_i}{\partial x_\alpha \partial t}. \tag{30}
$$

Here we have used the Einstein summation convention again.

For the functions a and b we assume: $a \geq a_0 > 0$, $b \in C^\infty(\mathbb{R})$, $b(\theta) = \theta + T_0$ for $|\theta| \leq T_0/2$, $0 < b_1 \leq b(\theta) \leq b_2 < \infty$, $-\infty < \theta < \infty$, and $T_0 > 0$ is the reference temperature.

For $\nabla u = 0$, $\theta = 0$ the medium should be isotropic, so we assume:

$$
\begin{aligned}
C_{i\alpha j\beta}(0,0) &= \lambda\delta_{i\alpha}\delta_{j\beta} + \mu(\delta_{ij}\delta_{\alpha\beta} + \delta_{\alpha j}\delta_{i\beta}) \\
\tilde{C}_{i\alpha}(0,0) &= -\beta\delta_{i\alpha} \\
\frac{\partial q_i(0,0,0)}{\partial(\partial\theta/\partial x_j)} &= \kappa\delta_{ij} \\
a(0,0) &= c.
\end{aligned} \tag{31}
$$

Furthermore, we assume:

$$
\begin{aligned}
C_{i\alpha j\beta}(\nabla u, \theta) &= C_{j\beta i\alpha}(\nabla u, \theta) \\
\frac{\partial q_i(\nabla u, \theta, \nabla\theta)}{\partial(\partial\theta/\partial x_j)} &= \frac{\partial q_j(\nabla u, \theta, \nabla\theta)}{\partial(\partial\theta/\partial x_i)} \\
\frac{\partial q_i(\nabla u, \theta, \nabla\theta)}{\partial\theta} &= 0 \\
\frac{\partial q_i(\nabla u, \theta, \nabla\theta)}{\partial(\partial u_\alpha/\partial x_\beta)} &= 0 \quad , 1 \leq i, j, \alpha, \beta \leq 3.
\end{aligned} \tag{32}
$$

The initial conditions are:

$$
u(0) = u_0, \; u_t(0) = u_1, \; \theta(0) = \theta_0.
$$

To formulate the boundary conditions we have to start with (29) and (30).
The easiest boundary condition (Dirichlet in u and θ) was considered in [3]. In this section we want to look at one of the more delicate boundary conditions: the Dirichlet condition for θ together with the Neumann condition for u. (Some ideas to handle the other possible boundary conditions are sketched in [10].) The difficulties we have to handle are mainly based on the nonlinearity of this boundary condition

$$
\vec{n}S(\nabla u, \theta) \mid_{\partial G} = 0. \tag{33}
$$

This difference to the Dirichlet case leads to some interesting technical problems.

Now we want to reformulate the problem (29)–(33) to apply the methods of the last section. For this aim we define the nonlinearities f and g as follows:

$$f_i := (C_{i\alpha j\beta}(\nabla u,\theta) - C_{i\alpha j\beta}(0,0))\frac{\partial^2 u_j}{\partial x_\alpha x_\beta} + (\tilde{C}_{i\alpha}(\nabla u,\theta) - \tilde{C}_{i\alpha}(0,0))\frac{\partial \theta}{\partial x_\alpha} \tag{34}$$

$$\begin{aligned} g := c&\left\{\frac{1}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial\theta/\partial x_j)} - \frac{1}{a(0,0)b(0)}\frac{\partial q_i(0,0,0)}{\partial(\partial\theta/\partial x_j)}\right\}\frac{\partial^2\theta}{\partial x_i\partial x_j} \\ +c&\left\{\frac{\tilde{C}_{i\alpha}(\nabla u,\theta)}{a(\nabla u,\theta)} - \frac{\tilde{C}_{i\alpha}(0,0)}{a(0,0)}\right\}\frac{\partial^2 u_u}{\partial x_\alpha \partial t} + \frac{c}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial\theta}\frac{\partial\theta}{\partial x_i} \\ &+\frac{c}{a(\nabla u,\theta)b(\theta)}\frac{\partial q_i(\nabla u,\theta,\nabla\theta)}{\partial(\partial u_\alpha/\partial x_\beta)}\frac{\partial^2 u_\alpha}{\partial x_i \partial x_\beta}. \end{aligned} \tag{35}$$

With this we get from (29) and (30):

$$u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla\mathrm{div}\, u + \beta\nabla\theta = f(\nabla u,\theta,\nabla^2 u,\nabla\theta) \tag{36}$$

$$c\theta_t - \kappa\Delta\theta + \beta\mathrm{div}\, u_t = g(\nabla u,\theta,\nabla^2 u,\nabla^2\theta,\nabla u_t). \tag{37}$$

(For (36) we will sometimes use the form $u_{tt} - \mathcal{D}'\mathcal{S}\mathcal{D}u + \beta\nabla\theta = f$.)
For technical reasons we have to restrict the tensor $C_{i\alpha j\beta}$ to a special form. We assume:

$$C_{i\alpha j\beta}\frac{\partial^2 u_j}{\partial x_\alpha \partial x_\beta} = A_{ij}(\nabla u,\theta)\Delta u_j, \quad i = 1,\cdots,n. \tag{38}$$

3.2. *Global existence.* We want to prove global existence of our problem for radially symmetric initial datas and boundary conditions by applying a local existence result and exponential decay of local solutions.

First we need the Poincaré and the Korn inequality for u, which we get in the same way as in the last section:

LEMMA 3.1 (Poincaré and Korn inequality for u). Assume for u_0, u_1: $\langle u_0, 1\rangle = \langle u_1, 1\rangle = 0$, then we have:

$$||u|| \leq C||\mathcal{D}u||, \quad ||u|| \leq C||\nabla u||. \tag{39}$$

LEMMA 3.2 (Estimate for the nonlinear boundary condition). For $m = 0,1,2,3$ we have, if u has rotation zero:

$$\left|\frac{d^m}{dt^m}\Big((\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u,\theta)\Big)\right| \leq C\sum_{i=1}^m |(\partial_t^i \nabla u, \partial_t^i\theta)|^2.$$

Proof. We start with $m = 0$ and expand $S(\nabla u,\theta) = (S_{ij}(\nabla u,\theta))_{ij}$ around $(0,0)$ in a Taylor expansion. With $S(0,0) = 0$ we get:

$$\begin{aligned} S(\nabla u,\theta) &= S_{\nabla u}(0,0)\nabla u + S_\theta(0,0)\theta + \mathcal{O}(|\nabla u|^2 + |\theta|^2) \\ &= (C_{i\alpha j\beta}(0,0)\partial_\alpha u_\beta + \tilde{C}_{ij}(0,0)\theta)_{ij} + \mathcal{O}(|\nabla u|^2 + |\theta|^2). \end{aligned}$$

Using the isotropy in $(0,0)$ — see (31) — we get:

(40)

$$|(\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u,\theta)| = |(n_j \underbrace{(C_{i\alpha j\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u,\theta))_i}_{=\mathcal{O}(|\nabla u|^2+|\theta|^2)}|.$$

This leads to the stated estimate.

In the case $m = 1$ this calculation is quite similar, so we only write down briefly the case $m = 2$ (the case $m = 3$ indeed is again analogous). — The arguments of the functions are omitted:

$$\begin{aligned} & \left|\frac{d^2}{d^2t}\left((\mathcal{N}'\mathcal{S}\mathcal{D}u - \mathcal{N}'\vec{\beta}\theta) - \vec{n}S(\nabla u,\theta)\right)\right| \\ = \ & |\mathcal{N}'\mathcal{S}\mathcal{D}u_{tt} - \mathcal{N}'\vec{\beta}\theta_{tt} - \vec{n}\,(S_{\nabla u}\nabla u_{tt} + S_{\nabla u\nabla u}\nabla u_t\nabla u_t + S_{\nabla u\theta}\nabla u_t\theta_t + S_\theta\theta_{tt} \\ & + S_{\nabla u\theta}\nabla u_t\theta_t + S_{\theta\theta}\theta_t\theta_t)\,| \\ = \ & |n_jC_{i\alpha j\beta}(0,0)\partial_\alpha\partial_t^2u_\beta - n_j\tilde{C}_{ij}(0,0)\theta_{tt} - n_jC_{i\alpha j\beta}(\nabla u,\theta)\partial_\alpha\partial_t^2u_\beta - n_j\tilde{C}_{ij}(\nabla u,\theta)\theta_{tt} \\ & -\vec{n}\,(S_{\nabla u\nabla u}\nabla u_t\nabla u_t + S_{\nabla u\theta}\nabla u_t\theta_t + S_{\nabla u\theta}\nabla u_t\theta_t + S_{\theta\theta}\theta_t\theta_t)\,| \\ \leq \ & C(|\nabla u|^2 + |\theta|^2 + |\nabla u_t|^2 + |\theta_t|^2 + |\nabla u_{tt}|^2 + |\theta_{tt}|^2). \end{aligned}$$

We just mention that the second order derivatives of $S(\nabla u,\theta)$ exist and are bounded because we have assumed that S is smooth, and we have small ∇u and small θ.

We now quote the local existence theorem. It is a special application of the theorem given in [2].

THEOREM 3.3 (Local existence theorem). Let $u_j \in H^{4-j}$, $j = 0,\ldots,4$, $\theta_j \in H^{4-j}$, $j = 0,1,2$, $\theta_3 \in \mathcal{L}^2$, where $u_j := \partial_t^j u\,|_{t=0}$, $\theta_j := \partial_t^j\theta\,|_{t=0}$.
Furthermore, there is a $K_0 \leq \min\{1, T_0/2\}$ with $|\nabla u_0(x)|, |u_1(x)|, |\theta_0(x)|, |\nabla\theta_0(x)| < K_0$ for all $x \in \bar{G}$. Then we have a unique solution (u,θ) of (36), (37), (33) defined on a maximal existence interval $[0,T)$, $T \leq \infty$, such that for all $\hat{t} \in [0,T)$:

$$\begin{aligned} u &\in \mathcal{C}^j([0,\hat{t}], H^{4-j}), \quad j = 0,\ldots,4 \\ \theta &\in \mathcal{C}^j([0,\hat{t}], H^{4-j}), \quad j = 0,1,2 \\ \theta_{ttt} &\in \mathcal{C}^0([0,\hat{t}],\mathcal{L}^2) \cap \mathcal{L}^2([0,\hat{t}], H^1). \end{aligned} \tag{41}$$

Furthermore, for all $(x,t) \in \bar{G}\times[0,T)$ we have:

$$|\nabla u(x,t)|, |u_t(x,t)|, |\theta(x,t)|, |\nabla\theta(x,t)| < K_0. \tag{42}$$

Furthermore, there is $T = \infty$, i.e., the solution is a global solution if:

(43)

$$\sup_{t\in[0,T)}\left(\sum_{j=0}^{4}||\partial_t^j u||^2_{H^{4-j}} + \sum_{j=0}^{2}||\partial_t^j\theta||^2_{H^{4-j}} + ||\theta_{ttt}||^2\right)(t) + \int_0^T ||\nabla\theta_{ttt}(s)||^2 ds < \infty$$

and

$$\sup_{x\in G, t\in[0,T)} (|\nabla u(x,t)|, |u_t(x,t)|, |\theta(x,t)|, |\nabla\theta(x,t)|) < K_0. \tag{44}$$

We use this to prove the following global existence theorem:

THEOREM 3.4 (Global existence theorem). Let (u, θ) be a local solution of theorem 3.3 being also radially symmetric. Then there exists a $\delta > 0$ with

$$\sum_{j=0}^{4} ||u_j||^2_{H^{4-j}} + \sum_{j=0}^{2} ||\theta_j||^2_{H^{4-j}} + ||\theta_3||^2 \leq \delta^2, \tag{45}$$

such that a global solution exists, which satisfies:

$$||u(t)||_{H^4}, ||\theta(t)||_{H^4} \to 0 \quad \text{exponentially as } t \to \infty.$$

REMARK 3.5. To get radially symmetric solutions we have to assume that for all $\Omega \in O(n)$, the group of orthonormal matrices of dimension n, for all $W \in \mathbb{R}^{n\times n}$ and all $x \in G$:

$$\left.\begin{array}{r} S(\Omega^T W \Omega, \cdot) = \Omega^T S(W, \cdot)\Omega \\ g(\nabla u, \theta, \nabla^2 u, \nabla^2\theta, \nabla u_t)(\Omega x) = g(\nabla u, \theta, \nabla^2 u, \nabla^2 \theta, \nabla u_t)(x) \end{array}\right\}. \tag{46}$$

In addition, all initial values must be radially symmetric. It follows the radial symmetry of the solution (u, θ). To see this, let (u, θ) be a solution and $v(x) := \Omega^T u(\Omega x), \varphi(x) := \theta(\Omega x)$, then (v, φ) satisfies the differential equations (36) and (37):

$$f(\nabla u, \theta, \nabla^2 u, \nabla\theta) \quad \doteq \quad \operatorname{div} S(\nabla u, \theta)_{|\nabla u=0, \theta=0} - \operatorname{div} S(\nabla u, \theta)$$

means that for all $\Omega \in O(n), W \in \mathbb{R}^{n\times n}$ and all $x \in G$ we have:

$$f(\nabla u, \theta, \nabla^2 u, \nabla\theta)(\Omega x) \quad = \quad f(\nabla u, \theta, \nabla^2 u, \nabla\theta)(x).$$

But (v, φ) satisfies the same initial values like (u, θ), because they are radially symmetric as we have assumed. Furthermore, we have:

$$\begin{aligned} \vec{n}S(\nabla v, \psi)(x) &= n_r S_{sr}(\Omega_{kj}\partial_m u_k(\Omega x)\Omega_{mi}, \theta(\Omega x)) \\ &= n_r S_{sr}(((\nabla u \Omega)^T_{jm}\Omega_{mi})(\Omega x), \theta(\Omega x)) \\ &= n_r \Omega^T_{is} S_{sr}((\nabla u)(\Omega x), \theta(\Omega x))\Omega_{rj} \\ &= \Omega^T \vec{n} S(\nabla u, \theta)(\Omega x)\Omega = 0. \end{aligned}$$

So (v, φ) satisfies the same boundary conditions as (u, θ). Using the uniqueness we have: $(u, \theta) = (v, \varphi)$, but that means especially that (u, θ) is radially symmetric.

There exist indeed functions satisfying our assumptions; for an example see [10].

Proof. First we define:

(47)

$$M(t) := e^{\gamma t}\left\{\sum_{j=0}^{4} ||\partial_t^j u||^2_{H^{4-j}} + \sum_{j=0}^{2} ||\partial_t^j \theta||^2_{H^{4-j}} + ||\theta_{ttt}||^2\right\}(t) + \int_0^t e^{\gamma s}||\nabla\theta_{ttt}(s)||^2 ds$$

(48)

$$\Lambda := 258[(\beta^2 + c^2 + 1)\tilde{\Gamma}]^3 \Gamma(1 + \kappa^{-2}) \sum_{j=0}^{2} \tau^{-2j}.$$

Here Γ is the constant of the exponential decay theorem in Sec. 2. $\tilde{\Gamma} > 1$ is given by the inequalities:

$$||h||^2_{H^{j+2}} \le \tilde{\Gamma}||\Delta h||^2_{H^j},$$
$$(\text{where } h \in H_0^{j+2}(G),\ \Delta h \in \mathcal{L}^2,\ j = 0,1,2),$$
$$||v||^2_{H^{j+2}} \le \tilde{\Gamma}||\Delta v||^2_{H^j} + C\delta^3,$$
$$(\text{where } v \in (H^{j+1}(G))^n,\ \vec{n}S(v,0) = 0 \text{ on } \partial G,\ \langle v, 1\rangle = 0,\ \Delta v \in (H^j)^n,$$
$$j = 0,1,2 \text{ and } v = u \text{ or } j = 0,1 \text{ and } v = u_t.) \tag{49}$$

(For the second one of these elliptic regularity properties, compare with Sec. 4.)

Then there exists a $t_0 \in (0,T]$, such that $M(t) \le \Lambda\delta^2$ for all $t \in [0,t_0)$, because we have assumed $M(0) \le \delta^2$, it is $\Lambda > 1$ (because $\tilde{\Gamma} \ge 1$), and M is continuous.

Now let

$$T^* := \sup\{t_1 > 0 \mid M(t) \le \Lambda\delta^2 \text{ for } t \in [0,t_1)\}, \tag{50}$$

then obviously we have $0 < T^* \le T$.

We consider the cases $T^* < T$ and $T^* = T$:

If $T = T^*$ the solution is in $\mathcal{L}^\infty(\bar{G})$, because we can use the Sobolev imbedding theorem $H^4(G) \hookrightarrow \mathcal{C}_b(G)$ having $4 > n/2$. Using lemma 3.3 we therefore have $T = \infty$, i.e., the solution is a global solution. So it is sufficent to find a contradiction to the second case.

Let $T^* < T$. For $t \in [0,T^*)$ we now obtain, using (50), that:

$$u(t), \theta(t) \in H^4(G),\ u_t(t), \theta_t(t) \in H^3(G),\ u_{tt}(t), \theta_{tt}(t) \in H^2(G).$$

Applying the Sobolev imbedding theorem $H^2(G) \hookrightarrow \mathcal{L}^\infty(G)$ (where we use explicitly $n < 4$) we get for all $t \in [0,T^*)$:

There are $C, \gamma > 0$, satisfying:

$$||u(t)||_{W^{2,\infty}}, ||\theta(t)||_{W^{2,\infty}}, ||u_t(t)||_{W^{1,\infty}}, ||\theta_t(t)||_{W^{1,\infty}}, ||u_{tt}(t)||_{\mathcal{L}^\infty}, ||\theta_{tt}(t)||_{\mathcal{L}^\infty} \le C\delta e^{-\gamma\frac{t}{2}}. \tag{51}$$

We now define an auxiliary energy similar to the energy $E(t)$ in Sec. 2:

$$\mathcal{E}(t;u,\theta) := e^{\gamma t}\left(\sum_{j=0}^{2} ||\partial_t^j u||^2_{H^{2-j}} + ||\theta_t||^2 + ||\theta||^2_{H^2}\right)(t) + \int_G e^{\gamma s}||\nabla\theta_t(s)||^2 ds. \tag{52}$$

In contrast to the linear case we will not be able to prove exponential decay for arbitrarily large initial data. Nevertheless, we apply the methods we have used in the linear case to get energy estimates. But these energy estimates contain certain terms with f and g. Utilizing the smallness of the initial data we can estimate these terms to get exponential decay at least for small initial data.

First we prove:

$$\begin{aligned}
\mathcal{E}(t;u,\theta) + \int_0^t e^{\gamma s} \int_{\partial G} \left|\frac{\partial u_t}{\partial \vec{n}}\right|^2 dx\, ds \leq \Gamma\mathcal{E}(0;u,\theta) + Ce^{\gamma t}||g(t)||^2 \\
+C\int_0^t e^{\gamma s}\left\{||f||(||u|| + ||u_t|| + ||\Delta u|| + ||f||) + ||g||(||\theta|| + ||\nabla\theta|| + ||\Delta\theta|| + ||g||)ds\right\} \\
+C\Big\{|\int_0^t e^{\gamma s}\langle f_t, u_{tt}\rangle ds| + |\int_0^t e^{\gamma s}\langle f, \Delta u_t\rangle ds| + |\int_0^t e^{\gamma s}\langle g_t, \theta_t\rangle ds| + |\int_0^t e^{\gamma s}\langle f_t, \sigma_k\partial_k u_t\rangle ds|\Big\} \\
+\int_0^t e^{\gamma s}\int_{\partial G}|\nabla u|^4 + \int_0^t e^{\gamma s}\int_{\partial G}|\nabla u_t|^4 \\
:= \Gamma\mathcal{E}(0;u,\theta) + \mathcal{P}(t;u,\theta,f,g), \quad t\in[0,T^*).
\end{aligned} \tag{53}$$

For the proof we use the energy method in the same manner as in the linear case. This is a rather long but straightforward calculation, so we only want to mention that we can take advantage of the generalized nature of corollary 2.10. (For a complete proof see [10].)

Using the auxiliary energies $F_i(t)$ defined in the previous section we finally get:

$$\begin{aligned}
\frac{d}{dt}(\eta F_1 + \eta F_2 + F_3) \leq\ & -\kappa\eta(||\nabla\theta||^2 + ||\nabla\theta_t||^2) - \kappa||\Delta\theta||^2 \\
& +\eta(\int_G f u_t + \int_G g\theta + \int_G f_t u_{tt} + \int_G g_t\theta_t) \\
& -\int_G g\Delta\theta - \int_G f\Delta u_t + \frac{C}{\eta}||(\nabla u, u_t, \nabla\theta, \nabla u_t, u_{tt}, \nabla\theta_t, f)||^2 \\
& +\frac{C}{\eta}\int_G f_t\sigma_k\partial_k u_t + \frac{K_2}{\eta}\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + \frac{K_3}{\eta}\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t \\
& +\frac{C}{\varepsilon^3}||(\nabla\theta,\theta_t)||^2 + \varepsilon||\mathrm{div}\, u_t||^2 + C\varepsilon||(\nabla u_t, u_t t, \nabla\theta_t, g)||^2 \\
& +\frac{C}{\varepsilon}||g||\,||\nabla\theta|| + \frac{C\eta^2}{\tau}\int_{\partial G}\left(|\nabla u_t|^4 + |\nabla u|^4\right) + C\varepsilon\int_G f_t\sigma_k\partial_k u_t \\
& +K_1\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t - C\varepsilon\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2,
\end{aligned} \tag{54}$$

where we have used this lemma:

LEMMA 3.6. Let $v = w(|x|)x$ be radially symmetric; then we have:

$$\frac{1}{2}\tau\int_{\partial G}\left|\frac{\partial v}{\partial\vec{n}}\right|^2 \leq \frac{1}{2}\tau\int_{\partial G}|\mathrm{div}\, v|^2 + C||(v,\nabla v)||^2.$$

Proof. The idea is similar to the proof of the theorems 2.10 and 2.11: Here we multiply $\Delta v = \mathrm{div}\,\nabla v$ with $\sigma_k\partial_k v$, integrate by parts and calculate the boundary integrals using explicitly the radial symmetry.

We now define similar to the linear case the energy

$$H(t) := \eta F_1 + \eta F_2 + F_3 - \left(\frac{K_2}{\eta^2} + \frac{K_5}{\alpha}\right)\int_G u_{tt}\sigma_k\partial_k u_t - \left(K_1\varepsilon + \frac{K_3}{\eta^2} + K_4\alpha\right)\int_G u_t\sigma_k\partial_k u$$
$$+\varepsilon^{1/4}\int_G u_t u + \varepsilon^{1/2}\int_G \operatorname{div} u \operatorname{div} u_t.$$

The constants K_4 and K_5 will be given later.

Again like in the linear case we show three auxiliary estimates:

LEMMA 3.7. Under the given assumptions we have:

$$\frac{d}{dt}\int_G u_t\, u \le -\frac{\tau s_1}{2}||\nabla u||^2 + C||(\theta, u_t)||^2 + C\int_{\partial G}|\nabla u|^4 + ||f||\,||u|| \tag{55}$$

$$\frac{d}{dt}\int_G \operatorname{div} u_t \operatorname{div} u \le -\frac{\tau}{2}||\Delta u||^2 + C||\nabla\theta||^2 + ||\Delta u||\,||f|| + ||\nabla u_t||^2$$
$$+\alpha\left\{K_4\frac{d}{dt}\int_G u_{tt}\sigma_k\partial_k u_t + ||(u_{tt}, \nabla u_t, \nabla\theta_t)||^2 + \int_G f_t\sigma_k\partial_k u_t\right\}$$
$$+\frac{C}{\alpha}\left\{K_5\frac{d}{dt}\int_G u_t\sigma_k\partial_k u + ||(u_t, \nabla u, \nabla\theta, f)||^2\right\} \tag{56}$$

$$-\frac{\tau}{2}||\Delta u||^2 \le -\frac{\tau}{4}||\Delta u||^2 - \frac{1}{8\tau}||u_{tt}||^2 + \frac{\beta^2}{4\tau}||\nabla\theta||^2 + \frac{1}{8\tau}||f||^2. \tag{57}$$

Proof. The proofs correspond to the linear case in the previous section. □

Now we can derive an energy estimate for $H(t)$. For this purpose we use (54), (55), (56), and (57) to arrive at:

(58)

$$\frac{d}{dt}H(t) \le -\kappa\eta(||\nabla\theta||^2 + ||\nabla\theta_t||^2) - \kappa||\Delta\theta||^2 + \eta\left\{\int_G fu_t + \int_G g\theta + \int_G f_t u_{tt} + \int_G g_t\theta_t\right\}$$
$$-\int_G g\Delta\theta - \int_G f\Delta u_t + \frac{C}{\varepsilon^3}||(\nabla\theta, \theta_t)||^2 + C\varepsilon||(\nabla u_t, u_{tt}, \nabla\theta, f, g)||^2$$
$$+\frac{C}{\varepsilon}||g||\,||\nabla\theta|| + \frac{C}{\eta}||(u_t, \nabla u, \nabla\theta, f, u_{tt}, \nabla u_t, \nabla\theta_t)||^2 + \frac{C}{\eta}\int_G f_t\sigma_k\partial_k u_t$$
$$+C\eta^2\int_{\partial G}|\nabla u|^4 + C\eta^2\int_{\partial G}|\nabla u_t|^4 + C\varepsilon\int_G f_t\sigma_k\partial_k u_t$$
$$+\varepsilon^{1/4}\left\{C||(\theta, u_t)||^2 + ||f||\,||u|| - \frac{\tau s_1}{2}||\nabla u||^2\right\}$$
$$+\varepsilon^{1/2}\left\{C||(\nabla\theta, \nabla u_t)||^2 + ||\Delta u||\,||f||\right\}$$
$$+\varepsilon^{1/2}\varepsilon^{1/8}\left\{C||(u_{tt}, \nabla u_t, \nabla\theta_t)||^2 + \int_G f_t\sigma_k\partial_k u_t\right\}$$
$$+\varepsilon^{1/2}\frac{C}{\varepsilon^{1/8}}||(u_t, \nabla u_t, \nabla\theta, f)||^2$$
$$-\varepsilon^{1/2}\frac{\tau}{4}||\Delta u||^2 - \varepsilon^{1/2}\frac{1}{8\tau}||u_{tt}||^2 + \varepsilon^{1/2}\frac{\beta^2}{4\tau}||\nabla\theta||^2 + \varepsilon^{1/2}\frac{1}{8\tau}||f||^2$$
$$-C\varepsilon\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2.$$

Using the Poincaré inequality, Eq. (37), and lemma 2.5, we get:

(59)
$$-\kappa\eta(||\nabla\theta||^2+||\nabla\theta_t||^2)-\kappa||\Delta\theta||^2\le -C\eta||(\nabla\theta,\theta,\nabla\theta_t,\theta_t)||^2-C||(\Delta\theta,\nabla u_t,u_t)||^2+C||g||^2.$$

We insert (59) into (58), choose a small $\varepsilon > 0$ and a large η, such that we arrive at:

$$\begin{aligned}\frac{d}{dt}H(t) \le\ & -C||(u_t,u_{tt},\nabla u,\nabla u_t,\Delta u,\theta,\theta_t,\nabla\theta,\nabla\theta_t)||^2+C||f||^2+C||g||^2\\ & +C\left\{\int_G fu_t+\int_G g\theta+\int_G f_t u_{tt}+\int_G g_t\theta_t\right\}\\ & -\int_G g\Delta\theta-\int_G f\Delta u_t+C||g||\,||\nabla\theta||+C\int_G f_t\sigma_k\partial_k u_t\\ & +C\int_{\partial G}|\nabla u|^4+C\int_{\partial G}|\nabla u_t|^4+C\int_G f_t\sigma_k\partial_k u_t\\ & +C||f||\,||u||+C||\Delta u||\,||f||+C\int_G f_t\sigma_k\partial_k u_t-C\int_{\partial G}\left|\frac{\partial u_t}{\partial\vec{n}}\right|^2.\end{aligned}$$

Now we can use the Gronwall inequality. To show the equivalence of $\mathcal{E}(t;u,\theta)$ and $H(t)$ for large η we have to use some kind of elliptic regularity property. For a nonlinear Neumann boundary condition, however, there is no such analogy. Nevertheless for the radially symmetric case we have a similar result proved in Sec. 4:

$$||u||_{H^2}^2\le C||\Delta u||^2+C\int_{\partial G}|\nabla u|^4. \tag{60}$$

Taking this all together we can prove (53).

If we consider the differential equations (36) and (37), we see that they are fulfilled by $\partial_t^l u$ and $\partial_t^l\theta$ if the nonlinearities f and g are replaced with $\partial_t^l f$ resp. $\partial_t^l g$. Using this and (53), we get:

$$\begin{aligned}&\mathcal{E}(t;\partial_t^l u,\partial_t^l\theta)+\int_0^t e^{\gamma t}\int_{\partial G}\left|\partial_t^{l+1}\frac{\partial u}{\partial\vec{n}}\right|^2 dxds\\ &\le\Gamma\mathcal{E}(0;\partial_t^l u,\partial_t^l\theta)+\mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g),\quad l=0,1,2.\end{aligned} \tag{61}$$

We now want to estimate $\mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g)$ to get a priori estimates for all derivatives of u and θ up to order four with not more than order two in space. It is useful to divide

$\mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g)$ into six terms T_0–T_5:

$$\begin{aligned}\mathcal{P}(t;\partial_t^l u,\partial_t^l\theta,\partial_t^l f,\partial_t^l g) &= \underbrace{C\int_0^t e^{\gamma s}\{||\partial_t^l f||(||\partial_t^l u||+||\partial_t^l u_t||+||\Delta\partial_t^l u||+||\partial_t^l f||)}\\
&+\underbrace{||\partial_t^l g||(||\partial_t^l\theta||+||\nabla\partial_t^l\theta||+||\Delta\partial_t^l\theta||+||\partial_t^l g||)ds\}}_{=:T_0}+C\left\{\underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l f,\Delta\partial_t^l u_t\rangle ds\right|}_{=:T_1}\right.\\
&+\underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l f,\Delta\partial_t^l u_t\rangle ds\right|}_{=:T_2}+\underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l f_t,\sigma_k\partial_k\partial_t^l u_t\rangle ds\right|}_{=:T_3}+\underbrace{\left|\int_0^t e^{\gamma s}\langle\partial_t^l g_t,\partial_t^l\theta_t\rangle ds\right|}_{=:T_4}\\
&\left.+\sum_{i=1}^{l+1}\underbrace{\left|\int_0^t e^{\gamma s}\int_{\partial G}|\nabla\partial_t^i u|^4 ds\right|}_{=:T_5}\right\}.\end{aligned}\tag{62}$$

We have $T_0\le C\varepsilon^3+C\varepsilon^4$ because using (38) and denoting

$$\begin{aligned}A:=A(\nabla u,\theta) &:= (A_{ij}(\nabla u,\theta)-A_{ij}(0,0))_{ij},\\ \tilde{C}:=\tilde{C}(\nabla u,\theta) &:= (\tilde{C}_{ij}(\nabla u,\theta)-\tilde{C}_{ij}(0,0))_{ij},\end{aligned}$$

we get:

$$\begin{aligned}f_t &= \partial_t(A(\nabla u,\theta))\Delta u+A(\nabla u,\theta)\Delta u_t+\partial_t(\tilde{C}(\nabla u,\theta))\nabla\theta+\tilde{C}(\nabla u,\theta)\nabla\theta_t\\ &= A_{\nabla u}(\nabla u,\theta)\nabla u_t\Delta u+A_\theta(\nabla u,\theta)\theta_t\Delta u+A(\nabla u,\theta)\Delta u_t\\ &\quad+\tilde{C}_{\nabla u}(\nabla u,\theta)\nabla u_t\nabla\theta+\tilde{C}_\theta(\nabla u,\theta)\theta_t\nabla\theta+\tilde{C}(\nabla u,\theta)\nabla\theta_t.\end{aligned}$$

And if we differentiate this with respect to t, we get:

$$\begin{aligned}f_{tt} &= \partial_t(A_{\nabla u})\nabla u_t\Delta u+A_{\nabla u}\nabla u_{tt}\Delta u+A_\nabla u\nabla u_t\Delta u_t\\ &\quad+\partial_t(A_\theta)\theta_t\Delta u+A_\theta\theta_{tt}\Delta u+A_\theta\theta_t\Delta u_t\\ &\quad+A_{\nabla u}\nabla u_t\Delta u_t+A_\theta\theta_t\Delta u_t+A\Delta u_{tt}\\ &\quad+\partial_t(\tilde{C}_\nabla u)\nabla u_t\nabla\theta+\tilde{C}_{\nabla u}\nabla u_{tt}\nabla\theta+\tilde{C}_{\nabla u}\nabla u_t\nabla\theta_t\\ &\quad+\partial_t(\tilde{C}_\theta)\theta_t\nabla\theta+\tilde{C}_\theta\theta_{tt}\nabla\theta+\tilde{C}_\theta\theta_t\nabla\theta_t\\ &\quad+\tilde{C}_{\nabla u}\nabla u_t\nabla\theta_t+\tilde{C}_\theta\theta_t\nabla\theta_t+\tilde{C}\nabla\theta_{tt}.\end{aligned}\tag{63}$$

Using that f is continuous and applying (51), we get: $||A(\nabla u,\theta)||_\infty\le C(||\nabla u||_\infty+||\theta||_\infty)\le C\delta e^{-\frac{\gamma}{2}s}$ and also $||\tilde{C}(\nabla u,\theta)||_\infty\le C(||\nabla u||_\infty+||\theta||_\infty)\le C\delta e^{-\frac{\gamma}{2}s}$. We therefore are able to estimate in each of the terms all functions but one with respect to the $\mathcal{L}_\infty$–norm. So we can prove:

$$||f_{tt}||\le C\delta^2e^{-\gamma s}.$$

A similar calculation for g leads to:

$$T_0\le C\delta^3.$$

We now consider T_1. Here we only want to discuss the case $l = 2$. The terms of lower order make fewer difficulties.

Using the special form (51) of f, the symmetry of $A(\nabla u, \theta)$, the Leibniz formula, the mean theorem of differentiation, and the Cauchy–Schwarz inequality, we have:

$$\begin{aligned} T_1 &\leq C\delta^3 + \left| \int_0^t e^{\gamma s} \langle f_{tt}, \Delta u_{ttt} \rangle ds \right| \\ &\leq C\delta^3 + \left| \int_0^t e^{\gamma s} \langle A(\nabla u, \theta) \partial_t^2 \Delta u, \partial_t^3 \Delta u \rangle ds \right| \\ &\leq C\delta^3 + \frac{1}{2} \left| \int_0^t e^{\gamma s} \frac{d}{dt} \langle A(\nabla u, \theta) \partial_t^2 \Delta u, \partial_t^2 \Delta u \rangle ds \right| \\ &\leq \frac{1}{2} e^{\gamma t} \left| \langle A(\nabla u, \theta) \partial_t^2 \Delta u, \partial_t^2 \Delta u \rangle \right| \\ &\quad + \frac{\gamma}{2} \left| \int_0^t e^{\gamma s} \langle A(\nabla u, \theta) \partial_t^2 \Delta u, \partial_t^2 \Delta u \rangle ds \right| \\ &\leq C\delta^3. \end{aligned}$$

To estimate T_2 we need the following lemma for radially symmetric functions:

LEMMA 3.8. Let $v = w(|x|)x$ be a radially symmetric function; then we have:

$$\int_{\partial G} |\nabla v|^2 \leq \int_{\partial G} \left| \frac{\partial v}{\partial \vec{n}} \right|^2 + C||v||_{H^1}^2.$$

Proof. Direct calculation and using the Sobolev trace theorem.

Now we can argue similarly to the estimate of T_1. Again we use the special form of f, integrate by parts, but now we use lemma 3.8 to estimate the boundary integrals. We arrive at:

$$T_2 \leq C\delta^3 + C\delta \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial u_{ttt}}{\partial \vec{n}} \right|^2 dx ds. \tag{64}$$

In a similar way we also estimate T_3 and get:

$$\begin{aligned} T_3 &\leq C\delta^3 + \left| \int_0^t e^{\gamma s} \langle A(\nabla u, \theta) \partial_t^3 \nabla u, \partial_t^4 \nabla u \rangle ds \right| \\ &\quad + C\delta \left| \int_0^t e^{\frac{\gamma}{2} s} \int_{\partial G} \partial_t^3 \nabla u \partial_t^4 u \vec{n} ds \right|. \end{aligned} \tag{65}$$

The boundary integral we have to estimate, using lemma 2.5 and corollary 2.12, as follows:

$$\begin{aligned} \int_{\partial G} \partial_t^3 \nabla u \partial_t^4 u \vec{n} &\leq C \left\{ \int_{\partial G} |\text{div}\, u_{ttt}|^2 + \int_{\partial G} |u_{tttt} \vec{n}|^2 \right\} + C||(\nabla u_{ttt}, u_{tttt})||^2 \\ &\leq C \frac{d}{dt} \int_G u_{tttt} \sigma_k \partial_k u_{ttt} + C||(u_{tttt}, \nabla u_{ttt}, \nabla \theta_{ttt})||^2 \\ &\quad + C \int_G f_{ttt} \sigma_k \partial_k u_{ttt}. \end{aligned} \tag{66}$$

Here we can use the estimate for T_2 and get, using Leibniz formula and the mean theorem of differentiation:

$$T_3 \leq C\delta^3 + C\delta^2 \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial}{\partial \vec{n}} u_{ttt} \right|^2 dx ds.$$

To estimate T_4 we use a similar method utilizing that θ vanishes on the boundary and get:

$$T_4 \leq C\delta^3.$$

To estimate the boundary terms of order four summarized in T_5, we have to use a very different and somewhat unusual method. The key observation is that in the linear and radially symmetric case we can estimate the derivatives of a function on the boundary by the function itself. Then we show that the nonlinear boundary condition for small data approximates the linear one, so we can get a similar estimate using the implicit function theorem.

For the proof we want to consider an easy case, so let $G := \mathcal{B}(0,1) \subset \mathbb{R}^2$. Let $v := \partial_t^m u$ be radially symmetric with $m = 0, 1, 2, 3$ and $v(x) = w(|x|)x$. Then on ∂G we have:

$$\partial_1 v_1 = w' x_1^2 + w, \qquad \partial_1 v_2 = w' x_1 x_2,$$
$$\partial_2 v_1 = w' x_1 x_2, \qquad \partial_2 v_2 = w' x_2^2 + w,$$

where we have defined $w' := w'(1)$ and $w := w(1)$.

We now consider the linear boundary condition $\mathcal{N}'\mathcal{S}\mathcal{D}u = 0$ on ∂G, e.g., in $x = (0, 1)$, where we have:

$$\mathcal{N}'\mathcal{S}\mathcal{D}v = \begin{pmatrix} n_1(\tau\partial_1 v_1 + \lambda\partial_2 v_2) + n_2(\mu\partial_1 v_2 + \mu\partial_2 v_1) \\ n_2(\lambda\partial_1 v_1 + \tau\partial_2 v_2) + n_1(\mu\partial_1 v_2 + \mu\partial_2 v_1) \end{pmatrix} = \begin{pmatrix} \tau(w' + w) + \lambda w \\ 0 \end{pmatrix}.$$

If v satisfies the linear boundary condition it follows:

$$\tau w' + (\tau + \lambda) w = 0. \tag{67}$$

So we could estimate w' by w. We now want to transfer this for the nonlinear case. Therefore we define the following functions for $x = (x_1, x_2) \in \partial G$:

$$g : \mathbb{R}^2 \to \mathbb{R}^4, \quad g(a, b) := \left(a x_1^2 + b, a x_1 x_2, a x_1 x_2, a x_2^2 + b\right),$$
$$f_i : \mathbb{R}^2 \to \mathbb{R}, \quad f_i(a, b) := (\vec{n} S(g(a, b), 0))_{i=1,2},$$

where $\vec{n}$ is the normal in x.

Obviously g is in $\mathcal{C}^\infty$ and f_i is also smooth, if we assume S to be smooth. We have g defined such that $g(w', w) = \nabla v$.

Remembering $M(t) \leq \Lambda\delta^2$ we have: $||v(t)||_{H^1}^2 = ||\partial_t^m u(t)||_{H^1}^2 \leq \Lambda\delta^2$ for $t \in [0, T^*)$. Using the Sobolev trace theorem we get: $\int_{\partial G} |v|^2 \leq C\Lambda\delta^2$.

Using the radial symmetry of u we have $|v|^2$ locally constant on ∂G (and constant for $G = \mathcal{B}(0, 1)$), so we have:

$$\sup_{x \in \partial G} |v(x,t)|^2 = ||\partial_t^m u(t)||_{\mathcal{L}^\infty(\partial G)}^2 \leq C\Lambda\delta^2 =: K\delta^2 \quad \text{for} \quad t \in [0, T^*).$$

Now we consider the nonlinear boundary condition $f_i(a, b) = 0$ in $x = (1, 0)$. We know:

$$|\mathcal{N}'\mathcal{S}\mathcal{D}v - \vec{n} S(v, 0)| = \mathcal{O}(|\nabla v|^2) \quad \text{for } |\nabla v| \to 0 \text{ and } \theta = 0.$$

Together with (67) we get:

$$\tau a + (\tau + \lambda)b - f_1(a, b) = \mathcal{O}(a^2 + b^2).$$

We calculate the derivative of f_1 with respect to the first argument in $(0,0)$. Let $\psi(a) := f_1(a,0)$, then we have:

$$\begin{aligned} \partial_1 f_1(a,b)\,|_{a=0,\,b=0} &= \psi'(0) = \lim_{h\to 0} \frac{\psi(h) - \psi(0)}{h} \\ &= \lim_{h\to 0} \frac{\tau h + \mathcal{O}(h^2)}{h} = \tau > 0. \end{aligned}$$

Using $f_1 \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ we know that there exists a $U(0) \subset \mathbb{R}$, such that: $\partial_1 f_1(0,b) \neq 0$ for $b \in U(0)$. Furthermore, we have: $f_1(0,0) = 0$. So we can use the implicit function theorem for f_1, and therefore there exists a $\phi \in \mathcal{C}^1(U(0), \mathbb{R})$, such that $\phi(b) = a$ with $f_1(a,b) = 0$ for $b \in U(0)$. If we choose δ such that $[-K\delta^2, +K\delta^2] \subset U(0)$, then we can estimate T_5 as follows:

$$\begin{aligned} \int_{\partial G} |\partial_t^m \nabla u|^4 &= \left| \int_{\partial G} |w'|^2 + 2w'w + |2w|^2 \right|^2 \leq C \left| \int_{\partial G} |\underbrace{w'}_{=\phi(w)}|^2 \right|^2 + C \left| \int_{\partial G} |w|^2 \right|^2 \\ &\leq C \left| \int_{\partial G} |w|^2 \right|^2 + C \left| \int_{\partial G} |w|^2 \right|^2 \leq C \left| ||\partial_t^m u||_{H^1}^2 \right|^2 \leq C ||\partial_t^m u||_{H^1}^4 \leq C\delta^4. \end{aligned}$$

One can check that this proof is also applicable for other radially symmetric domains.

Summarizing the estimates for T_0–T_5 we get using (45):

$$\begin{aligned} &\mathcal{E}(t; \partial_t^l u, \partial_t^l \theta) + \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial}{\partial \vec{n}} \partial_t^{l+1} u \right|^2 dx ds \\ &\leq \Gamma\delta^2 + C\delta^3 + (C\delta + C\delta^2) \int_0^t e^{\gamma s} \int_{\partial G} \left| \frac{\partial}{\partial \vec{n}} \partial_t^{l+1} u \right|^2 dx ds. \end{aligned}$$

For small δ we get with $t \in [0, T^*)$:

$$\sum_{l=0}^{2} \mathcal{E}(t; \partial_t^l u, \partial_t^l \theta) \leq 3\Gamma\delta^2 + C\delta^3. \tag{68}$$

So we have estimated all derivatives up to order four with order one or two in space. To conclude we have to estimate the missing derivatives of order three and four in space.

Using the elliptic regularity property (see (49)), the differential equation (36), and (68), we get:

$$\begin{aligned} ||u_t||_{H^3} &\leq \tilde{\Gamma} ||\Delta u_t(t)||_{H^1}^2 + C\delta^3 \\ &\leq \frac{3(\beta^2 + 1)\tilde{\Gamma}}{\tau^2} (||u_{ttt}||_{H^1}^2 + ||\nabla \theta_t||_{H^2}^2 + ||f_t||_{H^1}^2) + C\delta^3 \\ &\leq \frac{3}{86} 258\Gamma(\beta^2 + c^2 + 1)^3 \tilde{\Gamma} \sum_{j=0} 2\tau^{-2j}\delta^2 = \frac{3}{86}\Lambda\delta^2. \end{aligned} \tag{69}$$

Here we have also used that $f(\nabla u, \theta, \nabla^2 u, \nabla\theta) = \mathcal{O}(|(\nabla u, \theta, \nabla^2 u, \nabla\theta)|^2)$. Using (37) we get in a similar manner:

$$\begin{aligned} ||\theta||_{H^4}^2 + ||\theta_t||_{H^3}^2 &\leq \tilde{\Gamma}(||\Delta\theta||_{H^2}^2 + ||\Delta\theta_t||_{H^1}^2) \\ &\leq \frac{3\tilde{\Gamma}(c^2+\beta^2+1)}{\kappa^2}(||u_t|_{H^3}^2 + 3\Gamma\delta^2 + C\delta^3). \end{aligned}$$

Here we have used that $g(\nabla u, \theta, \nabla^2 u, \nabla^2\theta, \nabla u_t) = \mathcal{O}((\nabla u, \theta, \nabla^2 u, \nabla^2\theta, \nabla u_t)^2)$. We now use the inequality (69) and the definition of Λ to derive:

$$\begin{aligned} ||\theta||_{H^4}^2 + ||\theta_t||_{H^3}^2 &\leq \frac{3\tilde{\Gamma}(c^2+\beta^2+1)}{\kappa^2}\left(3\Gamma\delta^2\frac{3(\beta^2+1)\tilde{\Gamma}}{\tau^2} + 3\Gamma\delta^2\right) + C\delta^3 \\ &\leq \frac{9}{86}\Lambda\delta^2 + C\delta^3. \end{aligned} \tag{70}$$

For the last term we use (49), (36), and (70) to get:

$$||u||_{H^4}^2 \leq \frac{27}{86}\Lambda\delta^2 + C\delta^3. \tag{71}$$

Summarizing (68), (69), (70), and (71), we arrive at:

$$\begin{aligned} M(t) &\leq \frac{39}{86}\Lambda\delta^2 + 3\Gamma\delta^2 + C\delta^3 \\ &\leq \frac{1}{2}\Lambda\delta^2 + C\delta^3. \end{aligned}$$

Choosing $\delta > 0$ small, we finally get:

$$M(t) \leq \frac{5}{6}\Lambda\delta^2 < \Lambda\delta^2 \quad \text{for all } t \in [0, T^*).$$

Using the continuity of M, we get the inequality $M(T^*) < \Lambda\delta^2$ by taking the limit $t \to T^*$. This is a contradiction to the definition of T^* in (50). So we have proved 3.4. □

4. An elliptic regularity property. To use the elliptic regularity property for the Neumann boundary condition in (49), we apply theorem 4.4 of [12]. In our situation we get for $j = 0, 1, 2$: For all $v \in H^{j+2}(G)$ there exist $\tilde{\Gamma} > 0$ and $C > 0$ with:

$$||v||_{H^{j+2}}^2 \leq \tilde{\Gamma}||\Delta v||_{H^j}^2 + C||\mathcal{N}'\mathcal{S}\mathcal{D}v||_{H^{j+1/2}(\partial G)}^2. \tag{72}$$

In the nonlinear case the boundary term is non-zero. To estimate it we use (40) in lemma 3.2 and apply the trace theorem $H^{m+1}(G) \hookrightarrow H^{m+1/2}(\partial G)$ and Eq. 3.2 to derive:

$$\begin{aligned} ||\mathcal{N}'\mathcal{S}\mathcal{D}v||_{H^{j+1/2}(\partial G)}^2 &= ||\mathcal{N}'\mathcal{S}\mathcal{D}v - \mathcal{N}'\vec{\beta}\theta||_{H^{j+1/2}(\partial G)}^2 \\ &\leq ||n_j(C_{ij\alpha\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u, \theta))||_{H^{j+1/2}(\partial G)}^2 \\ &\leq C||\underbrace{C_{ij\alpha\beta}(0,0)\partial_\alpha u_\beta - \tilde{C}_{ij}(0,0)\theta - S_{ij}(\nabla u, \theta)}_{=:\phi(\nabla u,\theta) = \mathcal{O}(|\nabla u|^2 + |\theta|^2)}||_{H^{j+1/2}(\partial G)}^2 \\ &\leq C||\phi(\nabla u, \theta)||_{H^{j+1}(G)}^2 \\ &= C\sum_{m=0}^{j+1}\int_G |\nabla^m \phi(\nabla u, \theta)|^2. \end{aligned} \tag{73}$$

Using $|\phi(\nabla u,\theta)| = \mathcal{O}(|\nabla u|^2+|\theta|^2)$ we know that $\phi(0,0)=\phi'(0,0)=0$. Now we calculate the terms of (73) utilizing (47) and (51):
For the term with $m=0$ we get:

$$\begin{aligned}\int_G |\phi(\nabla u,\theta)|^2 &\leq C\int_G |\nabla u|^4 + C\int_G |\theta|^4\\ &\leq C||\nabla u||_\infty^2||\nabla u||^2 + C||\theta||_\infty^2||\theta||^2\\ &\leq C||u||_{H^3}^2||\nabla u||^2 + C||\theta||_{H^2}^2||\theta||^2\\ &\leq C\delta^4.\end{aligned}$$

The cases $m=1$ and $m=2$ are very similar. For $m=3$ we have:

$$\begin{aligned}\int_G |\nabla^3\phi(\nabla u,\theta)|^2 &\leq C\int_G |\phi'''(\nabla u,\theta)\nabla^2 u\nabla^2 u\nabla^2 u|^2 + C\int_G |3\phi''(\nabla u,\theta)\nabla^3 u\nabla^2 u|^2\\ &\quad +C\int_G |\phi'(\nabla u,\theta)\nabla^4 u|^2 + C\int_G |\phi'''(\nabla u,\theta)\nabla\theta\nabla\theta\nabla\theta|^2\\ &\quad +C\int_G |3\phi''(\nabla u,\theta)\nabla^2\theta\nabla\theta|^2 + C\int_G |\phi'(\nabla u,\theta)\nabla^3\theta|^2\\ &\leq C||\nabla^2 u||_\infty^4||\nabla^2 u||^2 + C||\nabla^2 u||_\infty^2||\nabla^3 u||^2 + C||\phi'||_\infty^2||\nabla^4 u||^2\\ &\quad +C||\nabla\theta||_\infty^4||\nabla\theta||^2 + C||\nabla\theta||_\infty^2||\nabla^2\theta||^2 + C||\phi'||_\infty^2||\nabla^3\theta||^2\\ &\leq C||u||_{H^4}^4||u||_{H^2}^2 + C||u||_{H^4}^2||u||_{H^3}^2 + C||u||_{H^1}^2||u||_{H^4}^2 + C||\theta||^2||u||_{H^4}^2\\ &\quad +C||\theta||_{H^3}^4||\theta||_{H^1}^2 + C||\theta||_{H^3}^2||\theta||_{H^2}^2 + C||u||_{H^1}^2||\theta||_{H^3}^2 + C||\theta||^2||\theta||_{H^3}^2\\ &\leq C\delta^4 + C\delta^6.\end{aligned}$$

Summarizing these estimates (for small $\delta>0$) we arrive at the following elliptic regularity property:

$$||u||_{H^{j+2}}^2 \leq \tilde{\Gamma}_1||\Delta u||_{H^j}^2 + C\delta^4, \quad j=0,1,2.$$

For u_t we derive in the same way:

$$||u_t||_{H^{j+2}}^2 \leq \tilde{\Gamma}_2||\Delta u_t||_{H^j}^2 + C\delta^4, \quad j=0,1.$$

We now define $\tilde{\Gamma} := \max\{\tilde{\Gamma}_1,\tilde{\Gamma}_2\}$ to get the estimates used in Sec. 3.

Now we prove (60); see Sec. 3. Starting with (72) we only have to show that in the radially symmetric case:

$$||\mathcal{N}'\mathcal{S}\mathcal{D}u||_{H^{1/2}(\partial G)}^2 \leq C\int_{\partial G}|\nabla u|^4.$$

We show this by explicitly parametrizing ∂G and calculating the $H^1(\partial G)$–norm of $\mathcal{N}'\mathcal{S}\mathcal{D}u$.

By an extensive calculation where we use the radial symmetry of u, we obtain the desired estimate. Together with (72), we finally arrive at:

$$||u||_{H^2}^2 \leq C||\Delta u||^2 + C\int_{\partial G}|\nabla u|^4, \tag{74}$$

proving (60). □

References

[1] D. E. Carlson, *Linear thermoelasticity*, Handbuch der Physik VIa/2, Springer-Verlag, Berlin, 1972, 297–346
[2] W. Dan, *On a Local in Time Solvability of the Neumann Problem of Quasilinear Hyperbolic Parabolic Coupled Systems*, Math. Meth. Appl. Sci. 18, 1053–1082, 1995
[3] S. Jiang, J. E. Muñoz Rivera, R. Racke, *Asymptotic stability and global existence in thermoelasticity with symmetry*, Quart. Appl. Math., LVI, 259, 1998
[4] S. Jiang, R. Racke, *Evolution Equations in Thermoelasticity*, Chapman & Hall/CRC, 2000
[5] H. Koch, *Slow decay in linear thermoelasticity*, Quart. Appl. Math. LVIII, no. 4, 601–612, 2000
[6] R. Leis, *Initial boundary value problems in mathematical physics*, Teubner-Verlag, Stuttgart; John Wiley & Sons, Chichester et al., 1986
[7] S. Mizohata, *The theory of partial differential equations*, Cambridge Univ. Press, 1973
[8] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, tome 1, Masson, Paris, 1988
[9] R. Racke, *Exponential Decay for a Class of Initial Boundary Value Problems in Thermoelasticity*, Comp. Appl. Math. 12, 67–80, 1993
[10] M. O. Rieger, *Asymptotisches Verhalten radialsymmetrischer Lösungen von Thermoelastizitätsgleichungen (Asymptotic Behaviour of Radially Symmetric Solutions for Thermoelasticity Equations)*, diploma thesis, University of Konstanz, 1998
[11] J. Smoller, *Shock Waves and Reaction–Diffusion Equations*, Springer Verlag, New York et al., 1994
[12] Y. Shibata, G. Nakamura, *On a Local Existence Theorem of Neumann Problem for Some Quasilinear Hyperbolic Systems of 2nd Order*, Mathematische Zeitschrift 202, 1–64, 1989