Exponential Stability and Local ISS for DC Networks

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Exponential Stability and Local ISS for DC Networks

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***Abstract*—In this letter, we consider the problem of regulating the voltage of an islanded Direct Current (DC) network subject to (i) *unknown* ZIP-loads, i.e., nonlinear loads with the parallel combination of constant impedance (Z), current (I) and power (P) components, and (ii) *unknown* time-varying disturbances. Using the port-Hamiltonian framework, two decentralized passivity-based control schemes are designed. It is shown that, using the proposed controllers, the desired equilibrium is exponentially stable and local input-to-state stable (LISS) with respect to unknown *time-varying* disturbances.**

***Index Terms*—Power systems, distributed control, stability of nonlinear systems, Lyapunov methods.**

I. Introduction

Power networks can generally be classified into Direct Current (DC) and Alternating Current (AC) networks. As a consequence of the wide use of renewable energy sources and the technological development in the field of power electronics, the design and operation of DC networks is nowadays attracting growing interest and receiving research attention. Indeed, DC networks are generally more efficient and reliable than AC networks [1], reducing lossy conversion stages and overcoming frequency and reactive power control. For these reasons, DC networks are (recently) deployed in aircraft, trains, ships, charging stations and data centres.

In order to guarantee a proper functioning of the connected loads, the main control objective in (islanded) DC networks is to stabilize the network voltage at the desired value [2]. Several controllers based on different techniques have been proposed in the literature, e.g., droop [3], plug-and-play [4], sliding mode [5] and passivity-based [6] controllers. Moreover, in [7], [8] an input-to-state stability (ISS)-like Lyapunov function is obtained and used for control design. However, these works do not include P-loads and, some of them, neglect the network or consider it purely resistive. In order to address the voltage destabilizing effect of the negative incremental impedance introduced by P-loads, several controllers have been proposed in the literature, including energy-based approaches [9]–[12]. More specifically, in [9], the authors provide a suitable port-Hamiltonian (pH) framework [13] to model electrical circuits including constant ZIP-loads and investigate their shifted passivity properties. In [10], the authors show that the controllers proposed in [4] passivate the generation and constant ZIP-load units of a DC network. In [11], the authors establish new passivity properties leading to the design of a voltage controller that is robust with respect to constant ZIP-loads. In [12], a systematic and constructive design based on the pH framework is proposed. We note that all these works provide stability guarantees only in presence of constant load components, while loads are in practice *time-varying*.

In this letter, inspired by [14], we propose a unified approach to control design and analysis based on the pH framework. This approach leads to a simple decentralized control structure. We nominally treat the unknown IP-loads as constant unmatched disturbances and the Z-loads as unknown constant parameters of the damping structure of the system. Then, in analogy with [9], we shift the considered DC network with respect to the desired voltage and propose two different decentralized voltage control schemes. The first controller employs an integral action that rejects the unknown and constant unmatched disturbances, ensuring that the desired equilibrium corresponding to zero steady-state voltage error is asymptotically stable. Then, motivated by the *time-varying* nature of loads in practice, we propose a second control scheme which differs from the first by the addition of a damping term into the integrator dynamics. This additional damping term ensures *exponential* stability of the desired equilibrium and local ISS (LISS) [15] of the closed-loop system with respect to unknown *time-varying* disturbances acting on any of the state dynamics. These disturbances may represent, for instance, *time-varying* components of the loads or imperfections in the control signal.

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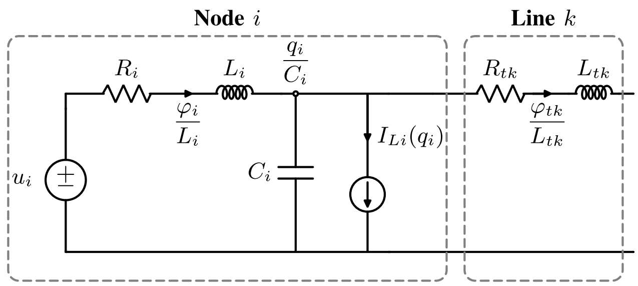

Fig. 1. Electrical scheme of node $i \in \mathcal{V}$ and transmission line $k \in \mathcal{E}$, where $I_{Li}(q_i) := G^{\star}_{Li}\frac{q_i}{C_i} + I^{\star}_{Li} + \frac{C_i}{q_i}P^{\star}_{Li}$.

The main contributions of this letter can be summarized as follows:

C.1 Differently from [10], we present a unified framework for control design and analysis whilst maintaining a simple control structure.

C.2 Differently from [12], where the control law comprises a model-based disturbance compensation that, if omitted, compromises the performance (see [12, Remarks 3 and 4]), we consider unknown load components.

C.3 Differently from [3]–[12] and other relevant works on the topic, we prove LISS of the closed-loop system with respect to unknown *time-varying* disturbances acting on any of the state dynamics. This could represent, for example, a time-varying component of a supplied ZIP-load.

Notation: Function arguments are declared upon definition and are omitted for subsequent use. $0_{n\times m}$ denotes a $n \times m$ zero matrix whereas I_n is a $n \times n$ identity matrix. For $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$, $P > 0$ we define $\|x\|^2_P := x^\top P x$. For mappings $H : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ we denote the transposed gradients as $\nabla H := (\frac{\partial H}{\partial x})^\top$ and $\nabla_x G(x, y) := (\frac{\partial G}{\partial x})^\top$. For a real matrix $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{symm}(A) = \frac{1}{2}(A + A^\top)$. For a symmetric matrix $B \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$ denote the smallest and largest eigenvalue of B, respectively. Given a vector $v \in \mathbb{R}^n$, $\mathrm{diag}(v)$ is a $n \times n$ diagonal matrix and the diagonal elements are equal to the elements of v. Definitions of comparison functions $\mathcal{K}_\infty$ and $\mathcal{KL}$ follows [16].

II. Background and Problem Formulation

In this section, we introduce the considered islanded DC power network model and formulate the voltage control problem.

A. System Model

We consider a network represented by a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} = \{1, \dots, m\}$ are the sets of nodes and edges, respectively. The network topology is described by its corresponding incidence matrix $B \in \mathbb{R}^{n\times m}$. Then, the overall network[1] can be written as a port-Hamiltonian system of the form

$$\begin{bmatrix}\dot\varphi\\ \dot q\\ \dot\varphi_t\end{bmatrix} = \begin{bmatrix}-R & -I_n & 0_{n\times m}\\ I_n & -G^{\star}_L & B\\ 0_{m\times n} & -B^\top & -R_t\end{bmatrix}\nabla H + \begin{bmatrix}u\\ 0_{n\times 1}\\ 0_{m\times 1}\end{bmatrix}$$

$$- \begin{bmatrix}0_{n\times 1}\\ I^{\star}_L + \mathrm{diag}(C^{-1}q)^{-1}P^{\star}_L\\ 0_{m\times 1}\end{bmatrix} + \delta(t),$$

$$H(\varphi, q, \varphi_t) = \frac{1}{2}\|\varphi\|^2_{L^{-1}} + \frac{1}{2}\|q\|^2_{C^{-1}} + \frac{1}{2}\|\varphi_t\|^2_{L_t^{-1}}, \tag{1}$$

[1]The control input u_i may represent for instance the output average voltage of a buck converter. Then, for the practical implementation, the duty cycle of the converter i can be obtained from u_i.

TABLE I
Description of the Used Symbols

φ	Flux (node)	$G^{\star}_L$	Load conductance
q	Charge	$I^{\star}_L$	Load current
φ_t	Flux (transmission line)	$P^{\star}_L$	Load power
u	Control input	R, L, C	Filter
δ	Disturbance	R_t, L_t	Transmission line

where $\varphi, q \in \mathbb{R}^n$, $\varphi_t \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$. R, R_t, L, L_t, C are constant positive definite diagonal matrices of appropriate dimensions. $G^{\star}_L \in \mathbb{R}^{n\times n}$ is an unknown positive definite constant impedance and $I^{\star}_L, P^{\star}_L \in \mathbb{R}^n$ are unknown constant current and power loads. The term $\delta(t) = [\delta^\top_\varphi(t), \delta^\top_q(t), \delta^\top_{\varphi_t}(t)]^\top \in \mathbb{R}^{2n+m}$ is an unknown *time-varying* disturbance.

As the considered time-varying disturbance is very general, it could be used to model the effects of many practice imperfections. For example, if the true ZIP load is time varying, it could be decomposed into the sum of a constant and time-varying load. The time-varying component could then be considered in the term $\delta(t)$. Similarly, the term $\delta(t)$ can be used to model *time-varying* renewable energy injections or imperfections in the control signal.

B. Problem Formulation

The objective of this letter is to design a control scheme that regulates the voltage of the DC power network (1). As the loads are assumed to be unknown, the scheme is required to adapt the input voltage to compensate for these loads and losses in the network.

Problem statement: Considering the system (1), design a dynamic control law

$$\begin{aligned} u &= \hat u(\varphi, q, x_c)\\ \dot x_c &= \hat x_c(\varphi, q, x_c), \end{aligned} \tag{2}$$

where $x_c \in \mathbb{R}^n$ is the state of the controller, which ensures that $C^{-1}q \to V^{\star}$ for some desired constant voltage level $V^{\star} \in \mathbb{R}^n$. This objective can be equivalently addressed by regulating the charge to the set-point $q^{\star} = CV^{\star}$.

Note that in this letter we focus only on voltage regulation, which is vital to guarantee the safety of the network and a proper functioning of the connected loads [2]–[12]. The achievement of also current (or power) sharing (see for example [17] and the references therein) is beyond the scope of this letter and left as a future research work.

III. Voltage Regulation Controller

In this section, we propose a solution to the control problem formulated in Section II-B. First, using techniques similar to those used in [9], the system (1) is shifted with respect to the desired voltage level $C^{-1}q^{\star}$. The shifted system can be written as a perturbed pH system affected by an unknown constant unmatched disturbance and an unknown *time-varying* disturbance. Then, to compensate for these disturbances, we design an integral action controller similar to the structure proposed in [14]. Finally, the proposed integral action controller is extended to include a damping term in the integrator dynamics, resulting in improved stability performance.

A. Shifted System

The first stage of the development is to shift the system (1) with respect to the nominal voltage level $C^{-1}q^\star$. To do this, we need to provide an alternate representation of the constant P-load. This representation can be derived from [9, Lemma 1].

Lemma 1: The current resulting from a constant P-load can be written as

$$\begin{aligned} -\operatorname{diag}(C^{-1}q)^{-1}P_L^\star = X(q)C^{-1}(q-q^\star) \\ -\operatorname{diag}(C^{-1}q^\star)^{-1}P_L^\star, \end{aligned} \tag{3}$$

where

$$X(q) := \operatorname{diag}(C^{-1}q^\star)^{-1}\operatorname{diag}(P_L^\star)\operatorname{diag}(C^{-1}q)^{-1}. \tag{4}$$

Proof: The proof directly follows from [9, Lemma 1]. ■

Using the representation (3), the system (1) will now be shifted about the desired voltage level $C^{-1}q^\star$.

Proposition 1: Let $G_L^{\mathrm{eq}}(q) := G_L^\star - X(q)$. Consider the system (1) with the input

$$u = C^{-1}q^\star + v. \tag{5}$$

Defining the vector $\Phi \in \mathbb{R}^m$ as

$$\Phi = -R_t^{-1}B^\top C^{-1}q^\star, \tag{6}$$

the closed-loop dynamics can be written as a perturbed port-Hamiltonian system, i.e.,

$$\begin{aligned} \begin{bmatrix}\dot\varphi\\ \dot q\\ \dot\varphi_t\end{bmatrix} &= \begin{bmatrix}-R & -I_n & 0_{n\times m}\\ I_n & -G_L^{\mathrm{eq}}(q) & B\\ 0_{m\times n} & -B^\top & -R_t\end{bmatrix}\nabla\tilde H + \begin{bmatrix} v\\ 0_{n\times 1}\\ 0_{m\times 1}\end{bmatrix} \\ &\quad - \begin{bmatrix}0_{1\times n} & \Lambda^\top & 0_{1\times n}\end{bmatrix}^\top + \delta(t), \\ \tilde H &= \frac12\|\varphi\|^2_{L^{-1}} + \frac12\left\|q-q^\star\right\|^2_{C^{-1}} + \frac12\|\varphi_t - L_t\Phi\|^2_{L_t^{-1}}, \\ \Lambda &= I_L^\star + \operatorname{diag}(C^{-1}q^\star)^{-1}P_L^\star + G_L^\star C^{-1}q^\star - B\Phi. \end{aligned} \tag{7}$$

Proof: The proof follows from direct matching between (7) and (1). Considering the dynamics of φ_t in (7) we have

$$\begin{aligned} \dot\varphi_t &= -B^\top C^{-1}(q-q^\star) - R_t(L_t^{-1}\varphi_t - \Phi) + \delta_{\varphi_t}(t) \\ &= -B^\top C^{-1}q - R_tL_t^{-1}\varphi_t + \delta_{\varphi_t}(t), \end{aligned} \tag{8}$$

which agrees with (1). Now, consider the dynamics of q in (7) which can be rearranged as

$$\begin{aligned} \dot q &= L^{-1}\varphi - G_L^\star C^{-1}(q-q^\star) + X(q)C^{-1}(q-q^\star) \\ &\quad + B(L_t^{-1}\varphi_t - \Phi) - I_L^\star - \operatorname{diag}(C^{-1}q^\star)^{-1}P_L^\star \\ &\quad - G_L^\star C^{-1}q^\star + B\Phi + \delta_q(t) \\ &= L^{-1}\varphi - G_L^\star C^{-1}q + BL_t^{-1}\varphi_t \\ &\quad - I_L^\star - \operatorname{diag}(C^{-1}q)^{-1}P_L^\star + \delta_q(t) \end{aligned} \tag{9}$$

which agrees with (1). Verification of the φ dynamics follows from similar argument. ■

The feed-forward control (5) simply sets the nominal voltage for the DC network. In the absence of losses or loads, this would be sufficient to regulate the network voltage. However, due to the network loads, it is not clear if the voltage $V^\star$ is achieved. Our focus is now to utilise the additional control input v to design a feedback controller to ensure that the network voltage is stabilised at the desired voltage level.

B. Integral Action Control Law

The shifted system (7) has a pH structure and is subject to the unknown constant unmatched disturbance Λ and the unknown *time-varying* disturbance $\delta(t)$. In this section, we design a simple feedback control law to asymptotically stabilise the network voltage at the desired voltage level in the absence of the *time-varying* disturbance, i.e., $\delta(t) = 0_{(2n+m)\times 1}, \forall t \geq 0$. For this development, we follow the integral action design for constant unknown unmatched disturbances proposed in [14].

Proposition 2: Consider the electrical system (7) with $\delta(t) = 0_{(2n+m)\times 1}$, in closed-loop with the control law

$$\begin{aligned} \dot x_c &= -\beta C^{-1}(q-q^\star) \\ v &= -\beta R K_i(\beta\varphi - x_c), \end{aligned} \tag{10}$$

where $\beta \in \mathbb{R}_+$ and $K_i = K_i^\top > 0$ are tuning parameters. The resulting closed-loop dynamics have the form

$$\begin{aligned} \begin{bmatrix}\dot\varphi\\ \dot q\\ \dot\varphi_t\\ \dot x_c\end{bmatrix} &= \begin{bmatrix}-R & -I_n & 0_{n\times m} & 0_{n\times n}\\ I_n & -G_L^{\mathrm{eq}}(q) & B & \beta I\\ 0_{m\times n} & -B^\top & -R_t & 0_{m\times n}\\ 0_{n\times n} & -\beta I & 0_{n\times m} & 0_{n\times n}\end{bmatrix}\nabla\mathcal{W} \\ \mathcal{W} &= \frac12\|\varphi - L\Lambda\|^2_{L^{-1}} + \frac12\left\|q-q^\star\right\|^2_{C^{-1}} \\ &\quad + \frac12\|\varphi_t - L_t\Phi\|^2_{L_t^{-1}} + \frac12\left\|\beta\varphi - x_c + \frac1\beta K_i^{-1}\Lambda\right\|^2_{K_i}. \end{aligned} \tag{11}$$

Proof: The closed-loop dynamics (11) can be verified to match (7) with the control law (10) by direct matching. First consider the dynamics of φ in (11) that can be rearranged as

$$\begin{aligned} \dot\varphi &= -R\left[L^{-1}(\varphi - L\Lambda) + \beta K_i(\beta\varphi - x_c + \frac1\beta K_i^{-1}\Lambda)\right] \\ &\quad - C^{-1}(q-q^\star) \\ &= -RL^{-1}\varphi - C^{-1}(q-q^\star) + v, \end{aligned} \tag{12}$$

which agrees with (7). Now, considering the behaviour of q in (11), we find

$$\begin{aligned} \dot q &= \left[L^{-1}(\varphi - L\Lambda) + \beta K_i(\beta\varphi - x_c + \frac1\beta K_i^{-1}\Lambda)\right] \\ &\quad - G_L^{\mathrm{eq}}(q)C^{-1}(q-q^\star) + BL_t^{-1}(\varphi_t - L_t\Phi) \\ &\quad - \beta K_i(\beta\varphi - x_c + \frac1\beta K_i^{-1}\Lambda) \\ &= L^{-1}\varphi - \Lambda - G_L^{\mathrm{eq}}(q)C^{-1}(q-q^\star) \\ &\quad + BL_t^{-1}(\varphi_t - L_t\Phi), \end{aligned} \tag{13}$$

which agrees with (7). The dynamics of φ_t and x_c can be easily seen to match by similar analysis. ■

Remark 1: Noting that R is diagonal, the control law (10) is decentralised provided that K_i is chosen to be diagonal.

Remark 2: It is clear that the control law (10) includes an integrator on the network voltage error. However, rather than directly using the integral state for feedback control, the control signal mixes the integral state with φ. It is precisely this coupling between states that allows for the closed-loop system to be written as a pH system.

The closed-loop system has now been expressed as an autonomous pH system (11). Moreover, the closed-loop energy $\mathcal{W}$ has a unique minimum at

$$(\varphi, q, \varphi_t, x_c) = (L\Lambda, q^\star, L_t\Phi, \beta L\Lambda + \frac{1}{\beta}K_i^{-1}\Lambda). \tag{14}$$

Stability of the point (14) depends on the positivity of the *equivalent* load conductance $G_L^{\mathrm{eq}}(q)$, which is formalised in the following proposition.

Proposition 3: Consider the closed-loop system (11) which has an equilibrium point (14). If there exists a neighbourhood of $q^\star$ such that $G_L^{\mathrm{eq}}(q) > 0$, the point (14) is asymptotically stable.

Proof: The derivative of $\mathcal{W}$ evaluated along the dynamics (11) satisfies

$$\begin{aligned}\dot{\mathcal{W}} = &-\nabla_\varphi^\top \mathcal{W} R \nabla_\varphi \mathcal{W} - \nabla_q^\top \mathcal{W} G_L^{\mathrm{eq}}(q)\nabla_q \mathcal{W} \\ &- \nabla_{\varphi_t}^\top \mathcal{W} R_t \nabla_{\varphi_t}\mathcal{W}.\end{aligned} \tag{15}$$

If there exists a neighbourhood of (14) such that $G_L^{\mathrm{eq}}(q) > 0$, then $\dot{\mathcal{W}} \le 0$ on the same neighbourhood. As $\mathcal{W}$ is minimised at (14), there exists positively invariant set containing (14). Asymptotic stability follows from application of LaSalle's theorem [16]. ■

C. Performance Improvement: Integral Action With Damping Injection

Although the closed-loop dynamics (11) are well-behaved, they are not necessarily robust against the unknown *time-varying* disturbance $\delta(t)$, which may represent imperfections in the control signal or *time-varying* load components. This lack of guaranteed robustness is due to the fact that $\mathcal{W}$ is not a strict Lyapunov function for the closed-loop dynamics (11). In this subsection, the integral action law is extended to add a damping term into the integrator dynamics, which will ensure that $\mathcal{W}$ is indeed a strict Lyapunov function. Consequently, this development allows verification of stronger stability properties of the closed-loop system. Specifically, we will prove *exponential* stability of the desired equilibrium point (14) if $\delta(t) = 0_{(2n+m)\times 1}, \forall t \ge 0$, and local input-to-state stability [15] of the closed-loop system with respect to any bounded *time-varying* disturbance $\delta(t)$.

Proposition 4: Consider the electrical system (7) in closed-loop with the control law

$$\begin{aligned}x_c &= z - \frac{1}{\beta}K_d(q - q^\star) \\ \dot{z} &= \frac{1}{\beta}K_d L^{-1}\varphi - \beta C^{-1}(q - q^\star) + K_d K_i(\beta\varphi - x_c) \\ v &= -\beta R K_i(\beta\varphi - x_c),\end{aligned} \tag{16}$$

where $\beta \in \mathbb{R}_+$, $K_i = K_i^\top > 0$ and $K_d = K_d^\top > 0$ are tuning parameters. The resulting closed-loop dynamics have the form

$$\begin{bmatrix}\dot{\varphi} \\ \dot{q} \\ \dot{\varphi}_t \\ \dot{x}_c\end{bmatrix} = F_{cl}\nabla\mathcal{W} + \delta_e(t),$$

$$F_{cl} = \begin{bmatrix} -R & -I_n & 0_{n\times m} & 0_{n\times n} \\ I_n & -G_L^{\mathrm{eq}}(q) & B & \beta I_n \\ 0_{m\times n} & -B^\top & -R_t & 0_{n\times n} \\ 0_{n\times n} & \frac{1}{\beta}K_d G_L^{\mathrm{eq}}(q) - \beta I_n & -\frac{1}{\beta}K_d B & -K_d \end{bmatrix}, \tag{17}$$

where $\mathcal{W}$ is as in (11), and $\delta_e(t) = [\delta^\top(t),\ -\frac{1}{\beta}\delta_q^\top(t)K_d]^\top$.

Proof: Verification of the φ, q and φ_t dynamics follows identically to Proposition 2. The control signal (16) can be rearranged as

$$\begin{aligned}\dot{x}_c &= \dot{z} - \frac{1}{\beta}K_d\dot{q} \\ &= -\beta C^{-1}(q - q^\star) + K_d K_i\Big(\beta\varphi - x_c + \frac{1}{\beta}K_i^{-1}\Lambda\Big) \\ &\quad + \frac{1}{\beta}K_d G_L^{\mathrm{eq}}(q) C^{-1}(q - q^\star) \\ &\quad - \frac{1}{\beta}K_d B L_t^{-1}(\varphi_t - L_t\Phi) - \frac{1}{\beta}K_d\delta_q(t) \\ &= \Big[\frac{1}{\beta}K_d G_L^{\mathrm{eq}}(q) - \beta I\Big]\nabla_q\mathcal{W} - K_d\nabla_{x_c}\mathcal{W} \\ &\quad - \frac{1}{\beta}K_d B\nabla_{\varphi_t}\mathcal{W} - \frac{1}{\beta}K_d\delta_q(t),\end{aligned} \tag{18}$$

which agrees with (17), completing the proof. ■

Notice that the closed-loop dynamics (17) now have an additional damping term in the x_c coordinate. The existence of this term allows the verification of some stronger stability properties of the equilibrium point (14). Let

$$x := \left[\varphi^\top, q^\top, \varphi_t^\top, x_c^\top\right]^\top, \tag{19}$$

and denote the equilibrium point (14) by $x^\star$. We now recall the definition of local input-to-state stability (LISS).

Definition 1 (LISS [18]): Let $x_0 := x(0)$. The system (17) is LISS with respect to the disturbance δ_e if there exists $\rho^\delta, \rho^0 > 0, \gamma \in \mathcal{K}_\infty, \eta \in \mathcal{KL}$ such that $\forall \|x_0 - x^\star\| \le \rho^0$, $\forall \|\delta_e(t)\|_\infty \le \rho^\delta$, $\|x - x^\star\| \le \eta(\|x_0 - x^\star\|, t) + \gamma(\|\delta_e(t)\|_\infty)$.

In lay terms, LISS ensures that in some neighbourhood of the equilibrium point, the system will have a bounded response to bounded perturbations. In DC networks, the LISS property will ensure that the system state remains in some neighbourhood of (14), even if the loads include bounded *time-varying* components.

Theorem 1: Consider the closed-loop dynamics (17) with equilibrium point (14). If there exists a neighbourhood of $q^\star$ such that $G_L^{\mathrm{eq}}(q) > 0$ and if the tuning parameters β and K_d are chosen such that

$$K_d - \frac{1}{4\beta^2}K_d G_L^{\mathrm{eq}}(q) K_d - \frac{1}{4\beta^2}K_d B R_t^{-1} B^\top K_d > 0 \tag{20}$$

on the same neighbourhood, then:

1) the equilibrium point (14) is exponentially stable if $\delta(t) = 0_{(2n+m)\times 1}, \forall t > 0$.
2) The closed-loop system is LISS with respect to the disturbance $\delta_e(t)$ on some neighbourhood of the equilibrium (14).

Proof: To verify both of the claims, we will need to verify that the symmetric part of F_{cl}, defined in (17), is negative definite. To verify this, we define $D(x)$ as

$$\begin{aligned} D(x) &:= -\operatorname{symm}(F_{cl}) \\ &= \begin{bmatrix} R & 0_{n\times n} & 0_{n\times m} & 0_{n\times n} \\ 0_{n\times n} & G_L^{\mathrm{eq}}(q) & 0_{n\times m} & -\frac{1}{2\beta} G_L^{\mathrm{eq}}(q) K_d \\ 0_{m\times n} & 0_{m\times n} & R_t & \frac{1}{2\beta} B^\top K_d \\ 0_{n\times n} & -\frac{1}{2\beta} K_d G_L^{\mathrm{eq}}(q) & \frac{1}{2\beta} K_d B & K_d \end{bmatrix}. \end{aligned} \tag{21}$$

As $R > 0$, $D(x)$ is positive definite in some neighbourhood of $x^\star$ if the block matrix

$$D_{2,2}(x) := \begin{bmatrix} G_L^{\mathrm{eq}}(q) & 0_{n\times m} & -\frac{1}{2\beta} G_L^{\mathrm{eq}}(q) K_d \\ 0_{m\times n} & R_t & \frac{1}{2\beta} B^\top K_d \\ -\frac{1}{2\beta} K_d G_L^{\mathrm{eq}}(q) & \frac{1}{2\beta} K_d B & K_d \end{bmatrix} \tag{22}$$

is positive definite in some neighbourhood of $x^\star$. To verify that $D_{2,2}(x)$ is positive definite, we consider its Schur complement, i.e.,

$$K_d - \frac{1}{4\beta^2} K_d [G_L^\star - X(q)] K_d - \frac{1}{4\beta^2} K_d B R_t^{-1} B^\top K_d, \tag{23}$$

which is positive precisely when (20) holds. Thus, $D(x)$ is positive definite in some neighbourhood $\mathcal{N}_{x^\star}$ of the equilibrium point (14) and satisfies $D(x) > \kappa I_{3n+m}$, for some $\kappa > 0$, on the same neighbourhood.

To verify claim 1, notice that the function $\mathcal{W}$ in (17) can be written as

$$\mathcal{W}(x) = \frac{1}{2}\left\| x - x^\star \right\|_Q^2 \tag{24}$$

where $Q := A^\top \operatorname{diag}(L^{-1}, C^{-1}, L_t^{-1}, K_i) A$ and

$$A := \begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times m} & 0_{n\times n} \\ 0_{n\times n} & I_n & 0_{n\times m} & 0_{n\times n} \\ 0_{m\times n} & 0_{m\times n} & I_m & 0_{n\times n} \\ \beta I_n & 0_{n\times n} & 0_{n\times n} & -I_n \end{bmatrix}. \tag{25}$$

As $\operatorname{diag}(L^{-1}, C^{-1}, L_t^{-1}, K_i) > 0$ and Q is defined as a quadratic form, $Q > 0$ and $\mathcal{W}$ satisfies $\frac{1}{2}\lambda_{\min}(Q)\|x - x^\star\|^2 < \mathcal{W}(x) < \frac{1}{2}\lambda_{\max}(Q)\|x - x^\star\|^2$. The time derivative of $\mathcal{W}$ along the trajectories of (17) satisfies

$$\begin{aligned} \dot{\mathcal{W}} &= -\nabla^\top \mathcal{W} D(x) \nabla \mathcal{W} \\ &\le -\kappa \left(x - x^\star\right)^\top Q Q \left(x - x^\star\right) \\ &\le -\kappa \lambda_{\min}(QQ) \left\| x - x^\star \right\|^2. \end{aligned} \tag{26}$$

Exponential stability of (14) follows by application of [16, Th. 4.10].

Now to verify claim 2, we consider $\mathcal{W}$ as an ISS Lyapunov function for the dynamics (17). The time derivative of $\mathcal{W}$ along the trajectories of the system satisfies

$$\begin{aligned} \dot{\mathcal{W}} &= -\nabla^\top \mathcal{W} D(x) \nabla \mathcal{W} + \nabla^\top \mathcal{W} \delta_e(t) \\ &\le -\frac{\kappa}{2} \nabla^\top \mathcal{W} \nabla \mathcal{W} + \frac{1}{2\kappa} \delta_e^\top(t) \delta_e(t) \\ &\le -\frac{\kappa}{2} \lambda_{\min}(QQ) \left\| x - x^\star \right\|^2 + \frac{1}{2\kappa} \|\delta_e(t)\|_\infty^2 \\ &\le -\kappa \frac{\lambda_{\min}(QQ)}{\lambda_{\max}(Q)} \mathcal{W} + \frac{1}{2\kappa} \|\delta_e(t)\|_\infty^2, \end{aligned} \tag{27}$$

on the neighbourhood $\mathcal{N}_{x^\star}$, where κ is an arbitrary constant from application of Young's inequality. Let $\mathcal{X}_{\mathcal{W}}$ be a sub-level set of $\mathcal{W}$ contained within $\mathcal{N}_{x^\star}$ and let $\mathcal{W}_L$ be the value of $\mathcal{W}$ on the boundary of $\mathcal{X}_{\mathcal{W}}$. From (27) we have that $\dot{\mathcal{W}} < 0$ on the boundary of $\mathcal{X}_{\mathcal{W}}$ provided that

$$\|\delta_e(t)\|_\infty^2 < 2\kappa^2 \frac{\lambda_{\min}(QQ)}{\lambda_{\max}(Q)} \mathcal{W}_L := \rho^\delta. \tag{28}$$

Therefore, the sub-level set $\mathcal{X}_{\mathcal{W}}$ is positively invariant. The value of ρ^0 can be computed as $\rho^0 = \sqrt{\frac{2\mathcal{W}_L}{\lambda_{\min}(Q)}}$, which ensures that any initial condition satisfying $\|x - x^\star\| < \rho^0$ is contained within the sub-level set $\mathcal{X}_{\mathcal{W}}$. Expressions for the functions η, γ can be derived from the expression (27), but are omitted in the interest of space. Thus, we conclude that the closed-loop system (17) is LISS in some neighbourhood of the point $x^\star$. ■

Remark 3: Theorem 1 holds on some neighbourhood of $x^\star$ where $G_L^{\mathrm{eq}}(q) > 0$ and (20) is satisfied. The condition (20) can always be satisfied on an arbitrarily large region for sufficiently large K_d, β. Thus, the region of applicability of Theorem 1 is limited only by the region on which $G_L^{\mathrm{eq}}(q) > 0$. Furthermore, we notice that $G_L^{\mathrm{eq}}(q) > 0$ is only a sufficient condition that in the literature is commonly assumed to be satisfied to establish local stability properties of the considered system in presence of P-loads (see [9], [10], [19]). In practice this condition is verified if the voltage trajectories evolve in a neighbourhood of the corresponding reference values and the power absorbed by the Z-load is higher than the one absorbed by the P-load.

Remark 4: In case of only ZI-loads, i.e., $P_L^\star = 0_{n\times 1}$, the results developed in this section can be extended to hold globally. Indeed, in this case $X(q)$ in (4) is equal to zero and, as a consequence, the equivalent load conductance $G_L^{\mathrm{eq}}(q) = G_L^\star - X(q) = G_L^\star$ is always positive definite. The inequality (20) can then be satisfied uniformly for constant K_d, β.

IV. Example

In this section, the control schemes proposed in Section III are assessed in simulation. We consider an islanded DC network composed of 4 nodes in ring topology as shown in [20, Fig. 2]. For the parameters of each node and line we refer to [20, Tabs. 2 and 3]. For notation simplicity, let $V_i := \frac{q_i}{C_i}$ denote the voltage at node $i = 1, \ldots, 4$, with initial condition $V(0) = [370, 370, 390, 390]^\top$ V. The desired voltage level at each node is chosen equal to 380 V. The load components are as follows: $G_L^\star = \operatorname{diag}(0.08, 0.04, 0.05, 0.07)$ S, $I_L^\star = [10, 15, 10, 15]^\top$ A, and $P_L^\star = [10, 2, 6, 10]^\top$ kW. We consider $\delta(t) = 0_{(2n+m)\times 1}$ in the interval $0 \le t \le 1$ s and a step variation of the P-loads equal to $\Delta P_L^\star = [1, -2, 2, -1]^\top$ kW at the time instant $t = 0.5$ s. Then, we consider $\delta_{q1}(t) = -\frac{C_1}{q_1}(1.9\,(P_{L1}^\star + \Delta P_{L1}^\star) + P_{L1}(t))$, $P_{L1}(t) = 1.5\sin(5t)$ kW,

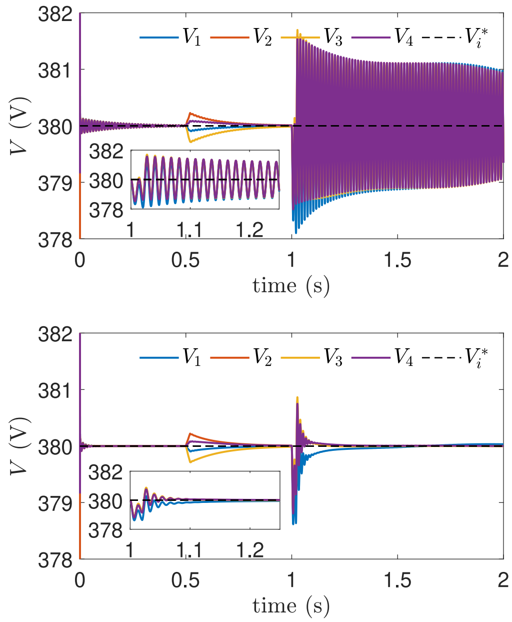

Fig. 2. Voltage at each node together with the desired value (dashed line). The performances of controllers (10) and (16) are illustrated in the top and bottom plot, respectively.

$\forall t \geq 1$ s, which is equivalent to consider a *time-varying* P-load in node 1. The controller parameters in (10) and (16) are $\beta = 10$, $K_i = 100 I_n$ and $K_d = 1000 I_n$. Figure 2 shows the time evolution of the voltage at each node of the network when (10) and (16) are applied, respectively. We can observe that in both cases the system response is bounded, even if we can guarantee this property only when (16) is applied (see Theorem 1). Moreover, according to Proposition 3 and Theorem 1, both the control laws are robust with respect to load step variations (see at $t = 0.5$ s). On the other hand, we notice that due to the difference between the voltage initial conditions and reference value and due to the *time-varying* P-load in node 1, the control law (10) induces an oscillatory behaviour, which is not desired in practical applications. Specifically, the *time-varying* P-load in node 1 induces large and high-frequency oscillations in all the nodes of the network when the control law (10) is applied. Differently, when the control law (16) is applied, it is evident that, due to the damping term injected into the integrator dynamics, the performance (e.g., in terms of oscillations and settling time) are excellent and definitely much better than the ones obtained by implementing (10).

V. Conclusion

In this letter, based on the port-Hamiltonian framework, we have proposed a simple decentralized control scheme to regulate the voltage in DC networks. We have proved exponential stability of the desired equilibrium and local input-to-state stability of the closed-loop system with respect to unknown *time-varying* disturbances. The limitations of this letter stem from the assumption that $G_L^\star(q) > 0$ and that the time-varying component of the disturbance satisfies (28). Future research will aim to relax these assumptions by considering injection of damping into the q coordinates and rejection of time-varying loads generated from known exo-systems.

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