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# EXPONENTIAL STABILITY OF THE SEMIGROUP ASSOCIATED WITH A THERMOELASTIC SYSTEM

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**Abstract.** In this paper it is proved that the semigroup associated with the one-dimensional thermoelastic system with Dirichlet boundary conditions is an exponentially stable $C_0$-semigroup of contraction on the space $H_0^1 \times L^2 \times L^2$. The technique of the proof is completely different from the usual energy method. It is shown that the exponential decay in $\mathscr{D}(\mathscr{A})$ recently obtained by Revira is a consequence of our main result. An important application of our main result to the Linear-Quadratic-Gaussian optimal control problem is also discussed.

**1. Introduction.** Consider the linear one-dimensional thermoelastic system

$$u_{tt} - u_{xx} + \gamma\theta_x = 0, \qquad (0, \pi) \times (0, +\infty), \tag{1.1}$$

$$\theta_t + \gamma u_{xt} - k\theta_{xx} = 0, \qquad (0, \pi) \times (0, +\infty), \tag{1.2}$$

subject to the Dirichlet-Dirichlet boundary conditions

$$u|_{x=0,\pi} = \theta|_{x=0,\pi} = 0 \quad \text{for } t > 0 \tag{1.3}$$

and initial conditions

$$u|_{t=0} = u_0(x), \qquad u_t|_{t=0} = u_1(x), \qquad \theta|_{t=0} = \theta_0(x), \tag{1.4}$$

where $u$ is the displacement, $\theta$ is the temperature deviation from the reference temperature, and $\gamma$, $k$ are positive constants depending on the material properties. By introducing the new variable (velocity)

$$v = u_t, \tag{1.5}$$

system (1.1), (1.2) is equivalently reduced to the abstract first-order evolution equation

$$\frac{dy}{dt} = \mathscr{A} y \tag{1.6}$$

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with

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \equiv \begin{pmatrix} u \\ v \\ \theta \end{pmatrix} \tag{1.7}$$

and

$$\mathscr{A} = \begin{pmatrix} 0 & I & 0 \\ D^2 & 0 & -\gamma D \\ 0 & -\gamma D & kD^2 \end{pmatrix}. \tag{1.8}$$

Here we have used the notation $D^i = \partial^i/\partial x^i$. Let

$$H = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \tag{1.9}$$

be equipped with the norm

$$\|y\|_H = (\|Dy_1\|^2 + \|y_2\|^2 + \|y_3\|^2)^{1/2}, \tag{1.10}$$

where $\|\cdot\|$ is the $L^2$ norm in $\Omega = (0, \pi)$.

Instead of dealing with Eqs. (1.1)–(1.4), we will consider Eq. (1.6) with the domain

$$\mathscr{D}(\mathscr{A}) = (H^2 \cap H_0^1) \times H_0^1 \times (H^2 \cap H_0^1). \tag{1.11}$$

Recall that for a $C_0$-semigroup $T(t)$ on Hilbert space $H$ if there exist positive constants $M$ and $\alpha$ such that $\|T(t)\|_{\mathscr{L}(H,H)} \le Me^{-\alpha t}$ for $t > 0$, then we say that $T(t)$ is exponentially stable. It was known (e.g., see [R]) that $\mathscr{A}$ generates a contraction $C_0$-semigroup $T(t)$ on $H$. We are interested in the following problem: Is $T(t)$ exponentially stable? In this paper we will give a positive answer to this question and prove the following main result.

THEOREM 1.1. The operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset H \to H$ defined in Eq. (1.8) generates a $C_0$-semigroup $T(t)$ of contraction on $H$. Moreover, $T(t)$ is exponentially stable, i.e., there exist constants $M > 0$, $\alpha > 0$ such that

$$\|T(t)\|_{\mathscr{L}(H,H)} \le Me^{-\alpha t} \quad \forall t > 0. \tag{1.12}$$

Before giving the proof of Theorem 1.1, we recall some related results. Dafermos [D] probably was the first to investigate the asymptotic behavior of the solution to the initial boundary value problem for the linear thermoelastic system. In 1981 Slemrod [S] considered system (1.1), (1.2) as well as the nonlinear thermoelastic system. He used the energy method to prove that if $u$ and $\theta$ satisfy Dirichlet-Neumann or Neumann-Dirichlet boundary conditions, i.e., $u$ satisfies the Dirichlet boundary condition and $\theta$ satisfies the Neumann boundary condition or vice versa, then the solution to system (1.1), (1.2) has exponential decay rate as time goes to infinity. More precisely, what he essentially proved is the following.

Suppose $u_0 \in H^2$, $u_1(x) \in H^1$, and $\theta_0(x) \in H^2$ satisfy the compatibility conditions; then there are positive constants $M$ and $\alpha$ such that

$$\begin{aligned} &\|u_t(x)\|^2 + \|u_x(x)\|^2 + \|u_{tt}(x)\|^2 + \|u_{xt}(x)\|^2 + \|u_{xx}(x)\|^2 + \|\theta(t)\|^2 + \|\theta_t(t)\|^2 \\ &\quad + \|\theta_x(t)\|^2 + \|\theta_{xx}(t)\|^2 \le M(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|\theta_0\|_{H^2}^2)e^{-\alpha t} \quad \forall t > 0. \end{aligned} \tag{1.13}$$

In the case of Dirichlet-Dirichlet boundary conditions the problem whether Eq. (1.13) holds remained open for a long time. Recently, Racke and Shibata [RS] considered system (1.1)–(1.4) as well as the nonlinear case and used the spectral analysis method to obtain the polynomial decay rate of the solution. Rivera [R] also considered system (1.1)–(1.4). He proved the estimate (1.13) by using the energy method again and a tricky way to deal with the boundary terms. Racke, Shibata, and Zheng [RSZ] extended Rivera's results to nonlinear thermoelastic systems with small initial data.

We would like to point out here that the estimate (1.13) does not imply the stronger statement (1.12). The latter estimate has important impact in control theory (see, e.g., [GRT, BLM]). In the cases that the boundary conditions are Dirichlet-Neumann or Neumann-Dirichlet type as stated before, Hansen [H] recently succeeded in proving Eq. (1.12) by simply expanding $u$ and $\theta$ into series of sine and cosine functions respectively, then decoupling the system, and directly calculating the corresponding eigenvalues of $\mathscr{A}$. However, the decoupling technique failed for the case of Dirichlet-Dirichlet boundary conditions.

In this paper we use a completely different method to prove Theorem 1.1. The following characteristic condition [Hu, Theorem $2'$, p. 51] for an exponentially stable $C_0$-semigroup on Hilbert space $H$ will play a very important role in our proof.

THEOREM 1.2. A $C_0$-semigroup $T(t) = e^{t\mathscr{A}}$ on Hilbert space $H$ is exponentially stable if and only if

$$\sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(\mathscr{A})\} < 0 \tag{1.14}$$

and

$$\sup_{\operatorname{Re}\lambda \geq 0} \|(\lambda - \mathscr{A})^{-1}\| < +\infty \tag{1.15}$$

hold. Here $\sigma(\mathscr{A})$ stands for the spectrum of $\mathscr{A}$.

In Sec. 2 we will use a contradiction argument and this theorem to prove Theorem 1.1. Two important remarks are given in Sec. 3.

**2. Proof of Theorem 1.1.** We first recall that the operator $\mathscr{A}$ defined in Eq. (1.8) is an infinitesimal generator of a $C_0$-semigroup of contraction on Hilbert space $H$. Indeed, by definitions (1.8) and (1.10) we have

$$\begin{aligned}(\mathscr{A}y, y)_H &= \int_0^\pi (Dy_2 \cdot Dy_1 + D^2 y_1 \cdot y_2 - \gamma Dy_3 \cdot y_2 - \gamma Dy_2 \cdot y_3 + kD^2 y_3 \cdot y_3)\, dx \\ &= -k\|Dy_3\|^2 \leq 0.\end{aligned} \tag{2.1}$$

Moreover, the fact that $\operatorname{Re}(I - \mathscr{A}) = H$ can be proved by a standard Faedo-Galerkin method or some other method (see [R], [BLM], or [GRT] for details). Thus the conclusion of $\mathscr{A}$ being an infinitesimal generator of a $C_0$-semigroup of contraction follows from the well-known Lumer-Phillips theorem. Let

$$\mathscr{A}_0 = \begin{pmatrix} 0 & I \\ D^2 & 0 \end{pmatrix} \tag{2.2}$$

and consider the eigenproblem

$$\mathscr{A}_0 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.3}$$

in $H_0^1 \times L^2$. It turns out from Eq. (2.3) that

$$y_2 = \lambda y_1, \tag{2.4}$$

$$D^2 y_1 = \lambda y_2. \tag{2.5}$$

A straightforward calculation yields that the eigenvalues are

$$\lambda_n = in, \quad \lambda_{-n} = -in \qquad (n = 1, 2, \dots) \tag{2.6}$$

and the corresponding eigenfunctions

$$e_n = \sqrt{\frac{1}{\pi}} \begin{pmatrix} \frac{1}{n} \sin nx \\ i \sin nx \end{pmatrix}, \qquad e_{-n} = \sqrt{\frac{1}{\pi}} \begin{pmatrix} \frac{1}{n} \sin nx \\ -i \sin nx \end{pmatrix}. \tag{2.7}$$

We are now in position to prove Theorem 1.1 using Theorem 1.2 and a contradiction argument. Suppose Theorem 1.1 is not true. Then by Theorem 1.2, one of Eqs. (1.14) and (1.15) must fail to hold.

(i) If Eq. (1.15) fails to hold, then there must exist a sequence of $\lambda_n \in C$ and a sequence of $h_n \in \mathscr{D}(\mathscr{A}) \subset H$ with $\operatorname{Re}\lambda_n \geq 0$, $\operatorname{Re}\lambda_n \to 0$ (as $n \to +\infty$) and $\|h_n\|_H = 1$ such that

$$\|(\lambda_n I - \mathscr{A})h_n\|_H \to 0. \tag{2.8}$$

As a result, we have

$$\begin{aligned} \operatorname{Re}((\lambda_n I - \mathscr{A})h_n, h_n)_H &= \operatorname{Re}\lambda_n \|h_n\|_H^2 - \operatorname{Re}(\mathscr{A} h_n, h_n)_H \\ &= \operatorname{Re}\lambda_n + k\|Dh_n^{(3)}\|^2 \to 0. \end{aligned} \tag{2.9}$$

Therefore

$$\|Dh_n^{(3)}\|^2 \to 0. \tag{2.10}$$

It follows from Eq. (2.8) that the first two components of $(\lambda_n I - \mathscr{A})h_n$ must also converge to zero. That is

$$\left\| (\lambda_n I - \mathscr{A}_0) \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma D h_n^{(3)} \end{pmatrix} \right\|_{H_0^1 \times L^2} \to 0. \tag{2.11}$$

Combining Eq. (2.11) with Eq. (2.10) gives

$$\left\| (\lambda_n I - \mathscr{A}_0) \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} \right\|_{H_0^1 \times L^2} \to 0. \tag{2.12}$$

Since the eigenfunctions $\{e_{\pm n}\}$ form a basis in $H_0^1 \times L^2$, we have

$$\begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} = \sum_{l=-\infty,\ l\neq 0}^{+\infty} \alpha_{nl} e_l. \tag{2.13}$$

By the Poincaré inequality and $\|h_n\|_H = 1$, we have

$$\|h_n^{(3)}\| \leq \text{const} \cdot \|Dh_n^{(3)}\| \to 0 \tag{2.14}$$

and

$$\left\| \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} \right\|_{H_0^1 \times L^2} = \sum_{l=-\infty,\ l\neq 0}^{+\infty} |\alpha_{nl}|^2 \to 1. \tag{2.15}$$

Therefore, it follows from Eq. (2.12) that

$$\begin{aligned} \left\| (\lambda_n I - \mathscr{A}_0) \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} \right\|^2_{H_0^1 \times L^2} &= \left\| \sum_{l=-\infty,\ l\neq 0}^{+\infty} (\lambda_n - il)\alpha_{nl} e_l \right\|^2_{H_0^1 \times L^2} \\ &= \sum_{l=-\infty,\ l\neq 0}^{+\infty} |\lambda_n - il|^2 |\alpha_{nl}|^2 \to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{2.16}$$

If for $n$ large enough, $|\lambda_n - il| \geq \delta > 0$ for all $l$, then $|\alpha_{n,l}| \to 0$, a contradiction with Eq. (2.15). Thus we derive from Eqs. (2.15) and (2.16) and $\operatorname{Re}\lambda_n \to 0$ that there exists $l(n) \in \{\pm 1, \pm 2, \dots\}$ such that as $n \to +\infty$

$$\begin{cases} \lambda_n - il(n) \to 0, \\ \alpha_{n,l} \to 0, \qquad (l \neq l(n)) \\ |\alpha_{n,l(n)}| \to 1, \end{cases} \tag{2.17}$$

and it follows from Eq. (2.13) that

$$\left\| \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} - \alpha_{n,l(n)} e_{l(n)} \right\|_{H_0^1 \times L^2} \to 0. \tag{2.18}$$

It turns out from Eq. (2.12) that

$$\|il(n)h_n^{(1)} - h_n^{(2)}\|_{H_0^1} \to 0, \qquad \|il(n)h_n^{(2)} - D^2 h_n^{(1)}\| \to 0. \tag{2.19}$$

Taking the third component of Eq. (2.8) into consideration, we have

$$\|\lambda_n h_n^{(3)} - kD^2 h_n^{(3)} + \gamma D h_n^{(2)}\| \to 0. \tag{2.20}$$

Therefore, from Eq. (2.17) follows

$$\|il(n)h_n^{(3)} - kD^2 h_n^{(3)} + \gamma D h_n^{(2)}\| \to 0. \tag{2.21}$$

Since $|l(n)| \geq 1$, it turns out from Eq. (2.21) that

$$\left\| ih_n^{(3)} - \frac{k}{l(n)} D^2 h_n^{(3)} + \frac{\gamma}{l(n)} D h_n^{(2)} \right\| \to 0. \tag{2.22}$$

Noticing that Eqs. (2.19) and (2.18) imply

$$\left\| \frac{1}{l(n)} D h_n^{(2)} - iDh_n^{(1)} \right\| \to 0, \qquad \left\| Dh_n^{(1)} - \sqrt{\frac{1}{\pi}} \alpha_{n,l(n)} \cos l(n)x \right\| \to 0, \tag{2.23}$$

we obtain

$$\left\| \frac{1}{l(n)} D h_n^{(2)} - i\sqrt{\frac{1}{\pi}} \alpha_{n,l(n)} \cos l(n)x \right\| \to 0. \tag{2.24}$$

Combining Eq. (2.24) with Eq. (2.22) yields

$$\left\| ih_n^{(3)} - \frac{k}{l(n)} D^2 h_n^{(3)} + i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \cos l(n)x \right\| \to 0. \tag{2.25}$$

Let

$$z_n = \frac{h_n^{(3)}}{l(n)}. \tag{2.26}$$

Then Eq. (2.25) can be rewritten as

$$\|g_n\| \equiv \left\| il(n)z_n - kD^2 z_n + i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \cos l(n)x \right\| \to 0. \tag{2.27}$$

Taking the real part of the inner product of $g_n$ with $-D^2 z_n$, we have

$$\begin{aligned} \operatorname{Re}(g_n, D^2 z_n) &= k\|D^2 z_n\|^2 + \operatorname{Re}\left( i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \cos l(n)x, -D^2 z_n \right) \\ &= k\|D^2 z_n\|^2 - \operatorname{Re}\left( i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} l(n) \sin l(n)x, Dz_n \right) \\ &\quad - \operatorname{Re}\left( i\gamma\sqrt{\frac{2}{\pi}}\alpha_{n,l(n)} \cos l(n)x \cdot Dz_n \right)\Bigg|_0^\pi . \end{aligned} \tag{2.28}$$

To estimate the last term on the right-hand side of Eq. (2.28), we use the well-known Nirenberg inequality to obtain

$$\begin{aligned} I &\equiv \left| \operatorname{Re}\left( i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \cos l(n)x \cdot Dz_n \right)\Bigg|_0^\pi \right| \\ &\le c_1\|Dz_n\|_{L^\infty} \\ &\le c_2\|D^2 z_n\|^{1/2}\|Dz_n\|^{1/2} + c_3\|Dz_n\| \end{aligned} \tag{2.29}$$

where $c_1$, $c_2$, and $c_3$ are positive constants depending only on $\gamma$ and $\Omega$. Therefore, applying Young's inequality yields

$$I \le \frac{k}{4}\|D^2 z_n\|^2 + \frac{3}{4}c_2^{4/3}k^{-1/3}\|Dz_n\|^{2/3} + c_3\|Dz_n\|. \tag{2.30}$$

The second term on the right-hand side of Eq. (2.28) can also be estimated as follows:

$$\begin{aligned} \left| \operatorname{Re}\left( i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} l(n) \sin l(n)x, Dz_n \right) \right| &= \left| \operatorname{Re}\left( i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \sin l(n)x, Dh_n^{(3)} \right) \right| \\ &\le |\gamma|\,|\alpha_{n,l(n)}|\,\|Dh_n^{(3)}\| \to 0. \end{aligned} \tag{2.31}$$

On the other hand,

$$|\operatorname{Re}(g_n, D^2 z_n)| \le \frac{1}{2k}\|g_n\|^2 + \frac{k}{2}\|D^2 z_n\|^2. \tag{2.32}$$

Combining Eq. (2.28) with Eqs. (2.30)–(2.32) and (2.26), we conclude that

$$\|D^2 z_n\|^2 \le \frac{2}{k^2}\|g_n\|^2 + 3c_2^{4/3}k^{-4/3}\|Dh_n^{(3)}\|^{2/3} + \frac{4}{k}(c_3 + |\gamma|\,|\alpha_{n,l(n)}|)\|Dh_n^{(3)}\| \to 0. \tag{2.33}$$

Let $w_n \in H^2 \cap H_0^1$ be a solution to the equation

$$il(n)w_n - kD^2 w_n + i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \cos l(n)x = 0, \tag{2.34}$$

which will be solved explicitly by a Fourier series expansion method later on. It is clear from Eqs. (2.27) and (2.34) that

$$\|il(n)(z_n - w_n) - kD^2(z_n - w_n)\| \to 0. \tag{2.35}$$

In the same way as before, we obtain

$$\|D^2(z_n - w_n)\| \to 0. \tag{2.36}$$

Hence

$$\|D^2 w_n\| \to 0. \tag{2.37}$$

In what follows we will show that this is a contradiction. Indeed, let

$$w_n = \sum_{j=1}^{\infty} \beta_{nj}\sqrt{\frac{2}{\pi}} \sin jx. \tag{2.38}$$

Substituting this expression of $w_n$ into Eq. (2.34), we obtain

$$\sum_{j=1}^{\infty} \beta_{nj}\sqrt{\frac{2}{\pi}}(il(n) + j^2) \sin jx = -i\gamma\sqrt{\frac{1}{\pi}}\alpha_{n,l(n)} \cos l(n)x. \tag{2.39}$$

We also expand $\cos l(n)x$ into the following Fourier series:

$$\cos l(n)x = \sum_{j=1}^{\infty} f_{nj}\sqrt{\frac{2}{\pi}} \sin jx, \tag{2.40}$$

with

$$f_{nj} = \begin{cases} \sqrt{\frac{2}{\pi}}\frac{2j}{j^2 - l^2(n)}, & |j - l(n)| = \text{odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.41}$$

Then it follows from Eq. (2.39) that

$$\beta_{nj} = \begin{cases} \frac{-i2\sqrt{2}\gamma j\alpha_{n,l(n)}}{\pi(j^2 - l^2(n))(j^2 + il(n))}, & |j - l(n)| = \text{odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.42}$$

$$\begin{aligned} \|D^2 w_n\|^2 &= \sum_{j=1}^{\infty} |\beta_{nj}|^2 j^4 \\ &= \sum_{j=1,\ |l(n)-j|=\text{odd}}^{\infty} \frac{8\gamma^2 |\alpha_{n,l(n)}|^2 j^6}{\pi^2 (j^2 - l^2(n))^2 (j^4 + l^2(n))} \\ &\geq \left(\frac{8\gamma^2}{\pi^2}|\alpha_{n,l(n)}|^2\right) \cdot \frac{(|l(n)| + 1)^6}{(2|l(n)| + 1)^2[(|l(n)| + 1)^4 + l^2(n)]}, \end{aligned} \tag{2.43}$$

where the last inequality is derived by taking $j = |l(n)| + 1$. Clearly, when $n$ is large enough, the term on the right-hand side of Eq. (2.43) is always larger than a positive constant $\delta$ independent of $n$, $l(n)$, i.e.,

$$\|D^2 w_n\|^2 \geq \delta > 0. \tag{2.44}$$

Thus it contradicts Eq. (2.37).

(ii) If Eq. (1.14) fails to hold, then there must exist a sequence of $\lambda_n \in C$ and a sequence of $h_n \in \mathscr{D}(\mathscr{A}) \subset H$ with $\lambda_n \in \sigma(\mathscr{A})$, $\operatorname{Re}\lambda_n \to 0$ (as $n \to +\infty$), and $\|h_n\|_H = 1$ such that

$$(\lambda_n I - \mathscr{A})h_n = 0. \tag{2.45}$$

Taking the real part of the inner product of Eq. (2.45) with $h_n$, we obtain

$$\operatorname{Re}\lambda_n + k\|Dh_n^{(3)}\|^2 = 0, \tag{2.46}$$

which also results in

$$\|Dh_n^{(3)}\|^2 \to 0. \tag{2.47}$$

Since the first two components of Eq. (2.45) are

$$\left\| (\lambda_n I - \mathscr{A}_0)\begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma D h_n^{(3)} \end{pmatrix} \right\|_{H_0^1 \times L^2} = 0, \tag{2.48}$$

it turns out from Eq. (2.47) that we again have Eq. (2.12). Repeating the same argument as before, we have a contradiction again. Thus the proof of Theorem 1.1 is complete.

**3. Remarks.** In this section we present two important remarks. In Remark 3.1 we show that our main result, Theorem 1.1, implies the inequality (1.13), which was recently obtained by Revira [R]. In Remark 3.2 an application of Theorem 1.1 to the Linear-Quadratic-Gaussian (LQG) optimal control problem is given.

REMARK 3.1. The inequality (1.13) is a consequence of Theorem 1.1.

Indeed, if $u_0 \in H^2 \cap H_0^1$, $u_1 \in H_0^1$, $\theta_0 \in H^2 \cap H_0^1$, then

$$y_0 \overset{\text{def}}{=} \begin{pmatrix} u_0 \\ u_1 \\ \theta_0 \end{pmatrix} \in \mathscr{D}(\mathscr{A}). \tag{3.1}$$

Let

$$z(t) = T(t)\mathscr{A} y_0 \in \mathscr{C}(R^+, H). \tag{3.2}$$

We have

$$\tilde{y}(t) = \int_0^t z(\tau)\,d\tau + y_0 \in \mathscr{C}(R^+, \mathscr{D}(\mathscr{A})) \cap \mathscr{C}^1(R^+, H) \tag{3.3}$$

satisfying

$$\frac{d\tilde{y}}{dt} = z(t). \tag{3.4}$$

On the other hand, by the well-known property of a $C_0$-semigroup (see [Pa, p. 4, Theorem 2.4]) we obtain

$$\begin{aligned} \mathscr{A}\tilde{y} &= \mathscr{A}\int_0^t T(\tau)\mathscr{A} y_0\,d\tau + \mathscr{A} y_0 \\ &= T(t)\mathscr{A} y_0 = z(t). \end{aligned} \tag{3.5}$$

Combining Eq. (3.5) with Eq. (3.4) yields that $\tilde{y}(t)$ is also a classical solution of the system (1.6) with initial datum $y_0$. By uniqueness we have

$$z(t) = \frac{dy}{dt} = \mathscr{A} y. \tag{3.6}$$

Therefore, it follows from Theorem 1.1 that

$$\left\| \frac{dy}{dt} \right\|_H = \|z(t)\|_H \leq Me^{-\alpha t} \|\mathscr{A} y_0\|_H \tag{3.7}$$

and

$$\|\mathscr{A} y\|_H \leq Me^{-\alpha t} \|\mathscr{A} y_0\|_H. \tag{3.8}$$

We now prove that there exist positive constants $c_1$, $c_2$ such that

$$c_1(\|y_1\|^2_{H^2} + \|y_2\|^2_{H^1} + \|y_3\|^2_{H^2}) \leq \|\mathscr{A} y\|^2_H \leq c_2(\|y_1\|^2_{H^2} + \|y_2\|^2_{H^1} + \|y_3\|^2_{H^2}). \tag{3.9}$$

Indeed, by the definitions (1.8) and (1.10), a straightforward application of Schwartz's inequality yields

$$\|\mathscr{A} y\|^2_H \leq c_2(\|y_1\|^2_{H^2} + \|y_2\|^2_{H^1} + \|y_3\|^2_{H^2}), \tag{3.10}$$

with $c_2$ being a positive constant depending only on $\gamma$ and $k$.

On the other hand,

$$\|\mathscr{A} y\|^2_H = \|y_2\|^2_{H^1} + \|D^2 y_1 - \gamma D y_3\|^2 + \|kD^2 y_3 - \gamma D y_2\|^2. \tag{3.11}$$

Applying Schwartz's inequality yields

$$\|kD^2 y_3\|^2 \leq 2(\|kD^2 y_3 - \gamma D y_2\|^2 + \|\gamma D y_2\|^2) \tag{3.12}$$

and

$$\|D^2 y_1\|^2 \leq 2(\|D^2 y_1 - \gamma D y_3\|^2 + \|\gamma D y_3\|^2). \tag{3.13}$$

By Eq. (2.1) we obtain

$$k\|Dy_3\|^2 = -(\mathscr{A} y, y)_H \leq \frac{\varepsilon}{2}\|y\|^2_H + \frac{2}{\varepsilon}\|\mathscr{A} y\|^2_H. \tag{3.14}$$

Therefore, from Eqs. (3.11)–(3.14) we have

$$\|kD^2 y_3\|^2 + \|D^2 y_1\|^2 + \|y_2\|^2_{H^1} \leq c_{3,\varepsilon}\|\mathscr{A} y\|^2_H + c_4\varepsilon\|y\|^2_H, \tag{3.15}$$

with $c_{3,\varepsilon} > 0$ depending on $\varepsilon$, $\gamma$, and $k$; $c_4 > 0$ depending only on $\gamma$, $k$. It follows from Poincaré's inequality and elliptic equation estimates that

$$\|y_2\| \leq c_5\|Dy_2\|, \tag{3.16}$$

$$\|y_1\|_{H^2} \leq c_6\|D^2 y_1\|, \tag{3.17}$$

$$\|y_3\|_{H^2} \leq c_6\|D^2 y_3\|. \tag{3.18}$$

Combining Eq. (3.15) with Eqs. (3.16)–(3.18) and taking $\varepsilon$ small enough, we obtain the left part of inequality (3.9):

$$c_1(\|y_1\|^2_{H^2} + \|y_2\|^2_{H^1} + \|y_3\|^2_{H^2}) \leq \|\mathscr{A} y\|^2_H. \tag{3.19}$$

Thus the desired inequality (1.13) follows from Eqs. (3.7)–(3.9).

REMARK 3.2. Our main result, Theorem 1.1, has an important application to the LQG optimal control problem.

Consider the control of the thermoelastic system

$$\frac{dy}{dt} = \mathscr{A} y + F \tag{3.20}$$

with

$$F = \begin{pmatrix} 0 \\ f_1 \\ f_2 \end{pmatrix}. \tag{3.21}$$

Usually, the input $F$ has the form

$$F = B\tilde{u}(t) + \widetilde{B}\mu(t), \qquad t > 0, \tag{3.22}$$

and output $z(t)$ has the form

$$z(t) = \mathscr{C} y + \nu(t), \qquad t > 0, \tag{3.23}$$

where $y$ is the mild solution to Eq. (3.20), $\tilde{u}(t) \in R^m$, $\mu(t) \in R^l$, $z(t), \nu(t) \in R^p$, $B \in \mathscr{B}(R^m, H)$, $\widetilde{B} \in \mathscr{B}(R^l, H)$, and $\mathscr{C} \in \mathscr{B}(H, R^p)$. Moreover, $\mu$ and $\nu$ are stationary zero-mean Gaussian white noise processes with covariance matrices $\Gamma$ and $\widehat{R}$, respectively, and $\widehat{R}$ is positive definite.

Then the LQG optimal control problem is to minimize the cost functional

$$J(\tilde{u}) = \lim_{t_f \to \infty} E\left\{ \frac{1}{t_f} \int_0^{t_f} [\langle \mathscr{Q} y(t), y(t)\rangle + \tilde{u}^{\mathrm{T}}(t) \mathscr{R} \tilde{u}(t)]\, dt \right\} \tag{3.24}$$

where $\mathscr{Q} \in \mathscr{B}(H, H)$ and $\mathscr{R} \in R^{m\times m}$ are selfadjoint with $\mathscr{Q}$ nonnegative and $\mathscr{R}$ positive definite.

The theory for the infinite-dimensional LQG optimal control problem with bounded input and output operator has been developed in [Ba, CP, G, GA]. As is the case in finite dimensions, the LQG problem separates into a deterministic linear-quadratic regulator problem on the infinite interval and a dual state estimator or filtering problem.

The existing theory shows that our Theorem 1.1 implies that the LQG optimal control problem admits a unique solution $\tilde{u}^*$ of the feedback in the form

$$\tilde{u}^* = -\mathscr{K} y(t) \tag{3.25}$$

and so on (see [GRT] for more details).

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