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EXPONENTIAL SUMS AND NEWTON POLYHEDRA

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Let p be a prime number and let k denote the field of $q = p^a$ elements. Fix a nontrivial additive character $\Psi\colon k \to \mathbf{Q}(\varsigma_p)^\times$. Given a variety V of dimension n and a regular function f on V, with both V and f defined over k, one can define an exponential sum

$$(1) \qquad S(V,f) = \sum_{x \in V(k)} \Psi(f(x)),$$

where $V(k)$ denotes the k-rational points of V. It is a classical problem to find conditions on V and f that will imply a good estimate for $|S(V,f)|$. By "good estimate" we mean an inequality of the form

$$(2) \qquad |S(V,f)| \le C\sqrt{q}^{\,n},$$

where C is a constant depending on V and f but not on q.

Deligne's fundamental theorem [**3**] reduces the problem of estimating the archimedean size of exponential sums to the problem of computing certain associated l-adic cohomology groups. Let $\mathbf{A}^n$ denote affine n-space over k and let $(\mathbf{G}_m)^n$ denote the product of n copies of the multiplicative group $\mathbf{G}_m$ over k. The purpose of this note is to report on some general criteria, when $V = (\mathbf{G}_m)^n$ or $\mathbf{A}^n$, that allow us to calculate this cohomology and hence obtain sharp archimedean estimates for the corresponding exponential sums. These same criteria allow us to obtain apparently sharp p-adic estimates for the exponential sums as well, although space limitations prevent us from describing them here. Connections between the p-adic theory and Newton polyhedra already appear in [**7** and **8**].

A novel feature of our work is the use of Dwork cohomology [**4**, **5**] to compute l-adic cohomology. The results of this note have not so far been obtainable by purely l-adic methods. Complete proofs and references will appear elsewhere. We are indebted to B. Dwork and N. Katz for many helpful discussions.

1. Statement of results. Let k_r denote the extension of k of degree r and let $\mathrm{Tr}_r\colon k_r \to k$ be the trace map. Let $\bar{k}$ denote the algebraic closure of k. Set

$$(3) \qquad S_r(V,f) = \sum_{x \in V(k_r)} \Psi(\mathrm{Tr}_r\, f(x)),$$

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where $V(k_r)$ denotes the k_r-rational points of V. Define the associated L-function $L(V, f; t)$ by

$$(4) \qquad L(V,f;t) = \exp\left(\sum_{r=1}^{\infty} S_r(V,f)t^r/r\right) \in \mathbf{Q}(\zeta_p)[[t]].$$

It is well known that for every prime number $l \neq p$ there is a lisse, rank-one, l-adic étale sheaf $\mathcal{L}_\Psi(f)$ on V whose associated L-function is identical to $L(V, f; t)$. By Grothendieck's Lefschetz trace formula and Deligne's fundamental theorem, if

$$(5) \qquad H^i_c(V \otimes_k \overline{k}, \mathcal{L}_\Psi(f)) = 0 \quad \text{for } i \neq n,$$

then one obtains the estimate

$$(6) \qquad |S_r(V,f)| \leq (\dim H^n_c(V \otimes_k \overline{k}, \mathcal{L}_\Psi(f)))\sqrt{q}^{\,rn}$$

(where $H^i_c(V \otimes_k \overline{k}, \mathcal{L}_\Psi(f))$ denotes l-adic cohomology with proper supports). When $V = (\mathbf{G}_m)^n$ or $\mathbf{A}^n$, we shall give conditions on f that allow us to deduce (5) and give a simple formula for $\dim H^n_c(V \otimes_k \overline{k}, \mathcal{L}_\Psi(f))$.

Consider first the case $V = (\mathbf{G}_m)^n$. The regular functions on V defined over k are the Laurent polynomials with coefficients in k, i.e., elements of $k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. For $j = (j_1, \dots, j_n) \in \mathbf{Z}^n$, let $x^j = x_1^{j_1} \cdots x_n^{j_n}$. A Laurent polynomial f over k can be written

$$(7) \qquad f = \sum_{j \in J} a_j x^j,$$

where J is a finite subset of $\mathbf{Z}^n$ and $a_j \in k^\times$. We define the *Newton polyhedron* $\Delta(f)$ of f to be the convex closure in $\mathbf{R}^n$ of the set $J \cup \{(0, \dots, 0)\}$. For each face σ of $\Delta(f)$, define a Laurent polynomial f_σ by

$$(8) \qquad f_\sigma = \sum_{j \in \sigma \cap J} a_j x^j.$$

Call f *nondegenerate with respect to* $\Delta(f)$ (Kouchnirenko [**6**]) if for every face σ of $\Delta(f)$ that does not contain the origin, $\partial f_\sigma/\partial x_1, \dots, \partial f_\sigma/\partial x_n$ have no common zero in $(\overline{k}^\times)^n$. The set of all nondegenerate polynomials having a given Newton polyhedron is Zariski open in the set of all polynomials having that Newton polyhedron, except possibly if the characteristic of k lies in a certain finite set which depends on the Newton polyhedron. We define the *dimension* of $\Delta(f)$ to be the dimension of the smallest subspace of $\mathbf{R}^n$ containing $\Delta(f)$. Let $V(f)$ denote the volume of $\Delta(f)$ with respect to Lebesgue measure on $\mathbf{R}^n$.

THEOREM 1. *Let Δ be an n-dimensional convex polyhedron in $\mathbf{R}^n$ with vertices in $\mathbf{Z}^n$ that contains the origin. There is a finite set of rational primes S_Δ such that the following holds: If* $\operatorname{char}(k) \notin S_\Delta$, *$f \in k[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ with $\Delta(f) = \Delta$, and f is nondegenerate with respect to $\Delta(f)$, then*

(i) *$H^i_c((\mathbf{G}_m)^n \otimes_k \overline{k}, \mathcal{L}_\Psi(f)) = 0$ if $i \neq n$;*

(ii) $\dim H^n_c((\mathbf{G}_m)^n \otimes_k \overline{k}, \mathcal{L}_\Psi(f)) = n!\,V(f)$.

If in addition the origin is an interior point of Δ, *then*
(iii) $H_c^n((\mathbf{G}_m)^n \otimes_k \bar{k}, \mathcal{L}_\Psi(f))$ *is pure of weight* n.

COROLLARY. *Under the hypotheses of Theorem* 1,

$$|S((\mathbf{G}_m)^n, f)| \leq n!\, V(f)\sqrt{q}^{\,n}.$$

PROOF. Using the ideal theory of [**6**], we are able to develop a cohomology theory along the lines of [**4**] and [**5**] to show that $L((\mathbf{G}_m)^n, f; t)^{(-1)^{n-1}}$ is a polynomial of degree $n!\,V(f)$ and obtain p-adic estimates for its roots. The proof then proceeds by induction on n. After an invertible change of coordinates, one may regard f as a one-parameter family of Laurent polynomials in $n-1$ variables, each satisfying the induction hypothesis and containing the origin in the interior of its Newton polyhedron. Applying basic theorems of l-adic cohomology shows that $H_c^i = 0$ except possibly in dimensions n and $n+1$. Corollaire 1.4.4 of [**3**] and the fact that $L((\mathbf{G}_m)^n, f; t)^{(-1)^{n-1}}$ is a polynomial show that $H_c^{n+1} = 0$. The p-adic estimate for the roots, Deligne's fundamental theorem [**3**], and the product formula for valuations then imply purity.

We conjecture that Theorem 1 remains true without restriction on the characteristic of k. This can be verified if $n = 2$ and in many other cases (see the examples at the end of this note).

We now turn to the case $V = \mathbf{A}^n$, $f \in k[x_1, \ldots, x_n]$. Since an ordinary polynomial may also be regarded as a Laurent polynomial, all our previous definitions concerning the Newton polyhedron make sense in this context. We call $f \in k[x_1, \ldots, x_n]$ *commode* if for each $i = 1, \ldots, n$, f contains a term $\gamma_i x_i^{d_i}$ with $\gamma_i \in k^\times$, $d_i > 0$. For each subset $A \subseteq \{1, \ldots, n\}$, let X_A be the subspace of $\mathbf{R}^n$ where $x_i = 0$ for all $i \notin A$. Let $V_A(f)$ be the volume of $\Delta(f) \cap X_A$, computed with respect to Lebesgue measure on X_A normalized so that a fundamental domain for $\mathbf{Z}^n \cap X_A$ has volume 1. Let $|A|$ denote the cardinality of A. Define the *Newton number* $\nu(f)$ by the formula

$$\nu(f) = \sum_{A \subseteq \{1, \ldots, n\}} (-1)^{n-|A|} |A|!\, V_A(f). \tag{9}$$

Let $\mathbf{R}_+$ denote the nonnegative real numbers.

THEOREM 2. *Let* Δ *be a convex polyhedron in* $(\mathbf{R}_+)^n$ *with vertices in* $\mathbf{Z}^n$ *that has a vertex at the origin and on each of the coordinate axes. There is a finite set of rational primes* S_Δ *such that the following holds: If* $\mathrm{char}(k) \notin S_\Delta$, $f \in k[x_1, \ldots, x_n]$ *with* $\Delta(f) = \Delta$, *and* f *is nondegenerate with respect to* $\Delta(f)$, *then* $L(\mathbf{A}^n, f; t)^{(-1)^{n-1}}$ *is a polynomial of degree* $\nu(f)$, *all of whose reciprocal roots are algebraic integers pure of weight* n.

COROLLARY. *Under the hypotheses of Theorem* 2, $|S(\mathbf{A}^n, f)| \leq \nu(f)\sqrt{q}^{\,n}$.

PROOF. The fact that $L(\mathbf{A}^n, f; t)^{(-1)^{n-1}}$ is a polynomial is a consequence of the p-adic theory. Theorem 2 then follows from Theorem 1 by the standard relations between exponential sums over $\mathbf{A}^n$ and $(\mathbf{G}_m)^n$.

We conjecture that Theorem 2 remains true without restriction on the characteristic of k. This can be verified if $n = 2$ and in many other cases (see Theorem 3 below). Of course, we believe that there is a cohomological explanation for this result:

CONJECTURE. *If $f \in k[x_1, \ldots, x_n]$ is commode and nondegenerate with respect to $\Delta(f)$, then*
(i) $H^i_c(\mathbf{A}^n \otimes_k \overline{k}, \mathcal{L}_\Psi(f)) = 0$ *if* $i \neq n$;
(ii) $\dim H^n_c(\mathbf{A}^n \otimes_k \overline{k}, \mathcal{L}_\Psi(f)) = \nu(f)$;
(iii) $H^n_c(\mathbf{A}^n \otimes_k \overline{k}, \mathcal{L}_\Psi(f))$ *is pure of weight* n.

We can prove this conjecture provided $\Delta(f)$ has a somewhat special form.

THEOREM 3. *Suppose $f \in k[x_1, \ldots, x_n]$ is commode and nondegenerate with respect to $\Delta(f)$. Assume in addition that for each codimension-one face σ of $\Delta(f)$ that does not contain the origin, all coordinates of the exterior normal vector to σ with respect to the standard basis are positive (where the* ***exterior*** *normal vector is the one pointing out of $\Delta(f)$). Then all conclusions of the Conjecture hold. In particular, we have*

$$|S(\mathbf{A}^n, f)| \leq \nu(f)\sqrt{q}^{\,n}.$$

PROOF. The proof is identical to the proof of Theorem 1, the point being that one can simply specialize one of the variables to regard f as a one-parameter family of polynomials, each satisfying the induction hypothesis.

EXAMPLES. The Laurent polynomial

$$f = \gamma_1 x_1^{d_1} + \cdots + \gamma_n x_n^{d_n} + \frac{\gamma_{n+1}}{x_1^{e_1} \ldots x_n^{e_n}}, \tag{10}$$

where the γ_i lie in $k^\times$ and the d_i and e_j are positive integers prime to p, satisfies the hypotheses of Theorem 1 (one can show in addition that no restriction on $\operatorname{char}(k)$ is necessary) and $n!\,V(f) = (\prod_{i=1}^n d_i)(1 + \sum_{i=1}^n e_i/d_i)$. Thus

$$\left| S\left((\mathbf{G}_m)^n, \gamma_1 x_1^{d_1} + \cdots + \gamma_n x_n^{d_n} + \frac{\gamma_{n+1}}{x_1^{e_1} \cdots x_n^{e_n}} \right) \right| \leq \left(\prod_{i=1}^n d_i \right) \left(1 + \sum_{i=1}^n \frac{e_i}{d_i} \right) \sqrt{q}^{\,n}. \tag{11}$$

See Carpentier [**1**] for a p-adic study of this exponential sum.

Consider the polynomial

$$f(x_1, \ldots, x_n) = \gamma_1 x_1^{d_1} + \cdots + \gamma_n x_n^{d_n} + g(x_1, \ldots, x_n), \tag{12}$$

where g is chosen subject to the restrictions that $\Delta(f)$ be the simplex with vertices at the origin and at $(d_1, 0, \ldots, 0), \ldots, (0, \ldots, 0, d_n)$ and that f be nondegenerate with respect to $\Delta(f)$. Then f satisfies the hypotheses of Theorem 3 and $\nu(f) = \prod_{i=1}^n (d_i - 1)$, hence

$$|S(\mathbf{A}^n, f)| \leq \left(\prod_{i=1}^n (d_i - 1) \right) \sqrt{q}^{\,n}. \tag{13}$$

It can be shown that this result includes Deligne's theorem [**2**, Théorème 8.4] as the special case where $d_1 = \cdots = d_n$.

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