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Exponential sums for symplectic groups and their applications

by

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1. Introduction. Let λ be a nontrivial additive character of the finite field $\mathbb{F}_q$, and let r be a positive integer. Then we consider the exponential sum

$$\sum_{w\in \mathrm{Sp}(2n,q)} \lambda((\operatorname{tr} w)^r), \tag{1.1}$$

where $\mathrm{Sp}(2n,q)$ is the symplectic group over $\mathbb{F}_q$, and $\operatorname{tr} w$ is the trace of w. Also, we consider

$$\sum_{w\in \mathrm{GSp}(2n,q)} \lambda((\operatorname{tr} w)^r), \tag{1.2}$$

where $\mathrm{GSp}(2n,q)$ denotes the symplectic similitude group over $\mathbb{F}_q$.

The main purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). It turns out that (1.1) is a polynomial in q times

$$\sum_{\gamma\in \mathbb{F}_q} \lambda(\gamma^r) \tag{1.3}$$

plus another polynomial in q involving certain exponential sums. On the other hand, the expression for (1.2) is similar to that for (1.1), except that the polynomial in q involving (1.3) is multiplied by $q-1$ and that the exponential sums appearing in the other polynomial in q are replaced by averages of those exponential sums.

In [8], the sums in (1.1) and (1.2) were studied for $r=1$ and the connection of the sum in (1.1) with Hodges' generalized Kloosterman sum over nonsingular alternating matrices was also investigated (cf. [4]–[6]). As the

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sum in (1.3) vanishes for $r = 1$, the polynomials involving (1.3) do not appear in that case. For $r = 1$, similar sums for other classical groups over a finite field have been considered ([7]–[14]).

The sums in (1.1) and (1.2) may be viewed as generalizations to the symplectic group case of the sum in (1.3), which was considered by several authors ([1]–[3]).

Another purpose of this paper is to find formulas for the number of elements w in $\mathrm{Sp}(2n, q)$ and $\mathrm{GSp}(2n, q)$ with $\operatorname{tr} w = \beta$, for each $\beta \in \mathbb{F}_q$. Although we derive those expressions from (5.2) based on a well-known principle, they can also be obtained from the expressions for (1.1) and (1.2) by specializing them to the $r = q - 1$ and $r = 1$ cases.

We now state the main results of this paper. For some notations here, one is referred to the next section.

Theorem A. *The sum* $\sum_{w\in \mathrm{Sp}(2n,q)} \lambda((\operatorname{tr} w)^r)$ *equals*

$$f(q) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r)$$

plus

$$q^{n^2-1} \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1} - 1) \times \sum_{l=1}^{[(n-2b+2)/2]} q^l MK_{n-2b+2-2l}(\lambda^r; 1, 1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \tag{1.4}$$

with

$$f(q) = q^{n^2-1} \Bigg\{ \prod_{j=1}^{n} (q^{2j} - 1) - \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1} - 1) \times \sum_{l=1}^{[(n-2b+2)/2]} q^{l-1} (q-1)^{n-2b+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \Bigg\}, \tag{1.5}$$

where both unspecified sums in (1.4) *and* (1.5) *run over the same set of integers* $j_1, \ldots, j_{l-1}$ *satisfying* $2l - 1 \le j_{l-1} \le \ldots \le j_1 \le n - 2b + 1$, *and* $MK_m(\lambda^r; a, b) = MK_m(\lambda^r; a, b; 0)$ *is the exponential sum defined in* (3.16) *and* (3.17) (cf. (3.19)).

Theorem B. *With* $f(q)$ *as in* (1.5), *the sum* $\sum_{w\in \mathrm{GSp}(2n,q)} \lambda((\operatorname{tr} w)^r)$ *is given by*

$$(q-1) f(q) \sum_{\gamma\in\mathbb{F}_q} \lambda(\gamma^r)$$

plus the expression in (1.4) *with* $MK_{n-2b+2-2l}(\lambda^r;1,1)$ *replaced by the average*

$$\sum_{\alpha\in\mathbb{F}_q} MK_{n-2b+2-2l}(\lambda^r;\alpha,1).$$

THEOREM C. *For each* $\beta\in\mathbb{F}_q$, *the number* $N_{\mathrm{Sp}(2n,q)}(\beta)$ *of* $w\in\mathrm{Sp}(2n,q)$ *with* $\mathrm{tr}\,w=\beta$ *is given by*

$$q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1)$$

plus

$$\begin{aligned} q^{n^2-1}&\sum_{b=0}^{[n/2]} q^{b(b+1)}\begin{bmatrix} n\\ 2b\end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1)\\ &\times\sum_{l=1}^{[(n-2b+2)/2]} q^l(\delta(n-2b+2-2l,q;\beta)-q^{-1}(q-1)^{n-2b+2-2l})\\ &\times\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),\end{aligned}$$

where the innermost sum runs over the same set of integers as in (1.4), *and* $\delta(m,q;\beta)$ *is as in* (5.4) *and* (5.5).

THEOREM D. *For each* $\beta\in\mathbb{F}_q$, *the number* $N_{\mathrm{GSp}(2n,q)}(\beta)$ *of* $w\in\mathrm{GSp}(2n,q)$ *with* $\mathrm{tr}\,w=\beta$ *is given by*

$$\begin{cases} (q-1)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1)-q^{-1}\sum_{w\in\mathrm{GSp}(2n,q)}\lambda(\mathrm{tr}\,w) & \text{if } \beta\neq 0,\\ (q-1)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1)+q^{-1}(q-1)\sum_{w\in\mathrm{GSp}(2n,q)}\lambda(\mathrm{tr}\,w) & \text{if } \beta=0,\end{cases}$$

where λ *is any nontrivial additive character of* $\mathbb{F}_q$ *as before and the last sum is the expression in Theorem* B *with* $r=1$ (cf. (5.9)).

Theorems A, B, C and D are respectively stated below as Theorems 4.2, 4.1, 5.2 and 5.3.

2. Preliminaries. In this section, we fix some notations and gather some elementary facts that will be used in the sequel.

Let $\mathbb{F}_q$ denote the finite field with q elements, $q=p^d$ (p a prime, d a positive integer). Let λ be an additive character of $\mathbb{F}_q$. Then $\lambda=\lambda_a$ for a unique $a \in \mathbb{F}_q$, where, for $\gamma \in \mathbb{F}_q$,

$$\lambda_a(\gamma) = \exp\left\{\frac{2\pi i}{p}(a\gamma + (a\gamma)^p + \ldots + (a\gamma)^{p^{d-1}})\right\}. \tag{2.1}$$

It is nontrivial if $a \neq 0$.

In the following, $\operatorname{tr} A$ denotes the trace of A for a square matrix A, and ${}^tB$ denotes the transpose of B for any matrix B.

Let $\mathrm{GL}(n,q)$ denote the group of all invertible $n \times n$ matrices with entries in $\mathbb{F}_q$. The order of $\mathrm{GL}(n,q)$ equals

$$g_n = \prod_{j=0}^{n-1}(q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1). \tag{2.2}$$

$\mathrm{Sp}(2n,q)$ is the symplectic group over $\mathbb{F}_q$ defined by

$$\mathrm{Sp}(2n,q) = \{w \in \mathrm{GL}(2n,q) \mid {}^twJw = J\},$$

where

$$J = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix}. \tag{2.3}$$

As is well known,

$$|\mathrm{Sp}(2n,q)| = q^{n^2} \prod_{j=1}^{n}(q^{2j} - 1). \tag{2.4}$$

$P(2n,q)$ indicates the maximal parabolic subgroup of $\mathrm{Sp}(2n,q)$ given by

$$P(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \in \mathrm{Sp}(2n,q) \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n,q), \\ {}^tB = B \end{array} \right\}. \tag{2.5}$$

The Bruhat decomposition of $\mathrm{Sp}(2n,q)$ with respect to $P(2n,q)$ can be expressed as a disjoint union of right cosets of $P = P(2n,q)$:

$$\mathrm{Sp}(2n,q) = \coprod_{b=0}^{n} P\sigma_b(A_b \backslash P), \tag{2.6}$$

where

$$A_b = A_b(q) = \{w \in P(2n,q) \mid \sigma_b w \sigma_b^{-1} \in P(2n,q)\}, \tag{2.7}$$

$$\sigma_b = \begin{bmatrix} 0 & 0 & 1_b & 0 \\ 0 & 1_{n-b} & 0 & 0 \\ -1_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-b} \end{bmatrix}. \tag{2.8}$$

From (3.10) and (5.7) of [8] (cf. (2.17)),

$$|A_b(q) \backslash P(2n,q)| = q^{\binom{b+1}{2}} \begin{bmatrix} n \\ b \end{bmatrix}_q, \tag{2.9}$$

and the number a_b of all $b \times b$ nonsingular alternating matrices over $\mathbb{F}_q$, for each positive integer b, is given by

$$a_b = \begin{cases} q^{(b/2)(b/2-1)} \prod_{i=1}^{b/2} (q^{2i-1} - 1) & \text{if } b \text{ is even,} \\ 0 & \text{if } b \text{ is odd.} \end{cases} \tag{2.10}$$

$\mathrm{GSp}(2n,q)$ denotes the symplectic similitude group over $\mathbb{F}_q$ given by

$$\mathrm{GSp}(2n,q) = \{w \in \mathrm{GL}(2n,q) \mid {}^t w J w = \alpha(w) J \text{ for some } \alpha(w) \in \mathbb{F}_q^\times\},$$

where J is as in (2.3). We have

$$|\mathrm{GSp}(2n,q)| = (q-1) q^{n^2} \prod_{j=1}^{n} (q^{2j} - 1). \tag{2.11}$$

$Q(2n,q)$ is the maximal parabolic subgroup of $\mathrm{GSp}(2n,q)$ defined by

$$Q(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & \alpha {}^t A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n,q), \\ \alpha \in \mathbb{F}_q^\times, {}^t B = B \end{array} \right\}. \tag{2.12}$$

The decomposition in (2.6) can be modified to give

$$\mathrm{GSp}(2n,q) = \coprod_{b=0}^{n} Q \sigma_b (A_b \backslash P), \tag{2.13}$$

where $Q = Q(2n,q)$ is as in (2.12).

We recall the following theorem from [17, Theorem 5.30]. For a nontrivial additive character λ of $\mathbb{F}_q$ and a positive integer r,

$$\sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) = \sum_{j=1}^{e-1} G(\psi^j, \lambda) \tag{2.14}$$

where ψ is a multiplicative character of $\mathbb{F}_q$ of order $e = (r, q-1)$ and $G(\psi^j, \lambda)$ is the usual Gauss sum given by

$$G(\psi^j, \lambda) = \sum_{\gamma \in \mathbb{F}_q^\times} \psi^j(\gamma) \lambda(\gamma). \tag{2.15}$$

For a nontrivial additive character λ of $\mathbb{F}_q$ and $a, b \in \mathbb{F}_q$, the usual Kloosterman sum is given by

$$K(\lambda; a, b) = \sum_{\gamma \in \mathbb{F}_q^\times} \lambda(a\gamma + b\gamma^{-1}). \tag{2.16}$$

We put, for integers n, b with $0 \le b \le n$,

$$\begin{bmatrix} n \\ b \end{bmatrix}_q = \prod_{j=0}^{b-1} (q^{n-j} - 1)/(q^{b-j} - 1), \tag{2.17}$$

and put

$$(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}),$$

for x an indeterminate and n a nonnegative integer. Then the q-binomial theorem says

$$\sum_{b=0}^{n} \begin{bmatrix} n \\ b \end{bmatrix}_q (-1)^b q^{\binom{b}{2}} x^b = (x;q)_n. \tag{2.18}$$

Finally, for a real number x, $[x]$ denotes the greatest integer $\le x$.

3. Certain exponential sums. For a nontrivial additive character λ of $\mathbb{F}_q$, r a positive integer, and for $a, b \in \mathbb{F}_q$, we define

$$K_{\mathrm{GL}(t,q)}(\lambda^r; a, b) := \sum_{w \in \mathrm{GL}(t,q)} \lambda((a \operatorname{tr} w + b \operatorname{tr} w^{-1})^r). \tag{3.1}$$

In [8], this sum was defined for $r = 1$ and its explicit expression in that case was derived.

As mentioned in (4.4)–(4.6) of [8] and (3.3)–(3.5) of [7], we have the following decomposition:

$$\mathrm{GL}(t,q) = P(t-1,1;q) \coprod P(t-1,1;q)\sigma(B(t,q) \backslash P(t-1,1;q)), \tag{3.2}$$

where

$$P(t-1,1;q) = \left\{ \begin{bmatrix} A & B \\ 0 & d \end{bmatrix} \in \mathrm{GL}(t,q) \;\middle|\; \begin{array}{l} A, B, d \text{ are respectively of sizes} \\ (t-1)\times(t-1), (t-1)\times 1, 1\times 1 \end{array} \right\},$$

$$B(t,q) = \{ w \in P(t-1,1;q) \mid \sigma w \sigma^{-1} \in P(t-1,1;q) \},$$

$$\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1_{t-2} & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

A recursive relation for (3.1) can be obtained by using an argument similar to that in Section 4 of [8]. For this, we need to consider a sum which is slightly more general than (3.1). Namely, for $\alpha \in \mathbb{F}_q^{\times}$, $\beta \in \mathbb{F}_q$, we define

$$K_{\mathrm{GL}(t,q)}(\lambda^r; \alpha, 1; \beta) := \sum_{w \in \mathrm{GL}(t,q)} \lambda((\alpha \operatorname{tr} w + \operatorname{tr} w^{-1} + \beta)^r). \tag{3.3}$$

Note that for $\alpha = ab$ $(a, b \in \mathbb{F}_q^{\times})$ and $\beta = 0$, this is the same as (3.1).

The sum in (3.3) can be written, in view of (3.2), as

$$\begin{aligned} K_{\mathrm{GL}(t,q)}&(\lambda^r; \alpha, 1; \beta) \\ &= \sum \lambda((\alpha \operatorname{tr} w + \operatorname{tr} w^{-1} + \beta)^r) \\ &\quad + |B(t,q)\backslash P(t-1,1;q)| \sum \lambda((\alpha \operatorname{tr} w\sigma + \operatorname{tr}(w\sigma)^{-1} + \beta)^r), \end{aligned} \tag{3.4}$$

where both sums are over $w \in P(t-1,1;q)$. Here one must observe that, for each $h \in P(t-1,1;q)$,

$$\sum_{w\in P(t-1,1;q)} \lambda((\alpha \operatorname{tr} w\sigma h + \operatorname{tr}(w\sigma h)^{-1} + \beta)^r)$$
$$= \sum_{w\in P(t-1,1;q)} \lambda((\alpha \operatorname{tr} hw\sigma + \operatorname{tr}(hw\sigma)^{-1} + \beta)^r)$$
$$= \sum_{w\in P(t-1,1;q)} \lambda((\alpha \operatorname{tr} w\sigma + \operatorname{tr}(w\sigma)^{-1} + \beta)^r).$$

The first sum in (3.4) is

$$\sum_{A,B,d} \lambda((\alpha \operatorname{tr} A + \operatorname{tr} A^{-1} + \alpha d + d^{-1} + \beta)^r) = q^{t-1} \sum_{d\in\mathbb{F}_q^\times} K_{\mathrm{GL}(t-1,q)}(\lambda^r; \alpha, 1; \alpha d + d^{-1} + \beta), \tag{3.5}$$

where we use the form, with A of size $(t-1)\times(t-1)$, d of size 1×1, etc.,

$$w = \begin{bmatrix} A & B \\ 0 & d \end{bmatrix} \in P(t-1,1;q).$$

Write $w \in P(t-1,1;q)$ as

$$w = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ 0 & 0 & d \end{bmatrix}, \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \tag{3.6}$$

where A_{11}, E_{11}, d are of size 1×1, and A_{22}, E_{22} are of size $(t-2)\times(t-2)$, etc. Then the second sum in (3.4) is

$$\sum \lambda((-\alpha B_1 + \alpha \operatorname{tr} A_{22} + \operatorname{tr} E_{22} - d^{-1}E_{11}B_1 - d^{-1}E_{12}B_2 + \beta)^r), \tag{3.7}$$

where the sum is over all $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, d$.

We separate the sum in (3.7) into the one with $A_{12} \neq 0$ and the other with $A_{12} = 0$. Note that $A_{12} = 0$ if and only if $E_{12} = 0$.

The subsum of (3.7) with $A_{12} \neq 0$ is

$$\sum_{\substack{A \text{ with } A_{12}\neq 0 \\ B_1, d}} \sum_{B_2} \lambda((-\alpha B_1 + \alpha \operatorname{tr} A_{22} + \operatorname{tr} E_{22} - d^{-1}E_{11}B_1 - d^{-1}E_{12}B_2 + \beta)^r). \tag{3.8}$$

Fix A with $A_{12} \neq 0, B_1, d$. Write $E_{12} = [\alpha_1 \ldots \alpha_{t-2}]$, $B_2 = {}^t[\beta_1 \ldots \beta_{t-2}]$. Then $\alpha_k \neq 0$ for some k $(1 \leq k \leq t-2)$.

Noting that, for $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$,

$$\sum_{\gamma \in \mathbb{F}_q} \lambda((a\gamma + b)^r) = \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r),$$

we see that the inner sum of (3.8) equals

$$\sum_{\substack{\text{all } \beta_i \\ \text{with } i \neq k}} \sum_{\beta_k} \lambda((-d^{-1}\alpha_k\beta_k + \ldots)^r) = q^{t-3} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r). \tag{3.9}$$

Combining (3.8) and (3.9), we see that the subsum of (3.7) with $A_{12} \neq 0$ is

$$(g_{t-1} - (q-1)q^{t-2}g_{t-2})q^{t-2}(q-1) \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r). \tag{3.10}$$

The subsum of (3.7) with $A_{12} = 0$ is

$$\sum \lambda((-(\alpha + d^{-1}A_{11}^{-1})B_1 + \alpha \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1} + \beta)^r), \tag{3.11}$$

where the sum is over $A = \left[\begin{smallmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, d.

Again, we separate the sum (3.11) into two subsums according as $\alpha + d^{-1}A_{11}^{-1} \neq 0$ or $\alpha + d^{-1}A_{11}^{-1} = 0$.

Assume that $\alpha + d^{-1}A_{11}^{-1} \neq 0$, i.e., $d \neq -\alpha^{-1}A_{11}^{-1}$. Proceeding just as when we were dealing with (3.8), we see that the subsum of (3.11) with $d \neq -\alpha^{-1}A_{11}^{-1}$ is

$$(q-1)(q-2)q^{2t-4}g_{t-2} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r). \tag{3.12}$$

On the other hand, it is easy to see that the subsum of (3.11) with $d = -\alpha^{-1}A_{11}^{-1}$ equals

$$(q-1)q^{2t-3}K_{\mathrm{GL}(t-2,q)}(\lambda^r; \alpha, 1; \beta). \tag{3.13}$$

As noted in (4.12) of [8],

$$|B(t,q)\backslash P(t-1,1;q)| = q(q^{t-1} - 1)/(q-1). \tag{3.14}$$

From (2.2), (3.4), (3.5), (3.10)–(3.14), we get the following recursive relation.

Lemma 3.1. *Let $K_{\mathrm{GL}(t,q)}(\lambda^r; \alpha, 1; \beta)$ be the sum defined by* (3.3). *Then, for integers $t \geq 2$, $\alpha \in \mathbb{F}_q^\times$ and $\beta \in \mathbb{F}_q$,*

$$(3.15)\quad K_{\mathrm{GL}(t,q)}(\lambda^r;\alpha,1;\beta) = q^{\binom{t}{2}}(q^{t-1}-2)\prod_{j=1}^{t-1}(q^j-1)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) + q^{2t-2}(q^{t-1}-1)K_{\mathrm{GL}(t-2,q)}(\lambda^r;\alpha,1;\beta) + q^{t-1}\sum_{\gamma\in\mathbb{F}_q^\times}K_{\mathrm{GL}(t-1,q)}(\lambda^r;\alpha,1;\alpha\gamma+\gamma^{-1}+\beta).$$

Here we understand that $K_{\mathrm{GL}(0,q)}(\lambda^r;\alpha,1;\beta)=\lambda(\beta^r)$.

For a nontrivial additive character λ, $a,b,c\in\mathbb{F}_q$, and a positive integer r, we define the exponential sum $MK_m(\lambda^r;a,b;c)$ as

$$(3.16)\quad MK_m(\lambda^r;a,b;c) = \sum_{\gamma_1,\dots,\gamma_m\in\mathbb{F}_q^\times}\lambda\big((a\gamma_1+b\gamma_1^{-1}+\ldots+a\gamma_m+b\gamma_m^{-1}+c)^r\big)$$

for $m\ge 1$, and

$$(3.17)\quad MK_0(\lambda^r;a,b;c)=\lambda(c^r).$$

Note that, for $r=1$,

$$(3.18)\quad MK_m(\lambda;a,b,c)=\lambda(c)K(\lambda;a,b)^m,$$

with $K(\lambda;a,b)$ the usual Kloosterman sum as in (2.16).

If $c=0$, then for brevity, we write

$$(3.19)\quad MK_m(\lambda^r;a,b)=MK_m(\lambda^r;a,b;0).$$

From the recursive relation in (3.15), one can prove the following theorem by induction on t.

THEOREM 3.2. *For a nontrivial additive character* λ, *integers* $t,r\ge 1$, *and for* $\alpha\in\mathbb{F}_q^\times$ *and* $\beta\in\mathbb{F}_q$, *the exponential sum* $K_{\mathrm{GL}(t,q)}(\lambda^r;\alpha,1;\beta)$ *defined by* (3.3) *is*

$$(3.20)\quad K_{\mathrm{GL}(t,q)}(\lambda^r;\alpha,1;\beta) = q^{(t+1)(t-2)/2} \times\Big\{\prod_{j=1}^{t}(q^j-1)-\sum_{l=1}^{[(t+2)/2]}q^{l-1}(q-1)^{t+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\} \times\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r) + q^{(t+1)(t-2)/2}\sum_{l=1}^{[(t+2)/2]}q^l MK_{t+2-2l}(\lambda^r;\alpha,1;\beta)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),$$

where both unspecified sums are over all integers $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le j_{l-2} \le \ldots \le j_1 \le t+1$. *Here we adopt the convention that the unspecified sums are* 1 *for* $l=1$.

4. Main theorems. In this section, we consider the sum in (1.2),

$$\sum_{w \in \mathrm{GSp}(2n,q)} \lambda((\operatorname{tr} w)^r),$$

for any nontrivial additive character λ of $\mathbb{F}_q$ and any positive integer r, and find an explicit expression for it by using the decomposition in (2.13). An explicit expression for the similar sum over $\mathrm{Sp}(2n,q)$ will then follow by a simple observation.

The sum in (1.2) can be written, using (2.13), as

$$\sum_{b=0}^{n} |A_b \backslash P| \sum_{w \in Q} \lambda((\operatorname{tr} w\sigma_b)^r), \tag{4.1}$$

where $P = P(2n,q), Q = Q(2n,q), A_b = A_b(q), \sigma_b$ are respectively as in (2.5), (2.12), (2.7), (2.8).

Here one has to observe that, for each $h \in P$,

$$\sum_{w \in Q} \lambda((\operatorname{tr} w\sigma_b h)^r) = \sum_{w \in Q} \lambda((\operatorname{tr} hw\sigma_b)^r) = \sum_{w \in Q} \lambda((\operatorname{tr} w\sigma_b)^r).$$

Write $w \in Q$ as

$$w = \begin{bmatrix} 1_n & 0 \\ 0 & \alpha 1_n \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \tag{4.2}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad {}^tA^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ {}^tB_{12} & B_{22} \end{bmatrix},$$

$${}^tB_{11} = B_{11}, \quad {}^tB_{22} = B_{22}. \tag{4.3}$$

Here $A_{11}, A_{12}, A_{21}, A_{22}$ are respectively of sizes $b \times b, b \times (n-b), (n-b) \times b, (n-b) \times (n-b)$, and similarly for ${}^tA^{-1}, B$. Then

$$\sum_{w \in Q} \lambda((\operatorname{tr} w\sigma_b)^r) \tag{4.4}$$

$$= \sum \lambda((-\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^tB_{12} + \operatorname{tr} A_{22} + \alpha \operatorname{tr} E_{22})^r), \tag{4.5}$$

where the sum is over $A, B_{11}, B_{12}, B_{22}, \alpha$, and B_{11}, B_{22} are subject to the conditions in (4.3).

Consider the sum in (4.5) first for the case $1 \le b \le n-1$ so that A_{12} does appear. We separate the sum into two subsums, with $A_{12} \ne 0$ and with $A_{12}=0$; the latter will be further divided into two subsums, with A_{11} alternating or not. So the sum in (4.5) is

$$\sum_{A_{12}\neq 0}\cdots+\sum_{\substack{A_{12}=0\\ A_{11}\text{ not alternating}}}\cdots+\sum_{\substack{A_{12}=0\\ A_{11}\text{ alternating}}}\cdots \tag{4.6}$$

The first sum in (4.6) is

$$\begin{aligned} & q^{\binom{n-b+1}{2}} \\ & \times \sum_{\substack{A \text{ with } A_{12}\neq 0\\ B_{11},\alpha}} \sum_{B_{12}} \lambda\big((-\operatorname{tr} A_{11}B_{11}-\operatorname{tr} A_{12}{}^{t}B_{12}+\operatorname{tr} A_{22}+\alpha \operatorname{tr} E_{22})^{r}\big). \end{aligned} \tag{4.7}$$

The inner sum of (4.7) can be treated just as that of (3.8), so that it equals

$$q^{b(n-b)-1}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r). \tag{4.8}$$

Combining (4.7) and (4.8), we see that the first sum of (4.6) equals

$$(q-1)q^{(n-1)(n+2)/2}(g_n-g_bg_{n-b}q^{b(n-b)})\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r). \tag{4.9}$$

The subsum of (4.5) with $A_{12}=0$ is

$$\begin{aligned} \sum_{A_{21},B_{12},B_{22}} & \sum_{A_{11},A_{22},B_{11},\alpha} \lambda\big((-\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{22}+\alpha \operatorname{tr} A_{22}^{-1})^{r}\big) \\ &= q^{\binom{n-b+1}{2}+2b(n-b)} \\ &\quad \times \sum_{A_{11},A_{22},B_{11},\alpha} \lambda\big((-\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{22}+\alpha \operatorname{tr} A_{22}^{-1})^{r}\big). \end{aligned} \tag{4.10}$$

Write $A_{11}=(\alpha_{ij})$ and $B_{11}=(\beta_{ij})$. Then $\operatorname{tr} A_{11}B_{11}=\sum_{1\le i\le j\le b}\gamma_{ij}\beta_{ij}$, where

$$\gamma_{ij}=\begin{cases}\alpha_{ii} & \text{if } i=j,\\ \alpha_{ij}+\alpha_{ji} & \text{if } i<j.\end{cases}$$

So A_{11} is alternating if and only if $\gamma_{ij}=0$ for all $1\le i\le j\le b$.

The subsum of the sum in (4.10) with A_{11} not alternating is

$$\sum_{\substack{A_{11}\text{ not alternating}\\ A_{22},\alpha}}\sum_{B_{11}}\lambda\big((-\operatorname{tr} A_{11}B_{11}+\operatorname{tr} A_{22}+\alpha\ \operatorname{tr} A_{22}^{-1})^{r}\big). \tag{4.11}$$

As A_{11} is not alternating, $\gamma_{st}\neq 0$ for some s,t. By the same argument as in the case of (3.8), we see that the inner sum of (4.11) equals

$$q^{\binom{b+1}{2}-1}\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r). \tag{4.12}$$

Combining (4.10)–(4.12) shows that the middle sum in (4.6) is

$$(q-1)q^{(n-1)(n+2)/2}q^{b(n-b)}g_{n-b}(g_b-a_b)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r), \tag{4.13}$$

where a_b denotes the number of all $b\times b$ nonsingular alternating matrices over $\mathbb{F}_q$ for each positive integer b.

The subsum of (4.10) with A_{11} alternating is

$$\begin{aligned}\sum_{\substack{A_{11}\text{ alternating}\\ B_{11}}}\sum_{\alpha}\sum_{A_{22}}\lambda((\operatorname{tr}A_{22}+\alpha\operatorname{tr}A_{22}^{-1})^r)\\ = a_b q^{\binom{b+1}{2}}\sum_{\alpha\in\mathbb{F}_q^\times}K_{\mathrm{GL}(n-b,q)}(\lambda^r;\alpha,1),\end{aligned} \tag{4.14}$$

where $K_{\mathrm{GL}(n-b,q)}(\lambda^r;\alpha,1)$ is as in (3.1). Combining (4.10) and (4.14), we see that the last sum in (4.6) is

$$q^{(n-1)(n+2)/2}q^{b(n-b)+1}a_b\sum_{\alpha\in\mathbb{F}_q^\times}K_{\mathrm{GL}(n-b,q)}(\lambda^r;\alpha,1). \tag{4.15}$$

Adding up (4.9), (4.13), and (4.15), we have shown that, for each $1\le b\le n-1$, the sum in (4.4) is

$$\begin{aligned}q^{(n-1)(n+2)/2}\Big\{(q-1)(g_n-q^{b(n-b)}g_{n-b}a_b)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)\\ +q^{b(n-b)+1}a_b\sum_{\alpha\in\mathbb{F}_q^\times}K_{\mathrm{GL}(n-b,q)}(\lambda^r;\alpha,1)\Big\}.\end{aligned} \tag{4.16}$$

Next, we consider the sum in (4.4) for $b=n$, which is given by

$$\sum_{w\in Q}\lambda((-\operatorname{tr}AB)^r) \tag{4.17}$$

with w as in (4.2). Just as when we were dealing with the subsum of (4.5) with $A_{12}=0$, we separate the sum in (4.17) into the one with A alternating and the other with A not alternating. Proceeding as above, we see that (4.17) equals

$$(q-1)q^{(n-1)(n+2)/2}\Big\{(g_n-a_n)\sum_{\gamma\in\mathbb{F}_q}\lambda(\gamma^r)+qa_n\Big\}. \tag{4.18}$$

So if we agree that $g_0=1, K_{\mathrm{GL}(0,q)}(\lambda^r;\alpha,1)=1$ then this is just (4.16) for $b=n$. Observe that $g_0=1$ is natural in view of the formula in (2.2). Further, $K_{\mathrm{GL}(0,q)}(\lambda^r;\alpha,1)=1$ is equivalent to saying that $MK_0(\lambda^r;\alpha,1)=1$ (cf. (3.19)), which is consistent with our convention in (3.17).

Finally, the sum in (4.4) for $b = 0$ is given by

$$\sum_{w \in Q} \lambda((\operatorname{tr} A + \alpha \operatorname{tr} A^{-1})^r) = q^{\binom{n+1}{2}} \sum_{\alpha \in \mathbb{F}_q^\times} K_{\mathrm{GL}(n,q)}(\lambda^r; \alpha, 1), \tag{4.19}$$

again with w as in (4.2). This agrees with (4.16) for $b = 0$ if we understand that $a_0 = 1$. Here again $a_0 = 1$ is natural in view of the formula in (2.10).

Putting everything together, we have shown so far that the sum in (4.1) can be written as

$$\begin{aligned}(q-1)q^{(n-1)(n+2)/2} \Big\{ \sum_{b=0}^{n} |A_b \backslash P| (g_n - q^{b(n-b)} g_{n-b}\, a_b) \Big\} \sum_{\gamma \in \mathbb{F}_q} \lambda(\gamma^r) \\ + q^{\binom{n+1}{2}} \sum_{b=0}^{n} |A_b \backslash P| q^{b(n-b)} a_b \sum_{\alpha \in \mathbb{F}_q^\times} K_{\mathrm{GL}(n-b,q)}(\lambda^r; \alpha, 1).\end{aligned} \tag{4.20}$$

From (2.2), (2.9), (2.10), (2.14), (2.15), (2.18) and from the explicit expression of $K_{\mathrm{GL}(t,q)}(\lambda^r; \alpha, 1)$ in (3.20) with $\beta = 0$ (cf. (3.19)), we have the following theorem.

Theorem 4.1. *For any nontrivial additive character λ of $\mathbb{F}_q$ and any positive integer r, the exponential sum*

$$\sum_{w \in \mathrm{GSp}(2n,q)} \lambda((\operatorname{tr} w)^r)$$

is given by

$$\begin{aligned}(q-1)q^{n^2-1} \Big\{ \prod_{j=1}^{n} (q^{2j} - 1) - \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1} - 1) \\ \times \sum_{l=1}^{[(n-2b+2)/2]} q^{l-1} (q-1)^{n-2b+2-2l} \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1) \Big\} \sum_{j=1}^{e-1} G(\psi^j, \lambda) \\ + q^{n^2-1} \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1} - 1) \\ \times \sum_{l=1}^{[(n-2b+2)/2]} q^{l} \sum_{\alpha \in \mathbb{F}_q^\times} MK_{n-2b+2-2l}(\lambda^r; \alpha, 1) \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1),\end{aligned} \tag{4.21}$$

where both unspecified sums run over the same set of integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 1 \le j_{l-1} \le \ldots \le j_1 \le n - 2b + 1, \psi$ is a multiplicative character of $\mathbb{F}_q$ of order $e = (r, q-1)$, and $MK_m(\lambda^r; \alpha, 1)$ is the exponential sum defined in (3.16), (3.17) (cf. (3.18), (3.19)).

As, with $d_\alpha = \left[\begin{smallmatrix} 1_n & 0 \\ 0 & \alpha 1_n \end{smallmatrix}\right]$,

$$\mathrm{GSp}(2n,q) = \coprod_{\alpha\in\mathbb{F}_q^\times} d_\alpha\,\mathrm{Sp}(2n,q),$$

we see that the sum $\sum_{w\in\mathrm{Sp}(2n,q)} \lambda((\mathrm{tr}\,w)^r)$ in (1.1) is the same as the expression in (4.21), except that the foremost term $q-1$ does not appear and that $\sum_{\alpha\in\mathbb{F}_q^\times} MK_{n-2b+2-2l}(\lambda^r;\alpha,1)$ is replaced by $MK_{n-2b+2-2l}(\lambda^r;1,1)$.

THEOREM 4.2. *For any nontrivial additive character λ of $\mathbb{F}_q$ and any positive integer r, the exponential sum*

$$\sum_{w\in\mathrm{Sp}(2n,q)} \lambda((\mathrm{tr}\,w)^r)$$

is given by

$$\begin{aligned}(4.22)\qquad & q^{n^2-1}\Big\{\prod_{j=1}^{n}(q^{2j}-1) - \sum_{b=0}^{[n/2]} q^{b(b+1)}\begin{bmatrix} n \\ 2b\end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1)\\ &\times \sum_{l=1}^{[(n-2b+2)/2]} q^{l-1}(q-1)^{n-2b+2-2l}\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)\Big\}\sum_{j=1}^{e-1}G(\psi^j,\lambda)\\ &+ q^{n^2-1}\sum_{b=0}^{[n/2]} q^{b(b+1)}\begin{bmatrix} n \\ 2b\end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1)\\ &\times \sum_{l=1}^{[(n-2b+2)/2]} q^{l} MK_{n-2b+2-2l}(\lambda^r;1,1)\sum\prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1),\end{aligned}$$

where both unspecified sums, ψ, and $MK_m(\lambda^r;1,1)$ are as in Theorem 4.1.

REMARK. If $r=1$, then Theorems 4.1 and 4.2 reduce respectively to Theorem 5.3 with χ trivial and Theorem 5.4 in [8].

5. Applications to certain countings. If $G(q)$ is one of the finite classical groups over $\mathbb{F}_q$, then, for each $\beta\in\mathbb{F}_q$, we put

$$(5.1)\qquad N_{G(q)}(\beta) = |\{w\in G(q) \mid \mathrm{tr}\,w = \beta\}|.$$

As applications, we will derive formulas for (5.1) in the case of $G(q) = \mathrm{Sp}(2n,q)$ and $\mathrm{GSp}(2n,q)$.

For λ a nontrivial additive character of $\mathbb{F}_q$, we have

$$(5.2)\qquad qN_{G(q)}(\beta) = |G(q)| + \sum_{\alpha\in\mathbb{F}_q^\times}\lambda(-\beta\alpha)\sum_{w\in G(q)}\lambda(\alpha\,\mathrm{tr}\,w).$$

Also, the following lemma can easily be proved.

LEMMA 5.1. *Let λ be a nontrivial additive character of $\mathbb{F}_q$, $\beta \in \mathbb{F}_q$, and let m be a nonnegative integer. Then*

$$\sum_{\alpha \in \mathbb{F}_q^\times} \lambda(-\beta\alpha) K(\lambda;\alpha,\alpha)^m = q\delta(m,q;\beta) - (q-1)^m, \tag{5.3}$$

where, *for $m \geq 1$*,

$$\delta(m,q;\beta) = |\{(\alpha_1,\ldots,\alpha_m) \in (\mathbb{F}_q^\times)^m \mid \alpha_1 + \alpha_1^{-1} + \ldots + \alpha_m + \alpha_m^{-1} = \beta\}| \tag{5.4}$$

and

$$\delta(0,q;\beta) = \begin{cases} 1 & \text{if } \beta = 0, \\ 0 & \text{otherwise}. \end{cases} \tag{5.5}$$

Observe that an explicit expression of $\sum_{w \in \mathrm{Sp}(2n,q)} \lambda(\alpha \operatorname{tr} w)$ for $\alpha \in \mathbb{F}_q^\times$ is given by [8, Theorem 5.4] with $K(\lambda;1,1)$ replaced by $K(\lambda;\alpha,\alpha)$. Now, this observation combined with (2.4), (5.2), (5.3) yields the following theorem.

THEOREM 5.2. *For each $\beta \in \mathbb{F}_q$, the number $N_{\mathrm{Sp}(2n,q)}(\beta)$, defined in* (5.1) *with $G(q) = \mathrm{Sp}(2n,q)$, is given by*

$$\begin{aligned} &q^{n^2-1} \prod_{j=1}^{n} (q^{2j}-1) \\ &+ q^{n^2-1} \sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b} (q^{2j-1}-1) \\ &\times \sum_{l=1}^{[(n-2b+2)/2]} q^l (\delta(n-2b+2-2l,q;\beta) - q^{-1}(q-1)^{n-2b+2-2l}) \\ &\times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \end{aligned} \tag{5.6}$$

where the innermost sum is over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-1 \leq j_{l-1} \leq \ldots \leq j_1 \leq n-2b+1$, and $\delta(m,q;\beta)$ is defined as in (5.4) *and* (5.5).

As $\alpha\mathrm{GSp}(2n,q) = \mathrm{GSp}(2n,q)$ for any $\alpha \in \mathbb{F}_q^\times$, we see from (5.2) that

$$N_{\mathrm{GSp}(2n,q)}(\beta) = q^{-1}|\mathrm{GSp}(2n,q)| + q^{-1} \sum_{w \in \mathrm{GSp}(2n,q)} \lambda(\operatorname{tr} w) \sum_{\alpha \in \mathbb{F}_q^\times} \lambda(-\beta\alpha). \tag{5.7}$$

So we get the following theorem from (2.11), (5.7), and [8, Theorem 5.3].

THEOREM 5.3. *For each $\beta \in \mathbb{F}_q$, the number $N_{\mathrm{GSp}(2n,q)}(\beta)$ is given by*

$$\begin{cases} (q-1)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1) - q^{-1}\sum_{w\in\mathrm{GSp}(2n,q)}\lambda(\operatorname{tr} w) & \text{if } \beta \neq 0, \\ (q-1)q^{n^2-1}\prod_{j=1}^{n}(q^{2j}-1) + q^{-1}(q-1)\sum_{w\in\mathrm{GSp}(2n,q)}\lambda(\operatorname{tr} w) & \text{otherwise}, \end{cases} \tag{5.8}$$

where

$$\begin{aligned} \sum_{w\in\mathrm{GSp}(2n,q)}\lambda(\operatorname{tr} w) &= q^{n^2-1}\sum_{b=0}^{[n/2]} q^{b(b+1)} \begin{bmatrix} n \\ 2b \end{bmatrix}_q \prod_{j=1}^{b}(q^{2j-1}-1) \\ &\times \sum_{l=1}^{[(n-2b+2)/2]} q^l \sum_{\alpha\in\mathbb{F}_q^\times} K(\lambda;\alpha,1)^{n-2b+2-2l} \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1). \end{aligned} \tag{5.9}$$

Here the innermost sum is over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-2b+1$.

REMARK. As we remarked in [8], the following average of tth powers of Kloosterman sums, appearing in (5.9),

$$\sum_{\alpha\in\mathbb{F}_q^\times} K(\lambda;\alpha,1)^t$$

was studied by some authors [15], [16], [18].

In particular, it can be shown that, for any nontrivial additive character λ of $\mathbb{F}_q$, we have

$$\sum_{\alpha\in\mathbb{F}_q^\times} K(\lambda;\alpha,1)^t = \sum_{\alpha\in\mathbb{F}_q^\times} K(\lambda_1;\alpha,1)^t,$$

and, for $t \ge 1$,

$$\sum_{\alpha\in\mathbb{F}_q^\times} K(\lambda_1;\alpha,1)^t = q^2 M_{t-1} - (q-1)^{t-1} + 2(-1)^{t-1}, \tag{5.10}$$

where M_t is the number of $\alpha_1,\ldots,\alpha_t \in \mathbb{F}_q^\times$ satisfying $\alpha_1+\ldots+\alpha_t = 1$ and $\alpha_1^{-1}+\ldots+\alpha_t^{-1} = 1$ for $t \ge 1$, $M_0 = 0$, and λ_1 is as in (2.1).

In [18], Salié showed (5.10) under the assumption that q is an odd prime. However, this assumption is not necessary and it holds true for any q.

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