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EXPONENTIALLY BOUNDED POSITIVE DEFINITE FUNCTIONS

BY

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Abstract

Equivalent conditions for scalar (or operator valued) positive definite functions, on a commutative semigroup S with identity e, to admit a disintegration with respect to a regular positive (operator valued) measure supported by an arbitrary compact subset of semicharacters are given. The theory links to the theory of τ-positive functions presented previously by the second author and comparisons between the two are given. Old and new theorems to classical and modern moment problems are obtained as a consequence.

Introduction

Lindahl-Maserick [7] and independently Berg, Christensen and Ressel [4] studied bounded positive definite functions on commutative semigroups with involution; the main result being an integral representation of these functions with respect to non-negative regular Borel measures supported by the semicharacters. The boundedness condition is a rather restrictive condition which guarantees that the measures be supported by compact sets of semicharacters with values in the unit disk. On the other hand, there exist unbounded positive definite functions which do not admit non-negative representing measures at all, cf. [3]. In this work, we show that the weaker notion of "exponentially bounded" characterizes those positive definite functions which admit (necessarily unique) non-negative representing measures supported by arbitrary compact subsets of semicharacters (Theorem 2.1).

In Section 1, we introduce the notion of an absolute value $|\cdot|$ on a semigroup S with involution and define boundedness with respect to it. The positive definite functions which are bounded with respect to a given absolute value form a convex cone with compact base, so that Choquet theory implies the desired integral representation for such functions.

Our main tool is the natural homomorphism $s \to E_s$ of S into the linear operators on the reproducing kernel prehilbert space H_f associated with the

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positive definite function f. It turns out that E_s is bounded in norm by $|s|$ if and only if f is bounded with respect to $|\cdot|$. Other conditions which are equivalent to and which imply boundedness of E_s are given in Theorem 1.3, Corollary 1.4 and Proposition 1.5 respectively.

In Section 2, we study the close relation between exponentially bounded positive definite functions and the τ-positive functions introduced by the second author [10]; our main result being that a function is τ-positive with respect to a "linearly admissible τ", if and only if Tf is positive definite for each $T \in \tau$ (Corollary 2.5). We conclude the section by giving a spectral resolution of positive definite operator-valued functions and by showing that these functions are exactly those of positive type in the sense of Nagy [15] (Theorem 2.6 and Corollary 2.7). The theory extends that given in Maserick [8] and implies the standard spectral theorems for positive, hermitian, unitary and normal operators.

When S is a group with $s^* = s^{-1}$, the theory presented here adds nothing new since every positive definite function is in fact bounded. However, general semigroups admit unbounded positive definite functions. In Section 3, we give an example where the exponentially bounded theory applies but the bounded theory is trivial. By considering special families of shift operators, old and new solutions to classical and modern moment problems are obtained.

1. Positive definite functions and representations of S

Throughout the sequel, $S = (S, \cdot, *)$ will denote a commutative semigroup with involution * and identity e. For the precise definition, see [7]. A generator set, $G \subset S$, is any subset such that every element $s \in S \setminus \{e\}$ is a finite product of elements of $G \cup G^*$. We equip the vector space $\mathcal{F}(S) \equiv \mathcal{F}$ of complex-valued functions on S with the topology of point-wise convergence. It follows that $\mathcal{F}$ is a completely regular topological space. A function $f : S \to \mathbf{C}$ is called *positive definite* if

$$(s, t) \to f(st^*)$$

is a positive definite kernel on $S \times S$, i.e., if $\Sigma_{i,j} c_i \overline{c_j} f(s_i s_j^*) \geq 0$ for all choices of $s_1, \ldots, s_n \in S$ and scalars $c_1, \ldots, c_n \in \mathbf{C}$. The set of positive definite functions on S is denoted by $\mathcal{P}(S) \equiv \mathcal{P}$. Clearly $\mathcal{P}$ is a closed convex cone in $\mathcal{F}$. Moreover, it is well known and follows easily that each $f \in \mathcal{P}$ has the following properties for $s, t \in S$:

$$f(s^*) = \overline{f(s)}, \quad f(ss^*) \geq 0, \tag{1.1}$$

$$|f(st^*)|^2 \leq f(ss^*) f(tt^*). \tag{1.2}$$

In particular,

$$|f(s)|^2 \leq f(e) f(ss^*), \tag{1.3}$$

which shows that f is identically zero whenever $f(e) = 0$.

DEFINITION 1.1. An *absolute value* on S is a function $|\cdot| : S \to [o, \infty[$ satisfying

$$|ss^*| \leq |s|^2 \quad \text{for all } s \in S, \tag{1.4}$$

and

$$|e| \geq 1. \tag{1.5}$$

DEFINITION 1.2. A function $f: S \to \mathbf{C}$ is called *bounded with respect to an absolute value* $|\cdot|$, if there exists a constant $K > 0$ such that $|f(s)| \leq K|s|$ for $s \in S$, and f is called *exponentially bounded* if f is bounded with respect to some absolute value.

LEMMA 1.1. *Let f be a positive definite function which is bounded with respect to the absolute value $|\cdot|$. Then $|f(s)| \leq f(e)|s|$ for all $s \in S$.*

Proof. Without loss of generality we may assume $f(e) = 1$. For $s \in S$ and $a = ss^*$, we find by p successive applications of (1.3) that $|f(s)|^{2^p} \leq f(a^{2^{p-1}})$. Since $a = a^*$ and $|f(s)| \leq K|s|$ for $s \in S$, (1.4) leads to

$$|f(s)|^{2^p} \leq K|a|^{2^{p-1}} \leq K|s|^{2^p}.$$

Hence $|f(s)| \leq (\lim_{p\to\infty} K^{2^{-p}})|s| = |s|$. □

Remark. If f is bounded then f is bounded with respect to the absolute value $|s| \equiv 1$, so the lemma implies that bounded positive definite functions are in fact bounded by $f(e)$. This particular case has been explored in detail; cf. [4], [7] and [9]. Observe that if S is a group with $s^* = s^{-1}$, or an indempotent semigroup with $s = s^*$ or more generally an inverse semigroup (cf. [5]) with $s^{-1} = s^*$, then (1.3) implies directly that every positive definite function is bounded (and hence exponentially bounded) by $f(e)$, so that nothing new is gained in such settings.

For each $s \in S$, consider the shift operator E_s defined on $\mathcal{F}$ by $E_s\phi(t) = \phi(st)$ for $\phi \in \mathcal{F}$ and $t \in S$. Clearly, $E_{st} = E_sE_t$, so that the complex linear span of these operators is a commutative algebra $\mathcal{A}$. Defining

$$\left(\sum_i c_i E_{s_i}\right)^* = \sum_i \overline{c_i} E_{s_i^*}$$

equips $\mathcal{A}$ with an involution. An element T of an algebra with involution is called *hermitian* if $T^* = T$.

For an arbitrary function $f: S \to \mathbf{C}$ the subspace $H_f \subset \mathcal{F}$ defined by

$$H_f = \{Tf \mid T \in \mathcal{A}\}$$

is invariant under each application of $R \in \mathcal{A}$. The restriction of $R \in \mathcal{A}$ to H_f is given by $Tf \to RT(f)$ and is still denoted by R. We denote the set of these restrictions by $\mathcal{A}_f$, i.e.,

$$\mathcal{A}_f = \{R|_{H_f} \mid R \in \mathcal{A}\}.$$

$\mathscr{A}_f$ is again a commutative algebra with involution.

Similarly, for $\tau \subset \mathscr{A}$ let τ_f be the set of restrictions of elements of τ to H_f.

Suppose f satisfies $f(s^*) = \overline{f(s)}$. Then $T^*f(s) = \overline{Tf(s^*)}$, so $T_1f = T_2f$ implies $T_1^*f = T_2^*f$ and therefore the expression

$$<Tf, Rf> := TR^*f(e), \quad T, R \in \mathscr{A},$$

yields a well defined sesquilinear form $<\cdot,\cdot>$ on H_f. For $T = \Sigma_{i=1}^n c_i E_{s_i}$ we find

$$<Tf, Tf> = \sum_{i,j=1}^{n} c_i \overline{c_j} f(s_i s_j^*),$$

thus showing that $<\cdot,\cdot>$ is a positive sesquilinear form if and only if f is positive definite.

Supposing that f is positive definite the Cauchy-Schwarz inequality gives

$$|Tf(s)|^2 = |<Tf, E_{s^*}f>|^2 \le <Tf, Tf> \cdot <E_{s^*}f, E_{s^*}f>$$

or

$$|(Tf)(s)| \le \|E_{s^*}f\| \cdot \|Tf\|. \tag{1.6}$$

Thus $\|Tf\| = 0$ implies $Tf \equiv 0$, which shows that $<\cdot,\cdot>$ is a non-degenerate inner product. Replacing T by $T_n - T_m$ in (1.6) shows that Cauchy sequences in H_f are always pointwise Cauchy, so that the completion $\overline{H}_f$ is continuously embedded in $\mathscr{F}$. Furthermore, the only possible continuous extension of an operator $T|_{H_f} \in \mathscr{A}_f$ to the completion $\overline{H}_f$ is given by $T\phi$ for $\phi \in \overline{H}_f \subset \mathscr{F}$. An application of the closed graph theorem shows that $T|_{H_f}$ admits a continuous extension to $\overline{H}_f$ if and only if $T(\overline{H}_f) \subset \overline{H}_f$. Notice that $\overline{H}_f$ is the classical reproducing kernel Hilbert space of Aronszajn associated with f.

Proposition 1.2. *Let $f \in \mathscr{P}$. Then $T \in \mathscr{A}_f$ is a positive operator on the prehilbert space H_f if and only if Tf is a positive definite function on S.*

Proof. Let $R = \Sigma_i c_i E_{s_i}$. Then

$$<TRf, Rf> = (TRR^*f)(e) = \sum_{i,j} c_i \overline{c_j}\, TE_{s_i s_j^*} f(e) = \sum c_i \overline{c_j} (Tf)(s_i s_j^*),$$

from which the assertion follows. □

If $f \in \mathscr{P}$ the map $s \to E_s$ is a *-homomorphism of S into the linear operators on H_f satisfying

$$f(s) = <E_s f, f>. \tag{1.7}$$

In general any *-homomorphism $E : S \to B(H)$ of S into the bounded operators $B(H)$ of a Hilbert space H such that $E_e = I$ will be called a *representation* of S. If S admits a representation E and $\xi \in H$, then the function defined by $f(s) = <E_s\xi, \xi>$ is positive definite and bounded with respect to the absolute value $|s| := \|E_s\|$, where $\|\cdot\|$ is the norm of the operator $E_s \in B(H)$.

THEOREM 1.3. *If $f \in \mathscr{P}$ and $|\cdot|$ is an absolute value on S, then the following are equivalent:*

(i) $\|E_s\| \leq |s|$ *for all* $s \in S$.

(ii) f *is bounded with respect to* $|\cdot|$.

(iii) (Nagy) $(|s|^2 I - E_{ss*})f$ *is positive definite for each* $s \in S$.

(iv) $(|s| I - \frac{1}{2} E_s - \frac{1}{2} E_{s*})f$ *is positive definite for each* $s \in S$.

(v) $(|s| I + \frac{1}{2} \sigma E_s + \frac{1}{2} \bar{\sigma} E_{s*})f$ *is positive definite for each* $s \in S$ *and* $\sigma \in \mathbf{C}$ *with* $|\sigma| \leq 1$.

Proof. (i) implies (ii) as mentioned above, so we will assume f bounded with respect to $|\cdot|$ and show that (iii) holds. Fix $T \in \mathscr{A}$. Then $TT^*f \in \mathscr{P}$ by Proposition 1.2, so that

$$TT^*f(ss^*) = \langle E_{ss*}f, TT^*f\rangle \leq \|E_{ss*}f\| \cdot \|TT^*f\| =$$
$$f^{1/2}((ss^*)^2)\|TT^*f\| \leq f^{1/2}(e)\|TT^*f\| \cdot |s|^2.$$

Thus (1.3) implies $|TT^*f(s)|^2 \leq K|s|^2$ for some constant K, so TT^*f is bounded with respect to $|\cdot|$. Applying Lemma 1.1 gives $TT^*f(ss^*) \leq TT^*f(e)|s|^2$, which shows

$$\langle(|s|^2 I - E_{ss*})Tf, Tf\rangle \geq 0.$$

Since T was arbitrary, $|s|^2 I - E_{ss*}$ is a positive operator on H_f, which is equivalent to positive definiteness of $(|s|^2 I - E_{ss*})f$ by Proposition 1.2. Therefore (iii) follows from (ii). As is well known and easy to see, positiveness of any operator $\alpha^2 I - AA^*$ implies $\|A\| \leq |\alpha|$ so that (i), (ii) and (iii) are all equivalent. Suppose that (v) holds. Clearly (iv) holds. Replacing s by ss^* we get that $(|ss^*| I - E_{ss*})f$ is positive definite, and using (1.4) we see that (iii) holds. Since $\alpha I + A$ is a positive operator whenever A is hermitian and $\|A\| \leq \alpha$, then (v) follows from (i) upon replacing A by

$$\frac{1}{2}\sigma E_s + \frac{1}{2}\bar{\sigma} E_{s*}$$

and α by $|s|$. □

COROLLARY 1.4. *Let $f \in \mathscr{P}$. Then f is exponentially bounded if and only if $E_s \in \mathscr{A}_f$ is a bounded operator on H_f for each $s \in S$.*

Remarks. If $E_s \in \mathscr{A}_f$ is bounded for each $s \in S$, then the operator norm $\|E_s\|$ is clearly the minimal absolute value which satisfies any of (ii) through (v). If $|s| = 0$ then $\|E_s\| = 0$ so that E_s and E_{s*} $(= E_s^*)$ are the zero operators on H_f. Also

$$|f(s^n)| \leq f(e)\|E_{s^n}\| \leq f(e)\|E_s\|^n \leq f(e)|s|^n,$$

which explains the terminology "exponentially bounded" in the title. Definition 1.2 is weaker than that given by Szafraniec [16]. Other equivalent definitions for $f \in \mathscr{P}$ to be exponentially bounded are

(a) $f(ss^*) \leq K|s|^2$ and

(b) $|f(s^n)| \leq K_s|s|^n$ for $n \in \mathbf{N}$,

where in the latter case the constant K_s may depend on s. It follows from the remark after Lemma 1.1, that if S is an inverse semigroup, then each positive definite function is bounded by $f(e)$, thus $\|E_s\| \leq 1$, without any further boundedness restrictions.

Other conditions which imply boundedness of the operator $E_s \in \mathscr{A}_f$ can easily be derived from known relations between positiveness and boundedness of operators by appealing to Proposition 1.2 as in the above proof. We list a somewhat general scenario below which we will appeal to later.

Let A be an arbitrary commutative algebra with involution and identity I. Following Maserick [10], we define a subset $\tau \subset$ U to be *admissible* if

(i) $T = T^*$ for every $T \in \tau$.

(ii) $I - T \in$ alg span$^+ \tau$ for every $T \in \tau$.

(iii) alg span $\tau =$ U.

If condition (ii) is replaced by the stronger condition

(ii)′ $I - T \in$ span$^+ \tau$ for every $T \in \tau$,

then the admissible subset τ will be called *linearly admissible.*

If τ is a (linearly) admissible subset of the algebra $\mathscr{A}$ of shift operators, then τ_f is (linearly) admissible in the algebra $\mathscr{A}_f$ of restrictions to H_f, but the converse is not in general true. If E denotes the unit shift operator on the semigroup $\mathbf{N} = \{0, 1, 2, \ldots\}$ under addition with $k^* = k$ for all $k \in \mathbf{N}$, then $\{I - E^k, I - E, E\}$ is an example of an admissible τ which is not linearly admissible for $n \geq 2$.

PROPOSITION 1.5. *Let $f \in \mathscr{P}$ and let $\tau \subset \mathscr{A}$ be such that τ_f is linearly admissible in $\mathscr{A}_f$. Then each of the following conditions implies that E_s is a bounded operator on H_f for each $s \in S$:*

(i) *Tf is positive definite for each $T \in \tau$.*

(ii) *$T(I - T)f$ is positive definite for each $T \in \tau$.*

Proof. Since τ_f is linearly admissible, then positiveness of each $T \in \tau_f$ implies positiveness of $I - T$. Using the identity $T(I - T) = T^2(I - T) + T(I - T)^2$, then it is easily seen that T and $I - T$ are positive if and only if their product $T(I - T)$ is positive, and the latter condition certainly implies boundedness of T. The assertion follows, since the algebraic span of τ_f is $\mathscr{A}_f$. □

If f is positive definite and bounded with respect to $|\cdot|$, then from the remarks following Theorem 1.3, the operators E_s and E_{s*} are both zero on H_f whenever $|s| = 0$. If $G \subset S$ is geneator set, then we set

$$\tau = \{T_{(\sigma,s)} = \frac{1}{2}(I + \frac{\sigma}{2|s|} E_s + \frac{\overline{\sigma}}{2|s|} E_{s*}) | s \in G, \sigma^4 = 1\}, \qquad (1.8)$$

with the understanding that

$$\frac{1}{|s|} E_s = \frac{1}{|s|} E_{s*} = 0 \quad \text{whenever } |s| = 0.$$

This subset τ, is linearly admissible in $\mathscr{A}_f$. Other examples will be mentioned in §3.

Remark. The representation of S in $B(\overline{H}_f)$ is similar to the so-called *GNS*-construction on groups. The special case of f being bounded is carried out in Lindahl and Maserick [7]; see also Schempp [14].

2. Disintegration of exponentially bounded positive definite functions

For our purpose we define $\varrho \in \mathscr{F}$ to be a *semicharacter* if

$$\varrho(e) \neq 0, \quad \varrho(st) = \varrho(s)\varrho(t) \quad \text{and} \quad \varrho(s^*) = \overline{\varrho(s)}$$

for all $s, t \in S$. We denote the set of all semicharacters on S by $\Gamma(S) \equiv \Gamma$ and equip Γ with the (completely regular) topology inherited from $\mathscr{F}$, i.e., the topology of pointwise convergence. Clearly $\Gamma \subset \mathscr{P}$. For a compact subset $K \subseteq \Gamma$ we define an absolute value $|\cdot|_K$ by

$$|s|_K = \sup\{|\varrho(s)| \mid \varrho \in K\}, \quad s \in S.$$

Clearly $|st|_K \leq |s|_K |t|_K$, $|ss^*|_K = |s|_K^2$ and $|e|_K = 1$.

We denote the class of all non-negative Borel measures on Γ which are inner regular with respect to compact subsets by $\mathscr{M}^+$ and the support of $\mu \in \mathscr{M}^+$ by $\text{supp}(\mu)$. If $\mu \in \mathscr{M}^+$ has compact support $K \subseteq \Gamma$, then the function

$$f(s) = \int \varrho(s) d\mu$$

is positive definite and bounded with respect to $|\cdot|_K$. Thus Theorem 1.3 implies that $s \to E_s$ is a representation of S. For each $T \in \mathscr{A}$, the map $\varrho \to T\varrho(e)$ is a continuous function on K. Since $T\varrho(e)R\varrho(e) = TR\varrho(e)$ and $T^*\varrho(e) = \overline{T\varrho(e)}$ for all $\varrho \in K$, then the functions $\varrho \to T\varrho(e)$, $T \in \mathscr{A}$, form a dense subalgebra of the algebra $C(K)$ of all continuous functions on K. The map $\Pi : H_f \to L_2(\mu)$ defined by

$$Tf \to [\varrho \to T\varrho(e)]$$

is well defined since $Tf = 0$ implies

$$0 = RTf(e) = \int R\varrho(e) T\varrho(e) d\mu \quad \text{for all } R \in \mathscr{A},$$

so that $T\varrho(e) = 0$ for all $\varrho \in K$. Since

$$<Tf, Rf> = TR^*f(e) = \int T\varrho(e)\overline{R\varrho(e)}\,d\mu,$$

Π densely embeds the prehilbert space H_f in the Hilbert space $L_2(\mu)$. Thus we have proved that if f admits a representing measure with compact support, then (i) $s \to E_s$ is a representation of S and (ii) $L_2(\mu)$ is the completion of H_f. Our main result is the converse of (i); namely that such a representing measure exists whenever $s \to E_s$ is a representation; i.e., $\|E_s\| < \infty$ for $s \in S$.

THEOREM 2.1. *If $f: S \to \mathbf{C}$ is positive definite and bounded with respect to an absolute value $|\cdot|$, then there is a unique $\mu \in \mathscr{M}^+$ satisfying*

$$f(s) = \int \varrho(s) d\mu(\varrho). \tag{2.1}$$

Moreover, supp(μ) *is a compact subset of the set of those semicharacters which are bounded with respect to $|\cdot|$.*

Proof. (i) Existence of μ. We assume f is bounded with respect to $|\cdot|$, and let $\mathscr{P}_{|.|}$ denote the set of all positive definite functions g which are bounded with respect to $|\cdot|$. Then $\mathscr{P}_{|.|}$ is a closed convex cone in $\mathscr{F}$ with base

$$B = \{g \in \mathscr{P}_{|.|} \,|\, g(e) = 1\}$$

by Lemma 1.1. But since B is a closed subset of the product $\Pi_{s \in S} D_s$ where

$$D_s = \{z \in \mathbf{C} \,|\, |z| \leq |s|\},$$

B is compact in the topology of pointwise convergence. Let ϕ be an extreme point of B. We will show ϕ to be a semicharacter. Using the notation of (1.8) and assuming that $|s| \neq 0$, we then have $\phi = T_{(\sigma,s)}\phi + T_{(-\sigma,s)}\phi$. But $T_{(\sigma,s)}\phi$ is positive definite by Theorem 1.3 (v). Since ϕ is positive definite, an application of (1.3) shows $|E_s\phi(t)|^2 = |\phi(st)|^2 \leq \phi(ss^*)\phi(tt^*) \leq K|t|^2$. Thus $T_{(\sigma,s)}\phi$ is bounded with respect to $|\cdot|$, so that $T_{(\sigma,s)}\phi \in \mathscr{P}_{|.|}$. Since ϕ is extreme, we must have $\lambda\phi = T_{(\sigma,s)}\phi$ for some $\lambda \geq 0$. Evaluating at $t \in S$ and setting $\sigma = 1$ and i gives

$$(\text{Re}\,\phi(s))\phi(t) = \frac{1}{2}\,\phi(st) + \frac{1}{2}\,\phi(s^*t)$$

and

$$(\text{Im}\,\phi(s))\phi(t) = \frac{1}{2i}\,\phi(st) - \frac{1}{2i}\,\phi(s^*t)$$

respectively. Thus $\phi(s)\phi(t) = \phi(st)$ provided $|s| \neq 0$. But $|s| = 0$ implies $|ss^*| = 0$ by (1.4), so that Definition 1.2 and formula (1.2) imply

$$\phi(st) = \phi(s)\phi(t) \quad \text{for all } s,\ t \in S.$$

Since ϕ is positive definite, it follows by (1.1) that ϕ is a semicharacter. The integral representation as well as the assertions about supp(μ) follow from the integral version of the Krein-Milman Theorem; cf. [12].

(ii) To establish uniqueness, we use that the set of all functions of the form

$$\varrho \to T\varrho(e) \quad (T \in \mathscr{A})$$

is dense in the space of continuous complex-valued functions on each compact subset of Γ. Assume the existence of $\nu \in \mathscr{M}^+$ such that $f(s) = \int \varrho(s) d\nu(\varrho)$ for each $s \in S$. Then we claim, $\text{supp}(\nu) \subseteq \text{supp}(\mu)$, where $\mu \in \mathscr{M}^+$ is any measure with compact support satisfying (2.1). For if not, there exists a compact set K, disjoint from $\text{supp}(\mu)$, such that $\nu(K) > 0$. Choose $\epsilon > 0$ such that

$$\epsilon^2 f(e) < \nu(K).$$

We can find a real-valued function $\varrho \to T\varrho(e)$ which is greater than 1 on K and is strictly between 0 and ϵ on $\text{supp}(\mu)$. Then $\varrho \to T^2\varrho(e)$ is non-negative and

$$\epsilon^2 f(e) < \int T^2(\varrho)(e) d\nu = \int T^2(\varrho)(e) d\mu < \epsilon^2 f(e),$$

which is a contradiction. Uniqueness now follows by density of the functions $\varrho \to T\varrho(e)$ in the space $C(\text{supp}(\mu))$ of continuous functions on $\text{supp}(\mu)$. □

Remarks. (a) The proof given in (i) is an adaptation of that given by either Berg et al. [4] or Maserick [10] to our setting. A shorter (but less motivated) approach is to appeal directly to Theorem 2.1 in Maserick [10] as follows. The family

$$\tau = \{T_{(\sigma,s)} \mid |s| \neq 0, \sigma^4 = 1\} \subset \mathscr{A}_f$$

is admissible, and each $T_{(\sigma,s)}$ is a positive bounded operator on H_f by Theorem 1.3, so that all finite products of members of τ are also positive operators. Therefore,

$$\Pi_j T_{(\sigma_j,s_j)} f(e) = \langle \Pi_j T_{(\sigma_j,s_j)} f, f \rangle \geq 0.$$

Hence the map $T \to Tf(e)$ defines a τ-positive linear functional on $\mathscr{A}_f$. Applying Theorem 2.1 of [10], one finds $Tf(e) = \int \chi(T) d\mu(\chi)$ which implies (2.1) above, since $\chi \to [s \to \chi(E_s)]$ is a homeomorphism of the compact set of τ-positive multiplicative linear functionals on $\mathscr{A}_f$ onto a compact subset of Γ. Still a third proof of existence of μ can be fashioned from the Spectral Theorem (cf. [13, p. 306]) by considering the commutative C^*-algebra X in $B(\overline{H}_f)$ generated by the operators E_s on $\overline{H}_f$. Then $T = \int \delta(T) dE(\delta)$, where E is a spectral measure on the compact spectrum of X and δ denotes a multiplicative linear functional on X. But

$$f(s) = \langle E_s f, f \rangle = \int \delta(E_s) d \langle E(\delta) f, f \rangle,$$

and (2.1) follows as above by considering the map $\delta \to [s \to \delta(E_s)]$.

(b) The product of two exponentially bounded positive definite functions is again of this type because $\mathscr{P}$ is stable under pointwise products, and the product of two absolute values on S is again an absolute value on S. The representing measure for the product is the convolution of the representing measures for the factors.

(c) The integral representation implies that an exponentially bounded positive definite function with representing measure μ is always bounded with respect to an absolute value of the form $|\cdot|_K$ with $K = \text{supp}(\mu)$. Assuming $f(e) = 1$, we have

$$\|E_s\| \geq \|E_s^n\|^{1/n} \geq |<E_s^n f, E_s^n f>|^{1/2n} = f^{1/2n}((ss^*)^n) = (\int |\varrho(s)|^{2n} d\mu)^{1/2n}.$$

But as is well known, this latter expression converges to the μ-essential supremum, $|s|_K$, of the function $\varrho \to |\varrho(s)|$. Since $|s|_K \geq \|E_s\|$ by Theorem 1.3, we have

$$\|E_s\| = \lim_n f^{1/2n}((ss^*)^n) = |s|_K.$$

(d) Since $f \in \mathscr{P}$ is bounded if and only if f is bounded with respect to the absolute value $|s| \equiv 1$, Theorem 2.1 subsumes the integral representations in [4], [7] and [9].

(e) The set $B = \{g \in \mathscr{P}_{|.|} \mid g(e) = 1\}$ is compact and convex, and the set ext B of extreme points of B is contained in $\Gamma_{|.|} = \{\varrho \in \Gamma \mid |\varrho(s)| \leq |s|, s \in S\}$, which is compact. Actually ext $B = \Gamma_{|.|}$, which can be seen as in [4] or [9]. It follows that B is a Bauer simplex.

COROLLARY 2.2. *Let f admit a representing measure $\mu \in \mathscr{M}^+$ with compact support K and let $T \in \mathscr{A}$. Then Tf is positive definite if and only if $T\varrho(e) \geq 0$ for all $\varrho \in K$. In this case, Tf is exponentially bounded with representing measure $T\varrho(e)d\mu(\varrho)$.*

Proof. Clearly,

$$Tf(s) = \int_K (T\varrho)(s) d\mu = \int_K \varrho(s) T\varrho(e) d\mu. \tag{2.2}$$

Thus, if $T\varrho(e) \geq 0$ for $\varrho \in K$, then Tf is positive definite. Conversely, suppose $Tf \in \mathscr{P}$. By (2.2), Tf is bounded with respect to $|\cdot|_K$. From Theorem 2.1, there exists a unique measure $\nu \in \mathscr{M}^+$ with compact support such that

$$Tf(s) = \int \varrho(s) d\nu.$$

Hence $\int R\varrho(e) d\nu = RTf(e) = \int R\varrho(e) T\varrho(e) d\mu$ for all $R \in \mathscr{A}$. Denseness in $C(K)$ of the functions $\varrho \to R\varrho(e)$, $R \in \mathscr{A}$, implies $T\varrho(e)d\mu = d\nu$, so that $T\varrho(e) \geq 0$ for all $\varrho \in K$. □

We cannot conclude that $T_1 T_2 f \in \mathscr{P}$ whenever f and $T_j f \in \mathscr{P}$ for $j = 1,2$. However, we can prove:

COROLLARY 2.3. *If f and $T_j f \in \mathscr{P}$ and $T_j f$ is exponentially bounded for $j = 1,2$, then $T_1 T_2 f$ is positive definite and exponentially bounded.*

Proof. From Theorem 2.1, we have the existence of $\mu_j \in \mathscr{M}^+$ with compact support K_j such that

$$T_j f(s) = \int_{K_j} \varrho(s) d\mu_j, \quad j = 1, 2. \tag{2.3}$$

But since f is positive definite, $T_j^* T_j f$ is positive definite from Proposition 1.2, and exponentially bounded from (2.3). Thus there exists $\nu_j \in \mathscr{M}^+$ with compact support such that $\int \varrho(s) d\nu_j = T_j^* T_j f(s) = \int_{K_j} \varrho(s) T_j^* \varrho(e) d\mu_j$. Therefore $T_j \varrho(e) \geq 0$ on K_j. But

$$\int_{K_1} \varrho(s) T_2 \varrho(e) d\mu_1 = T_1 T_2 f(s) = \int_{K_2} \varrho(s) T_1 \varrho(e) d\mu_2$$

implies

$$T_2 \varrho(e) d\mu_1 = T_1 \varrho(e) d\mu_2.$$

Setting $d\mu = T_1 \varrho(e) d\mu_2$, we have $\operatorname{supp}(\mu) \subset K_1 \cap K_2$ and $T_1 T_2 f(s) = \int \varrho(s) d\mu$. But μ is non-negative with compact support, so that $T_1 T_2 f$ is positive definite and exponentially bounded. □

Let $\tau \subset \mathscr{A}$ be any subset of the algebra $\mathscr{A}$ of shift operators. A function $f: S \to \mathbf{C}$ is called *τ-positive* if

$$Tf(e) \geq 0 \quad \text{for all } T \in \text{alg span}^+ \tau.$$

For f to be τ-positive it clearly suffices to verify that $T_1 \cdots T_n f(e) \geq 0$ for all choices of $T_1, \ldots, T_n \in \tau$. The set of τ-positive functions is a closed convex cone in $\mathscr{F}$, and if f is τ-positive so is Tf for any $T \in \text{alg span}^+ \tau$.

Linear functionals $L: \mathscr{A} \to \mathbf{C}$ are in one-to-one correspondence with functions $f: S \to \mathbf{C}$ via the formula $L(E_s) = f(s)$. In case of an admissible $\tau \subset \mathscr{A}$ this correspondence makes it possible to apply Theorem 2.1 of [10] to show that f is τ-positive if and only if f admits a disintegration $f(s) = \int \varrho(s) d\mu(\varrho)$, where $\operatorname{supp}(\mu)$ is a compact subset of the τ-positive semicharacters. Note that a semicharacter $\varrho \in \Gamma$ is τ-positive if and only if $T\varrho(e) \geq 0$ for all $T \in \tau$, because

$$T_1 T_2 \varrho(e) = T_1 \varrho(e) \cdot T_2 \varrho(e) \quad \text{for } T_1, T_2 \in \mathscr{A}.$$

In particular, any τ-positive function f, with τ admissible, is positive definite and exponentially bounded. Moreover Tf is positive definite for each $T \in \tau$.

It follows from Corollary 2.2 that the converse is also true: If f is positive definite and exponentially bounded and if Tf is positive definite for each $T \in \tau$, then f is τ-positive. We suspect the hypothesis of exponentially bounded can be eliminated when τ is admissible. However the best we can prove is:

COROLLARY 2.4. *Suppose $\tau \subset \mathscr{A}$ is admissible and f is positive definite. Then Tf is positive definite and exponentially bounded for each $T \in \tau$ if and only if f is τ-positive.*

Proof. If f is τ-positive, then the integral representation mentioned above, which appears in [10], implies that f satisfies the stated conditions. Conversely, assume f is positive definite and Tf is both positive definite and exponentially bounded for each $T \in \tau$. Then Corollary 2.3 and the definition of admissible implies that f is exponentially bounded. □

When τ is linearly admissible, we can waive any boundedness restrictions, as well as the positive definiteness requirement for f.

COROLLARY 2.5. *If $\tau \subset \mathscr{A}$ is linearly admissible, then f is τ-positive if and only if Tf is positive definite for each $T \in \tau$.*

Proof. Assume Tf is positive definite for each $T \in \tau$. Since τ is linearly admissible it follows that $(I - T)f$ is positive definite as well as Tf. But then

$$f = (I - T)f + Tf$$

is positive definite and T is a bounded operator on H_f. But $\mathscr{A}$ is the algebraic span of τ, so that E_s is bounded for each $s \in S$. Hence f is exponentially bounded by Theorem 1.3, and it follows that f is τ-positive. The converse follows from Corollary 2.4. □

Remark. The conclusion of Corollary 2.4 (2.5) holds for the weaker hypothesis that τ_f be admissible (linearly admissible) in $\mathscr{A}_f$.

Let H be an arbitrary Hilbert space. Following Maserick [8], we call an operator-valued function $F : S \to B(H)$ *positive definite* if for all $s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in \mathbf{C}$, the sum $\Sigma_{i,j} F(s_i s_j^*) c_i \bar{c}_j$ is a positive operator. Every representation $E : S \to B(H)$ is positive definite. It is easily verified that F is positive definite if and only if all of the scalar-valued functions

$$F_\xi(s) = \langle F(s)\xi, \xi \rangle$$

for $\xi \in H$ are positive definite. An operator-valued function F is called *exponentially bounded* if there exists an absolute value $|\cdot|$ on S and a constant $K > 0$ such that $\|F(s)\| \leq K|s|$ for each $s \in S$. Assume F is both positive definite and exponentially bounded. Since $|\langle F(s)\xi, \xi \rangle| \leq \|F(s)\| \cdot \|\xi\|^2 \leq K|s| \cdot \|\xi\|^2$, then $F_\xi(s)$ is also exponentially bounded for each $\xi \in K$. Using the integral representation for each F_ξ along the lines of [8], it is not hard to obtain the following generalization of Theorem 3.2 therein.

THEOREM 2.6. *A function $F : S \to B(H)$ is exponentially bounded and positive definite if and only if there exists a (necessarily unique) positive operator-valued measure E on Γ with compact support such that*

$$F(s) = \int_\Gamma \varrho(s) dE. \tag{2.4}$$

Moreover, F is a representation of S if and only if E is a spectral measure.

Remark. In [8], boundedness of the positive (definite) operator-valued functions considered must be assumed. This was clearly done for the semigroup case but was omitted in the definition of "positive" preceding Theorem 2.1 therein for the algebraic setting. With this insertion, it follows easily that $\|U_f\| \leq \|U_1\|$ (for $\|f\| \leq 1$) as stated at the top of p. 498; otherwise this claim is false.

Nagy [15] characterizes those operator-valued functions F on a semigroup S which are the projections, PU, of representations U of S in an extension space $\mathbf{H}$. We now give an alternate description of those F.

COROLLARY 2.7. *If $F : S \to B(H)$ is positive definite, exponentially bounded and normalized by the condition $F(e) = I$, then there exist a Hilbert space $\mathbf{H} \supset H$, a projection $P : \mathbf{H} \to H$ and a representation $U : S \to B(\mathbf{H})$ such that $F(s) = PU_s|_H$. Moreover, $\mathbf{H}$ can be chosen to be densely spanned by*

$$\{U_s\xi \mid \xi \in H, s \in S\}.$$

Proof. Let E be the positive operator-valued measure satisfying (2.4). Then E is normalized so that Neumark's Theorem (cf. [15, p. 29]) implies the existence of a Hilbert space $\mathbf{H}$ extending H and a $B(\mathbf{H})$-valued normalized spectral measure $\mathbf{E}$ such that $E = P\mathbf{E}|_H$, where P is the projection of $\mathbf{H}$ onto H. Thus,

$$F(s) = (P\textstyle\int_\Gamma \varrho(s)d\mathbf{E})|_H.$$

The assertion follows since $s \to \int_\Gamma \varrho(s)d\mathbf{E}$ is a representation of S. □

Remark. Nagy [15] proves Corollary 2.7 for the case where F is what he defines to be of "positive type", and satisfies a boundedness condition analogous to that given in Theorem 1.3 (iii). It is elementary to see that F is positive definite whenever it is of positive type. Conversely, if F is normalized, positive definite and exponentially bounded, then Corollary 2.7 implies that F is of positive type.

3. Applications and examples

The classical moment problem, in modern terminology, asks for a description of those scalar-valued functions on a classical semigroup S which admits an integral representation of the form

$$f(s) = \int_{\Gamma'} \varrho(s)d\mu; \quad \Gamma' \subset \Gamma,\ \mu \in \mathcal{M}^+. \tag{3.1}$$

We list below some classical semigroups with a description of their semicharacters for easy reference. We define $0^0 = 1$, $\mathbf{N} = \{0, 1, \ldots\}$ and $\mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

	$(S,\cdot)$	Involution	Semicharacters	Related Moment Problems
(3a)	$(\mathbf{N},+)$	$n^* = n$	$n \to t^n$ $(t \in \mathbf{R})$	"Power"
(3b)	$(\mathbf{N}^k,+)$	$(n_j)^* = (n_j)$	$(n_j) \to (\prod_{j=1}^{k} t_j^{n_j})$ $(t_j \in \mathbf{R})$	"Multidimensional power"
(3c)	$(\mathbf{N}^2,+)$	$(\mathrm{m},\mathrm{n})^* = (\mathrm{n},\mathrm{m})$	$(\mathrm{m},\mathrm{n}) \to z^m \cdot \bar{z}^n$ $(z \in \mathbf{C})$	"Complex"
(3d)	$(\mathbf{Z},+)$	$n^* = -n$	$n \to z^n$ $(z \in \mathbf{C}, \lvert z \rvert = 1)$	"Trigonometric" (Herglotz)
(3e)	$(\mathbf{Z},+)$	$n^* = n$	$n \to t^n$ $(t \in \mathbf{R} \setminus \{0\})$	

The following proposition generalizes and unifies all solutions known to us of the power moment problem (3a) for $\Gamma' = [-1,1]$, the complex moment problem (3c), where Γ' is the unit disc, and the trigonometric moment problem (3d) to arbitrary semigroups with involution.

PROPOSITION 3.1. *Let G be a generator set for S and f be a scalar-valued function on S. The following are equivalent:*

(i) *The function f is bounded and positive definite.*

(ii) *The function $T_{(\sigma,s)}f$ is positive definite for each*

$$T_{(\sigma,s)} = \frac{1}{2}(I + \frac{1}{2}\sigma E_s + \frac{1}{2}\bar{\sigma}E_{s^*}),$$

$s \in S$ and fourth root of unity σ.

(iii) *The function f is τ-positive, where $\tau = \{T_{(\sigma,s)} \mid s \in S, \sigma^4 = 1\}$.*

(iv) *The functions f and $(I - E_{ss^*})f$ are positive definite for each $s \in G$.*

(v) *The function f admits a representing measure μ with*

$$\operatorname{supp}(\mu) \subset \{\varrho \in \Gamma \mid \lvert\varrho\rvert \leq 1\}.$$

Proof. If f satisfies (i) it is bounded with respect to the absolute value $\lvert s\rvert \equiv 1$, so Theorem 2.1 implies the equivalence of (i) and (v). That (i) and (ii) are equivalent follows from Theorem 1.3; also (i) implies (iv) by Theorem 1.3. By Proposition 1.2, (iv) implies that $(I - E_{ss^*})$ is a positive operator on H_f. Thus $\lVert E_s\rVert \leq 1$ for all $s \in G$ and hence for all $s \in S$, so that (iv) implies (i) by Theorem 1.3. The equivalence of (ii) and (iii) follows from Corollary 2.5. □

Remarks. The equivalence of (iii) and (v) is due to Maserick [9]. For the classical cases of (3a), (3b) and (3c), the equivalence of (i), (iv) and (v) seems to have been known for many years; cf. Akhiezer [1], Atzmon [2] or Haviland [6]. An example where S cannot be replaced by G in (ii) and (iii) is given in Maserick [9]. That (iv) is equivalent to the other conditions seems not to have been noticed in the above generality.

The next proposition generalizes all solutions known to us of Hausdorff's "little" moment problem. Hausdorff's original solution asserts that $f: \mathbf{N} \to \mathbf{R}$ satisfies (3.1) for $\Gamma' = [0,1]$ if and only if f is "completely monotonic". For an arbitrary S with $s^* = s$ and generator set G, a function $f: S \to \mathbf{R}$ is called *completely monotonic* (with respect to G) whenever f is τ-positive for the linearly admissible subset $\tau = \{E_s, I - E_s \mid s \in G\}$. The equivalence of (i) and (v) of the following proposition shows complete monotonicity to be independent of the choice of G.

PROPOSITION 3.2. *Let G be a generator set for S and f be a real-valued function on S. The following are equivalent:*

(i) *The function f is completely monotonic (with respect to G).*

(ii) *The functions $E_s f$ and $(I - E_s)f$ are positive definite for each $s \in G$.*

(iii) *The functions f and $E_s(I - E_s)f$ are positive definite for each $s \in G$.*

(iv) *The function $E_s f$ is bounded and positive definite for each $s \in G \cup \{e\}$.*

(v) *The function f admits a representing measure μ with*

$$\operatorname{supp}(\mu) \subset \{\varrho \in \Gamma \mid 0 \leq \varrho \leq 1\}.$$

Proof. Assertion (i) is equivalent to (ii) by Corollary 2.5, and (ii) is equivalent to (iii) by Proposition 1.5, because (ii) implies that $f \in \mathscr{P}$. Assertion (v) implies (ii) by Corollary 2.2, and (ii) implies (iv) because if (ii) holds, then f is positive definite and E_s and $I - E_s$ are positive operators on H_f for $s \in G$. It follows that $\|E_s\| \leq 1$ for all $s \in G$, and finally for all $s \in S$ since G is a generator set. Hence $|f(s)| \leq f(e)$ and in particular $E_s f$ is bounded for each $s \in G \cup \{e\}$.

Finally assume (iv) holds. Then f is positive definite and bounded with respect to $|\cdot| \equiv 1$, so Theorem 2.1 implies the existence of a representing measure μ such that $|\varrho(t)| \leq 1$ for each $\varrho \in \operatorname{supp}(\mu)$ and $t \in S$. Since $E_s f$ is positive definite for each $s \in G$, Corollary 2.2 implies $0 \leq E_s \varrho(e) = \varrho(s)$ for each $\varrho \in \operatorname{supp}(\mu)$ and $s \in G$. Hence (v) follows from (iv) since G is a generator set for S. □

Remark. Various special cases of the above equivalences can be found in Akhiezer [1], Atzmon [2], Lindahl-Maserick [7], Nussbaum [11] and Widder [17] among other places.

We now consider applications of the theory to cases where the exponentially bounded positive definite functions discussed are in general unbounded. In sharp contrast to the group case, we first make the following observation.

Example 3.3. For the classical semigroup in (3e) the only bounded positive definite functions are $n \to a + b(-1)^n$ for $a, b \geq 0$. On the other hand every semicharacter is exponentially bounded.

Next, consider the classical semigroup $S = (\mathbf{N}^k, +)$ of example (3b). By way of motivation, we view the simplex

$$K = \{(t_1, t_2) \in \mathbf{R}^2 \mid t_1 \geq 0, t_2 \geq 0 \text{ and } (1 - t_1 - t_2) \geq 0\}$$

as a subset of the semicharacters on $(\mathbf{N}^2, +)$ via the map $(m, n) \to t_1^m t_2^n$. Let E_1 and E_2 denote the unit coordinate shift operators as defined by

$$E_1 f(m, n) = f(m + 1, n) \quad \text{and} \quad E_2 f(m, n) = f(m, n + 1).$$

Set

$$\tau = \{E_1, E_2, I - E_1 - E_2\}.$$

Then τ is linearly admissible in $\mathscr{A}$ so that Corollary 2.5 implies that

$$f : \mathbf{N}^2 \to \mathbf{R}$$

admits a representing measure μ supported by the simplex K if and only if $E_1 f, E_2 f$ and $(I - E_1 - E_2) f$ are each positive definite. We generalize as follows. Let K be a bounded convex polytope with non-void interior in Euclidean k-dimensional space $\mathbf{R}^k$. Then

$$K = \bigcap_{i=1}^{m} \{t \in \mathbf{R}^k \mid P_i(t) \geq 0\}.$$

where p_i is a polynomial in the variables $t_1, \ldots, t_k$ of degree 1. For each $j = 1, 2, \ldots, k$, let E_j denote the unit shift operator $E_{(0,\ldots,0,1,0,\ldots,0)}$ in the j-th coordinate. Then each polynomial p in k variables defines a shift operator

$$p(E) = p(E_1, \ldots, E_k) \in \mathscr{A}.$$

Since K has a non-void interior and is bounded, the defining polynomials, p_i, can be chosen so that $\tau = \{p_i(E) \mid i = 1, \ldots, m\}$ is linearly admissible in $\mathscr{A}$; cf. Maserick [10, p. 145].

Proposition 3.4. *Let τ and K be as above and f be a real-valued function on $\mathbf{N}^k$, then the following statements are equivalent:*

(i) *The function f is τ-positive.*

(ii) *Each $p_i(E)f$ is positive definite for $i = 1, \ldots, m$.*

(iii) *The function f admits a representing measure supported by K.*

Proof. The proof is a consequence of Theorem 2.1, Corollary 2.2 and Corollary 2.5, since $t \in \mathbf{R}^k$ defines a τ-positive semicharacter if and only if $t \in K$.

The equivalence of (i) and (iii) has previously been established by Maserick. In fact the above can be formulated more generally for arbitrary convex bodies in $\mathbf{R}^k$ and the reader should consult [10].

Remark. The τ-positive functions are not all bounded unless K is contained in $\{t \in \mathbf{R}^k | \ |t_i| \leq 1\}$.

The standard spectral theorems for bounded positive, hermitian, unitary and normal operators follow as an elementary application of Theorem 2.6. In the first two cases we take S as in (3a) and consider the representation $n \to A^n$ where A is either a positive or Hermitian operator. In the third case we take S as in (3d) and consider the representation $n \to U^n$ where U is unitary. Finally we take S as in (3c) and consider the representation $(m, n) \to A^m(A^*)^n$, where A is a normal operator. The details are worked out in [8].

Added in proof. In a forthcoming paper by G. Cassier, *Problème des moments sur un compact de* $\mathbf{R}^n$ *et décomposition de polynômes à plusieurs variables,* the moment problem for an arbitrary compact set K with non-empty interior in $\mathbf{R}^n$ is solved by construction of a linearly admissible family τ of polynomials of degree ≤ 2 such that

$$K = \bigcap_{p \in \tau} p^{-1}(0, \infty].$$

An application of our Corollary 2.5 then leads to Theorems 2 and 3 in the paper by Cassier.

References

1. N. I. Akhiezer, *The Classical moment problem,* Oliver and Boyd, Edinburgh, 1965.
2. A. Atzmon, *A moment problem for positive measures on the unit disc.,* Pacific J. Math., vol. 59(1975), pp. 317-325.
3. C. Berg, J. P. R. Christensen and C. U. Jensen, *A remark on the multi-dimensional moment problem,* Math. Ann., vol. 243(1979), pp. 163-169.
4. C. Berg, J. P. R. Christensen and P. Ressel, *Positive definite functions on abelian semigroups,* Math. Ann., vol. 223(1976), pp. 253-272.
5. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups,* vol. I, Math. Surveys, no. 7, Amer. Math. Soc., Providence, R. I., 1961.
6. E. K. Haviland, *On the momentum problem for distribution functions in more than one dimension* II, Amer. J. Math., vol. 58(1936), pp. 164-168.
7. R. J. Lindahl and P. H. Maserick, *Positive-definite functions on involution semigroups,* Duke Math. J., vol. 38(1971), pp. 771-782.
8. P. H. Maserick, *Spectral theory of operator-valued transformations,* J. Math. Anal. Appl., vol. 41(1973), pp. 497-507.
9. ———, *BV-functions, positive-definite functions and moment problems,* Trans. Amer. Math. Soc., vol. 214(1975), pp. 137-152.
10. ———, *Moments of measures on convex bodies,* Pacific J. Math., vol. 68(1977), pp. 135-152.
11. A. E. Nussbaum, *The Hausdorff-Bernstein-Widder theorem for semi-groups in locally compact abelian groups,* Duke Math. J., vol. 22(1955), pp. 573-582.
12. R. R. Phelps, *Lectures on Choquet's Theorem,* Van Nostrand, Princeton, N. J., 1966.
13. W. Rudin, *Functional Analysis,* McGraw-Hill, New York, 1973.
14. W. Schempp, *On functions of positive type on Commutative monoids,* Math. Zeitschr., vol. 156(1977), pp. 115-121.
15. B. Sz.-Nagy, *Extension of linear transformations in Hilbert space which extend beyond this space,* Ungar, New York, 1960.

16. F. H. SZAFRANIEC, *Dilations on involution semigroups,* Proc. Amer. Math. Soc., vol. 66(1977), pp. 30–32.
17. D. V. WIDDER, *The Laplace transform,* Princeton University Press, Princeton, 1946.

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