Exponentially Enhanced Quantum Metrology

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We show that when a suitable entanglement-generating unitary operator depending on a parameter is applied on N qubits in parallel, a precision of the order of 2^{-N} in estimating the parameter may be achieved. This exponentially improves the precision achievable in classical and in quantum nonentangling strategies.

The Mandelstam-Tamm version of the time energy uncertainty relation [1] and its rigorous developments [2] form the basis of quantum enhanced methods for parameter estimation such as those used in evolving frequency standards [3]. Giovannetti, Lloyd, and Maccone [4] have done beautiful work to classify methods of such quantum enhancement. A unitary transformation depending on the parameter to be estimated is applied to a suitably prepared system of N probes in parallel and then an appropriate observable is measured. Their result is that quantum metrology parallel strategies can achieve a precision of order $N^{-1/2}$ times the classically attainable precision if the probes are prepared in a suitable entangled state.

We propose here a parallel strategy which exploits the Hilbert space of N probes more fully than in previous work [4] and thereby attains an exponentially enhanced precision. Previous work only considered applying on the prepared probes a unitary operator which is a direct product of N unitary operators each acting on a single probe. We will show here that if instead we consider applying on the probes, an entanglement-generating unitary operator $U = e^{-i\theta H}$ which *cannot* be written as a direct product of one-probe operators, then we can obtain an exponentially enhanced precision in estimating the parameter θ. The fundamental reason for this improvement is that there are an exponentially large number of mutually commuting observables for the N-probe system whereas the number of mutually commuting single-probe operators is only of order N. For instance, for the N-qubit system each qubit cannot have more than one commuting observable and hence there are exactly N commuting observables of the form $\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A^j \otimes \mathbb{1} \otimes \ldots \mathbb{1}$ each of which acts nontrivially on only the jth qubit, with $j = 1, 2, \ldots, N$. We construct sets of 2^{N-1} mutually commuting Hermitian N-body Hamiltonians each of the form $A^1 \otimes A^2 \otimes \cdots \otimes A^N$ which acts nontrivially on all the N qubits. The unitary entanglement-generating operator is chosen to be a product (but not a direct product) of the 2^{N-1} commuting unitaries which are exponentials of these commuting Hermitian operators times $-i\theta$. Thus we fully exploit the quantum parallelism which is at the heart of exponential violations

of local realism [5] and of the well-known exponential speed-up achieved in certain quantum computation tasks [6]. The parameter θ is physically the product of the time and a coupling constant. For the probe system of N qubits we obtain the best possible precision in θ dictated by the uncertainty principle for such entanglement-generating unitary operators, viz., a quantum precision $2^{-(N-1)/2}$ times the best possible classical precision. In traditional quantum metrology with direct product unitary operators, the ratio of quantum to classical precision was at best of order $N^{-1/2}$. We are thus led to a new characterization of eight parallel quantum metrology strategies as XYZ where each of X, Y, and Z can be Q or C. See Fig. 1. Here X and Z specify the presence of entanglement (Q) or the absence of entanglement (C) in the probes and observables, respectively, as in [4], and the new label Y specifies whether the unitary operator U applied on the probes is an entanglement-generating operator (Q) or not (C).

We show that the parallel strategy CQC using an entanglement-generating operator U just described is enough to obtain the announced enhanced precision; entanglement at the probe preparation stage and measurement stage is inessential. However, practical realization of the entangling U which is a product of 2^{N-1} commuting unitaries is an experimental challenge. Here again, quan-
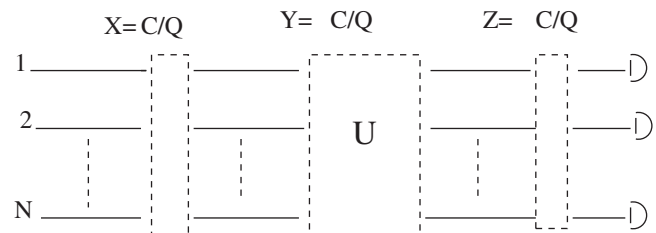

FIG. 1. The eight parallel strategies XYZ for estimation of a parameter occurring in an unitary operator U applied to N probes. Each of X, Y, Z takes values C or Q. $Y = Q$ if U is an entanglement-generating operator (as considered in this Letter) and $Y = C$ otherwise. X, Z specify the presence of entanglement (Q) or the absence of entanglement (C) in the prepared probe state and in the measured operator, respectively.

tum mechanics might come to the rescue. We propose a
quantum-optics model of laser light interacting with an
N-qubit system, say a polyatomic molecule, via a gener-
alized Jaynes-Cummings interaction which, in principle,
could achieve the exponentially enhanced precision. The
practical implementability of this model would be an
interesting subject for further theoretical and experimental
investigation.

Quantum limits on precision.—We recall first the
Mandelstam-Tamm uncertainty relations. Suppose we ap-
ply a unitary operator $U = e^{-i\theta H}$ to a probe state $|\phi\rangle$
where H is a self-adjoint operator and θ a parameter to
be estimated. We obtain the state

$$|\psi(\theta)\rangle = U|\phi\rangle, \tag{1}$$

and then measure the observable X on this state with a view
to estimating the parameter θ. Schwarz inequality yields

$$\Delta X \Delta H \geq |\langle\psi(\theta)|[H, X]/(2i)|\psi(\theta)\rangle|$$
$$= \frac{1}{2}\left|\frac{\partial\langle\psi(\theta)|X|\psi(\theta)\rangle}{\partial\theta}\right|, \tag{2}$$

where ΔX, ΔH denote the dispersions in X, H respectively.
The resulting precision in estimating θ is thus given by the
uncertainty relation

$$\delta\theta \equiv \Delta X \Big/ \left|\frac{\partial\langle\psi(\theta)|X|\psi(\theta)\rangle}{\partial\theta}\right| \geq \frac{1}{2\Delta H}. \tag{3}$$

In the case when the parameter θ is estimated using an
estimator θ_{est} and the estimation is repeated ν times, the
Cramer-Rao bound [7] was used in Ref. [2] to prove the
generalized uncertainty relation for the error estimate

$$\delta_\nu\theta \equiv \left\langle\left(\theta_{\text{est}}\Big/\left|\frac{\partial\langle\theta_{\text{est}}\rangle_{\text{av}}}{\partial\theta}\right| - \theta\right)^2\right\rangle_{\text{av}}^{1/2} \geq \frac{1}{2\Delta H\sqrt{\nu}}, \tag{4}$$

where $\langle\cdots\rangle_{\text{av}}$ denotes statistical average. The quantum
limits on precision are obtained by noting that

$$\Delta H \leq \tfrac{1}{2}(\lambda_{\max} - \lambda_{\min}), \tag{5}$$

where $\lambda_{\max}$, $\lambda_{\min}$ denote, respectively, the maximum and
minimum eigenvalues of H which we assume to be finite.
Hence, the Mandelstam-Tamm [1] and Braunstein-Caves
[2] quantum limits on precision are given, respectively, by

$$\delta\theta \geq \frac{1}{(\lambda_{\max} - \lambda_{\min})}, \tag{6}$$

and

$$\delta_\nu\theta \geq \frac{1}{(\lambda_{\max} - \lambda_{\min})\sqrt{\nu}}. \tag{7}$$

It is clear that quantum parallel strategies to improve
precision should aim to maximize ΔH on the N-probe
quantum state.

Parallel strategies for N qubits.—Consider first, as in [4]
the operator H to be a direct sum of the operators H^j acting

on the jth probe, each H^j having the same maximum
dispersion $\Delta H^j \leq d$. Then

$$\Delta\bigoplus_{j=1}^{N} H^j \leq Nd. \tag{8}$$

As noted in Ref. [4] this maximum dispersion is in fact
reached when we choose the N-qubit state to be an equally
weighted superposition of the eigenvectors of H with
maximum and minimum eigenvalues. With $\lambda_{\max} = N\lambda_M$
and $\lambda_{\min} = N\lambda_m$ where λ_M and λ_m are, respectively, the
maximum and minimum eigenvalues of each H^j we get the
above equation with $d = (\lambda_M - \lambda_m)/2$. Note for purposes
of comparison, that the dispersion of a sum of K classical
variables H^j_{cl} with a factorized joint probability distribution
is given by

$$\Delta\sum_{j=1}^{K} H^j_{\text{cl}} = \sqrt{\sum_{j=1}^{K}(\Delta H^j_{\text{cl}})^2}. \tag{9}$$

The maximum quantum dispersion is thus $\sqrt{N}$ times the
classical value when $K = N$ and $\Delta H^j_{\text{cl}} = d$. This has been
exploited in Ref. [4]. We now show that the dispersion of H
for N qubits can be made exponentially larger by appro-
priate choice of H. Consider the operator identity

$$\bigotimes_{j=1}^{N}(\sigma_x + i\sigma_y)^j = H + iA, \tag{10}$$

where σ_x^j and σ_y^j are Pauli matrices for the jth qubit, and H
and iA denote, respectively, the Hermitian and anti-
Hermitian parts of the operator on the left-hand side of
the equation. Explicit expressions for H and A are conven-
iently stated in terms of the matrices $\sigma^j(\pm1)$ defined by

$$\sigma^j(+1) \equiv \sigma_x^j, \qquad \sigma^j(-1) \equiv \sigma_y^j. \tag{11}$$

We obtain

$$H = \sum_{\substack{r_1, r_2, \ldots, r_N = \pm1 \\ N_- = \text{even}}} H(r_1, r_2, \ldots, r_N), \tag{12}$$

$$H(r_1, r_2, \ldots, r_N) \equiv (-1)^{N_-/2}\bigotimes_{j=1}^{N}\sigma^j(r_j), \tag{13}$$

and

$$A = \sum_{\substack{r_1, r_2, \ldots, r_N = \pm1 \\ N_- = \text{odd}}} A(r_1, r_2, \ldots, r_N), \tag{14}$$

$$A(r_1, r_2, \ldots, r_N) \equiv (-1)^{(N_- - 1)/2}\bigotimes_{j=1}^{N}\sigma^j(r_j), \tag{15}$$

and finally

$$N_- = \sum_{j=1}^{N} \frac{1}{2}(1 - r_j), \tag{16}$$

is just the number of j's with $r_j = -1$ or the number of σ_y^j's in the N-fold product of Pauli matrices in H and A. Both H and A are sums of 2^{N-1} products of Pauli matrices, each product having eigenvalues ± 1 and hence maximum dispersion

$$\Delta \bigotimes_{j=1}^{N} \sigma^j(r_j) \le 1. \tag{17}$$

The standard anticommutation rules between Pauli matrices lead to

$$\sigma^j(r_j)\sigma^j(r'_j) = \sigma^j(r'_j)\sigma^j(r_j) r_j r'_j, \tag{18}$$

for $r_j, r'_j = \pm 1$. Note that $r_1 r_2 \ldots r_N = (-1)^{N_-}$. The anticommutation rules then imply that the set of 2^{N-1} products of Pauli matrices occurring in H (or A) constitutes a set of mutually commuting observables. Hence

$$e^{-i\theta H} = \prod_{\substack{r_1, r_2, \ldots, r_N = \pm 1 \\ N_- = \text{even}}} e^{-i\theta H(r_1, r_2, \ldots, r_N)}, \tag{19}$$

and

$$e^{-i\theta A} = \prod_{\substack{r_1, r_2, \ldots, r_N = \pm 1 \\ N_- = \text{odd}}} e^{-i\theta A(r_1, r_2, \ldots, r_N)}. \tag{20}$$

Further any of these 2^{N-1} observables in H anticommutes with any of the 2^{N-1} observables occurring in A. In contrast with the parallel strategy in Ref. [4], we now have

$$\Delta H \le 2^{N-1}, \qquad \Delta A \le 2^{N-1}. \tag{21}$$

Interestingly, and in contrast with Ref. [4] the maximum dispersions of H and A are now reached in the separable N-qubit states

$$|\uparrow\uparrow \ldots \uparrow\uparrow\rangle, \qquad |\downarrow\downarrow \ldots \downarrow\downarrow\rangle, \tag{22}$$

each of which has

$$\Delta H = 2^{N-1}, \qquad \Delta A = 2^{N-1}, \tag{23}$$

where we have denoted the eigenstates of σ_z with eigenvalues $+1$ and -1 by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

For comparison, the maximum classical dispersion for the sum of 2^{N-1} classical variables with factorized probability distribution, each variable having maximum dispersion 1, would only be $2^{(N-1)/2}$.

Parallel strategy CQC with exponentially enhanced precision.—We start from one of the factorized N-qubit states given above and apply one of the unitary operators $U = e^{-i\theta H}$ or $U = e^{-i\theta A}$ given above. In particular, we obtain

$$|\psi_H(\theta)\rangle = e^{-i\theta H}|\uparrow\uparrow \ldots \uparrow\uparrow\rangle = \cos(2^{N-1}\theta)|\uparrow \ldots \uparrow\rangle$$
$$- i\sin(2^{N-1}\theta)|\downarrow \ldots \downarrow\rangle; \tag{24}$$

$$|\psi_A(\theta)\rangle = e^{-i\theta A}|\uparrow\uparrow \ldots \uparrow\uparrow\rangle = \cos(2^{N-1}\theta)|\uparrow \ldots \uparrow\rangle$$
$$+ \sin(2^{N-1}\theta)|\downarrow \ldots \downarrow\rangle. \tag{25}$$

We may now measure the probability that all qubits are in the up-state given by the expectation value of the direct product of projection operators

$$X = \bigotimes_{j=1}^{N} \frac{1}{2}(\mathbf{1} + \sigma_z^j). \tag{26}$$

We obtain, for example, for the state $|\psi_H(\theta)\rangle$

$$\langle X \rangle = \tfrac{1}{2}[1 + \cos(\theta 2^N)], \qquad \Delta X = \tfrac{1}{2}|\sin(\theta 2^N)|. \tag{27}$$

Hence, the quantum precision of estimating θ is given by

$$\delta\theta = 2^{-N}, \tag{28}$$

which achieves the best allowed by the uncertainty relation since $\Delta H = 2^{N-1}$. (A similar precision is obtained by using the state $|\psi_A(\theta)\rangle$). This improves by a factor $2^{-(N-1)/2}$ the best classical precision $2^{-(N+1)/2}$. We see that $\langle X \rangle$ is periodic in θ, the period decreasing exponentially as N becomes large. This period and the absolute precision, however, depend on scale changes in the definition of θ. They are not physically important for comparisons between classical and quantum precisions. If we allow an arbitrary possibly N-dependent scale factor α in the definition of the coupling constant g_N, and set $\theta = t\alpha g_N$, the corresponding precisions in measuring time or coupling constant become $\delta t = \delta\theta/(\alpha g_N)$, or $\delta g_N = \delta\theta/(\alpha t)$. The important point is that both the classical and quantum precisions scale by the same factor. Our final result is, both for the time parameter and for the coupling constant g_N, quantum precisions can be $2^{-(N-1)/2}$ times the best possible classical precision.

A generalized Jaynes-Cummings model.—We propose a quantum-optics model of interaction of laser light with N qubits which could, in principle, be used to obtain exponentially enhanced precision. We apply a unitary operator to the infinite-dimensional vector space which implies an entanglement-generating "operation" on the reduced density operator for N qubits.

Suppose we shine laser light of frequency $\Omega \approx N\omega$ on an N-atom molecule each atom of which can be approximated to be a two-level atom or qubit with level separation $\hbar\omega$. We assume that on irradiation, the molecule does not dissociate but gets excited to a higher level in which each atom is excited to the higher level. The second quantized interaction Hamiltonian is assumed to be

$$\hat{H} = \hat{H}_0 + \hat{H}_I, \qquad \hat{H}_0 = \frac{\omega}{2}\sum_{j=1}^{N} \hat{\sigma}_z^j + \Omega\hat{a}^\dagger\hat{a}, \tag{29}$$

$$\hat{H}_I = \frac{g}{2}\left[\hat{a}\bigotimes_{j=1}^{N}(\hat{\sigma}_x + i\hat{\sigma}_y)^j + \text{H.c.}\right]. \tag{30}$$

Here H.c. denotes Hermitian conjugate, $\hat{a}$ and $\hat{a}^{\dagger}$ denote annihilation and creation operators for photons of frequency Ω, $\frac{1}{2}(\hat{\sigma}_x + i\hat{\sigma}_y)^j$ denotes the level raising operators for the jth qubit and $g = g_N$ is a real N-dependent coupling constant. It is an exactly solvable Hamiltonian with eigenvalues

$$\lambda_{\pm} = \Omega\left(n + \frac{1}{2}\right) \pm \frac{\Omega_1}{2}, \qquad (31)$$

where $n = 0, 1, 2, \ldots$ and

$$\Omega_1 = \sqrt{(\omega N - \Omega)^2 + g^2(n+1)2^{2N}}. \qquad (32)$$

The corresponding eigenstates are

$$\frac{1}{\sqrt{2}}(\alpha_{\pm}|n\rangle|\phi_0\rangle + \beta_{\pm}|n+1\rangle|\phi_1\rangle), \qquad (33)$$

$$|\phi_0\rangle \equiv |\uparrow \ldots \uparrow\rangle, \qquad |\phi_1\rangle \equiv |\downarrow \ldots \downarrow\rangle, \qquad (34)$$

$$\begin{aligned} \alpha_{\pm} &\equiv \sqrt{1 \pm (\omega N - \Omega)/\Omega_1} \\ \beta_{\pm} &\equiv \pm\sqrt{1 \mp (\omega N - \Omega)/\Omega_1}, \end{aligned} \qquad (35)$$

and $\hat{a}^{\dagger}\hat{a}|n\rangle = n|n\rangle$. Interesting physics about Rabi oscillations between up and down qubit states may be read off from these equations even off-resonance, i.e., $\Omega \neq N\omega$. For the present we specialize to $\Omega = N\omega$ which implies $[\hat{H}_0, \hat{H}_I] = 0$. We deduce that for $r = 0$ and $r = 1$

$$\begin{aligned} e^{-it\hat{H}}|n+r\rangle|\phi_r\rangle = e^{-it(n+1/2)N\omega}[\cos(2^{N-1}\theta)|n+r\rangle|\phi_r\rangle \\ - i\sin(2^{N-1}\theta)|n+1-r\rangle|\phi_{1-r}\rangle], \end{aligned} \qquad (36)$$

where $\theta = tg\sqrt{n+1}$. Thus the reduced density operator for the N qubits obtained by tracing over the photon states undergoes the positivity and trace preserving transformation

$$\hat{\rho}(t) = \sum_{r=0,1} \hat{h}_r \hat{\rho}(0)\hat{h}_r^{\dagger}, \qquad \hat{h}_r = e^{-i\theta H}|\phi_r\rangle\langle\phi_r|, \quad (37)$$

where H is the N-qubit spin operator given by Eq. (10). For measurement of the observable X of Eq. (26) with $\langle X\rangle = \text{tr}\hat{\rho}(t)X$ we obtain exactly the same results on precision as in Eqs. (27) and (28) of the previous section. Hence, for measurement of t or g, the quantum precision is $2^{-(N-1)/2}$ times the best possible classical precision.

In conclusion, we have shown that the parameter estimation associated with suitable entanglement-generating unitary operators may lead to an exponential enhancement of accuracy over both classical schemes and nonentangling quantum schemes. The simple physical origin is that the magnitude of expectation value of the product (10) in a maximally entangled state is 2^{N-1}, whereas it could not exceed $2^{N/2}$ if the Pauli spin operators were replaced by classical variables with values ± 1 and arbitrary probability distribution. Boixo *et al.* [8] have recently made a nice application of our ideas of using many-body Hamiltonians to the measurement of N-independent coupling constants by replacing our N-body interactions by k-body interactions, with k less than N. They obtain improvements of order $N^{-k/2}$ on the classical precision. Our N-body interactions arising from coupling to a rank N tensor field have an N-dependent coupling constant. This causes no difficulty since the ratio of quantum to classical precisions in time or coupling constant $2^{-(N-1)/2}$ can be verified by measurements at just one large value of N. The exciting thing is that unlike quantum computation, quantum metrology might not face formidable problems of fighting decoherence.

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