Dias *et al. Journal of Statistical Distributions and Applications* (2016) 3:15
DOI 10.1186/s40488-016-0051-2

Journal of Statistical Distributions and Applications

REVIEW Open Access

Exponentiated Marshall-Olkin family of distributions

Cícero R. B. Dias[1], Gauss M. Cordeiro[1], Morad Alizadeh[3], Pedro Rafael Diniz Marinho[1*] and Hemílio Fernandes Campos Coêlho[2]

*Correspondence: pedro.rafael.marinho@gmail.com
[1]Department of Statistics, Federal University of Pernambuco, Recife, Brazil
Full list of author information is available at the end of the article

Abstract

We study general mathematical properties of a new class of continuous distributions with three extra shape parameters called the exponentiated Marshal-Olkin family of distributions. Further, we present some special models of the new class and investigate the shapes and derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions and probability weighted moments. We discuss the estimation of the model parameters by maximum likelihood and show empirically the potentiality of the family by means of two applications to real data.

Keywords: Generalized exponential geometric distribution, Generated family, Maximum likelihood, Moment

1 Introduction

In the past few years, several ways of generating new distributions from classic ones were developed and discussed. Eugene et al. (2002) defined a class of beta-generated distribution. Jones (2004) studied a family of distributions that arises naturally from the distribution of the order statistics and introduced general properties of the proposed class of distributions. Zografos and Balakrishnan (2009) proposed the gamma-generated family of distributions. Later, Cordeiro and de Castro (2011) defined the Kumaraswamy family. Recently, Alzaatreh et al. (2013) proposed a new technique to derive wider families by using any probability density function (pdf) as a generator. This generator called the T-X family of distributions has cumulative distribution function (cdf) defined by

$$F(x) = \int_{a}^{W[G(x)]} r(t)dt, \tag{1}$$

where $G(x)$ is the cdf of a random variable X, $r(t)$ is the pdf of a random variable T defined on $[a, b]$ and $W[G(x)]$ is a function of $G(x)$, which satisfies the following conditions:

- $W[G(x)] \in [a, b]$;
- $W[G(x)]$ is differentiable and monotonically non-decreasing;
- $W[G(x)] \to a$ as $x \to -\infty$ and $W[G(x)] \to b$ as $x \to \infty$.

Following Alzaatreh et al. (2013) and replacing $r(t)$ by the generalized exponential-geometric (GEG) density function (Silva et al. 2010), where $T \in [0, \infty)$, and using $W[G(x)] = -\log[1 - G(x)]$, we define the cdf of a new wider family by

$$F(x) = \int_0^{-\log[1-G(x;\boldsymbol{\xi})]} \frac{\alpha\lambda(1-p)\mathrm{e}^{-\lambda t}[\,1-\mathrm{e}^{-\lambda t}]^{\alpha-1}}{[\,1-p\,\mathrm{e}^{-\lambda t}]^{\alpha+1}}\mathrm{d}t = \left\{\frac{1-[\,1-G(x;\boldsymbol{\xi})]^{\lambda}}{1-p[\,1-G(x;\boldsymbol{\xi})]^{\lambda}}\right\}^{\alpha}, \tag{2}$$

where $G(x;\boldsymbol{\xi})$ is the baseline cdf depending on a parameter vector $\boldsymbol{\xi}$ and $\alpha > 0$, $\lambda > 0$ and $p < 1$ are three additional shape parameters. For each baseline G, the *exponentiated Marshall-Olkin-G* ("EMO-G" for short) distribution is defined by the cdf (2). The EMO family includes as special cases the *exponentiated generalized class of distributions* (Cordeiro et al. 2013), the proportional and reversed hazard rate models, the Marshall-Olkin family and other sub-families. Some special models are listed in Table 1, where $G(x) = G(x;\boldsymbol{\xi})$.

Furthemore, the basic motivations for using the EMO-G family in practice are the following:

i. to make the kurtosis more flexible compared to the baseline model;
ii. to produce a skewness for symmetrical distributions;
iii. to construct heavy-tailed distributions for modeling real data;
iv. to generate distributions with symmetric, left-skewed, right-skewed or reversed-J shape;
v. to define special models with all types of the hrf;
vi. to provide consistently better fits than other generated models under the same baseline distribution.

This paper is organized as follows. In Section 2, we define the new family of distributions and provide a physical interpretation. Five of its special distributions are discussed in this section. In Section 3, some properties of the EMOG family are presented. The shape of the density and hazard rate functions are described analytically, two useful linear mixtures are provided. We derive a power series for the quantile function (qf) and we provide two general formulae for the moments. The incomplete moments are investigated and we derive the moment generating function (mgf) and determine the mean deviations. Estimation of the model parameters by maximum likelihood is performed in Section 4. Applications to two real data sets illustrate the performance of the EMO family in Section 5. The paper is concluded in Section 6.

Table 1 Some special models

α	λ	p	$G(x)$	Reduced distribution
-	-	0	$G(x)$	Exponentiated Generalized Class of Distributions (Cordeiro et al. 2013)
1	-	-	$G(x)$	Marshal-Olkin family of distributions (Marshall and Olkin 1997)
1	-	0	$G(x)$	Proportional hazard rate model (Gupta and Gupta 2007)
-	1	0	$G(x)$	Proportional reversed hazard rate model (Gupta and Gupta 2007)
1	1	0	$G(x)$	$G(x)$
1	-	-	$1-\mathrm{e}^{-x}$	Exponential - Geometric distribution (Adamidis and Loukas 1998)
-	-	-	$1-\mathrm{e}^{-x}$	Generalized Exponential - Geometric distribution (Silva et al. 2010)
1	-	-	$1-\mathrm{e}^{-\beta x^{\gamma}}$	Weibull-Geometric distribution (Barreto-Souza et al. 2011)
-	-	-	$1-\mathrm{e}^{-\beta x^{\gamma}}$	Exponentiated Weibull-Geometric distribution (Mahmoudi and Shiran 2012)

2 The new family

The density function corresponding to (2) is given by

$$f(x) = \alpha\,\lambda\,(1-p)\,g(x;\boldsymbol{\xi})\,[\,1-G(x;\boldsymbol{\xi})]^{\lambda-1}\,\frac{\left\{1-[\,1-G(x;\boldsymbol{\xi})]^{\lambda}\,\right\}^{\alpha-1}}{\left\{1-p[\,1-G(x;\boldsymbol{\xi})]^{\lambda}\,\right\}^{\alpha+1}}, \quad (3)$$

where $g(x;\boldsymbol{\xi})$ is the baseline pdf. This density function will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions. Hereafter, a random variable X with density function (3) is denoted by $X \sim$ EMO-G$(p,\alpha,\lambda,\boldsymbol{\xi})$. Henceforth, we can omit sometimes the dependence on the baseline vector $\boldsymbol{\xi}$ of parameters and write simply $G(x) = G(x;\boldsymbol{\xi}), f(x) = f(x;p,\alpha,\lambda,\boldsymbol{\xi})$, etc.

A physical interpretation of the EMO-G distribution can be given as follows. Consider a system formed by α independent components having the Marshall-Olkin cdf given by

$$H(x) = \frac{1-[\,1-G(x)]^{\lambda}}{1-p[\,1-G(x)]^{\lambda}}.$$

Suppose the system fails if all of the α components fail and let X denote the lifetime of the entire system. Then, the cdf of X is

$$F(x) = H(x)^{\alpha} = \left\{\frac{1-[\,1-G(x;\boldsymbol{\xi})]^{\lambda}}{1-p[\,1-G(x;\boldsymbol{\xi})]^{\lambda}}\right\}^{\alpha}.$$

The hazard rate function (hrf) of X becomes

$$h(x) = \frac{\alpha\,\lambda\,(1-p)\,g(x;\boldsymbol{\xi})\,[\,1-G(x;\boldsymbol{\xi})]^{\lambda-1}\left\{1-[\,1-G(x;\boldsymbol{\xi})]^{\lambda}\,\right\}^{\alpha-1}}{\left\{\left\{1-p[\,1-G(x;\boldsymbol{\xi})]^{\lambda}\,\right\}^{\alpha}-\left\{1-[\,1-G(x;\boldsymbol{\xi})]^{\lambda}\,\right\}^{\alpha}\right\}\left\{1-p[\,1-G(x;\boldsymbol{\xi})]^{\lambda}\,\right\}}. \quad (4)$$

The EMO family of distributions is easily simulated by inverting (2): if u has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$X = G^{-1}\left[1-\left(\frac{1-u^{1/\alpha}}{1-p\,u^{1/\alpha}}\right)^{1/\lambda}\right] \quad (5)$$

follows the density function (3).

2.1 Special EMO distributions

For $p = 0$, we obtain, as a special case of (3), the exponentiated generalized class (Cordeiro et al. 2013) of distributions, which provides greater flexibility of its tails and can be applied in many areas of engineering and biology. Here, we present some special cases of the EMO family since it extends several useful distributions in the literature. For all cases listed below, $p \in (0,1)$, $\alpha > 0$ and $\lambda > 0$. These cases are defined by taking $G(x)$ and $g(x)$ to be the cdf and pdf of a specified distribution. The general form of the pdf of the special EMO distributions can be expressed as:

$$f(x) = q(\theta_1,\theta_2,\ldots,\theta_m)g(x)\,[1-G(x)]^{\lambda-1}\,\frac{\left\{1-[1-G(x)]^{\lambda}\right\}^{\alpha-1}}{\left\{1-p\,[1-G(x)]^{\lambda}\right\}^{\alpha+1}},$$

where $q(\theta_1,\theta_2,\ldots,\theta_m)$ is defined as a function of m parameters of the special EMO distribution. We list some special EMO distributions in Table 2, where the letters N, Fr, Ga, B, and Gu stand for the normal, Fréchet, gamma, beta and Gumbel baselines, respectively.

Table 2 Special EMO distributions

Distribution	$q(\cdot)$	$G(x)$	$g(x)$
$EMON(p,\alpha,\lambda,\mu,\sigma^2)$	$\frac{\alpha\lambda(1-p)}{\sigma}$	$\phi\left(\frac{x-\mu}{\sigma}\right)$	$\Phi\left(\frac{x-\mu}{\sigma}\right)$
$EMOFr(p,\alpha,\lambda,\beta,\sigma)$	$\alpha\lambda(1-p)\beta$	$\exp\left\{-\left(\frac{\sigma}{x}\right)^{\beta}\right\}$	$\sigma^{\beta}x^{-\beta-1}\exp\left\{-\left(\frac{\sigma}{x}\right)^{\beta}\right\}$
$EMOGa(p,\alpha,\lambda,a,b)$	$\frac{\alpha\lambda(1-p)b^a}{\Gamma(a)}$	$\frac{\gamma(a,bx)}{\Gamma(a)}$	$\frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}$
$EMOB(p,\alpha,\lambda,a,b)$	$\frac{\alpha\lambda(1-p)}{B(a,b)}$	$\frac{\int_0^x w^{a-1}(1-w)^{b-1}dw}{B(a,b)}$	$\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$
$EMOGu(p,\alpha,\lambda,\mu,\sigma)$	$\frac{\alpha\lambda(1-p)}{\sigma}$	$\exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\}$	$\exp\left\{-\exp\left[-\frac{(x-\mu)}{\sigma}\right]-\frac{(x-\mu)}{\sigma}\right\}$

For the EMON distribution, the parameter σ has the same dispersion property as in the normal density. For the EMOB distribution, the beta distribution corresponds to the limiting case: $p \to 0$ and $\alpha = \lambda = 1$. For the EMOFr distribution, we have the classical Fréchet distribution when $p = 0$ and $\alpha = \lambda = 1$. The Kumaraswamy beta (KwB) and Kumaraswamy-gamma (KwGa) distributions can be obtained from the EMOB and EMOGa models when $p \to 0$. Plots of these EMOG density functions are displayed in Figs. 1, 2, 3, 4 and 5.

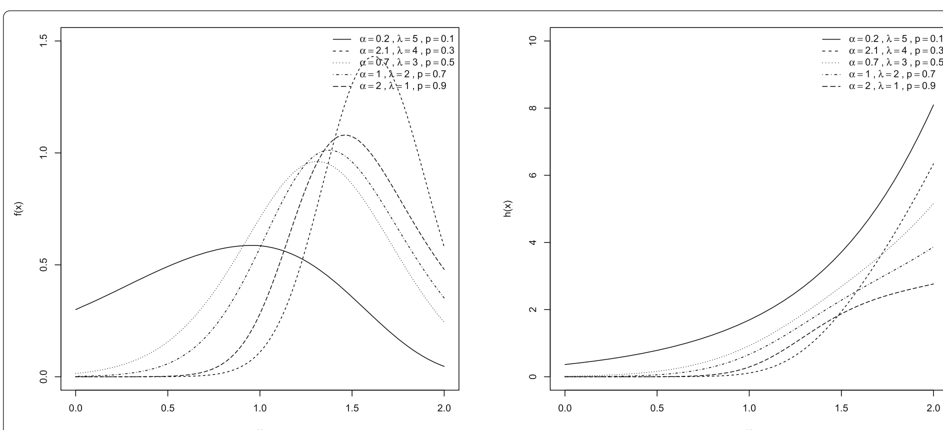
3 Some properties of the EMOG family

We investigate some properties of the EMOG in this section.

3.1 Asymptotic and shapes

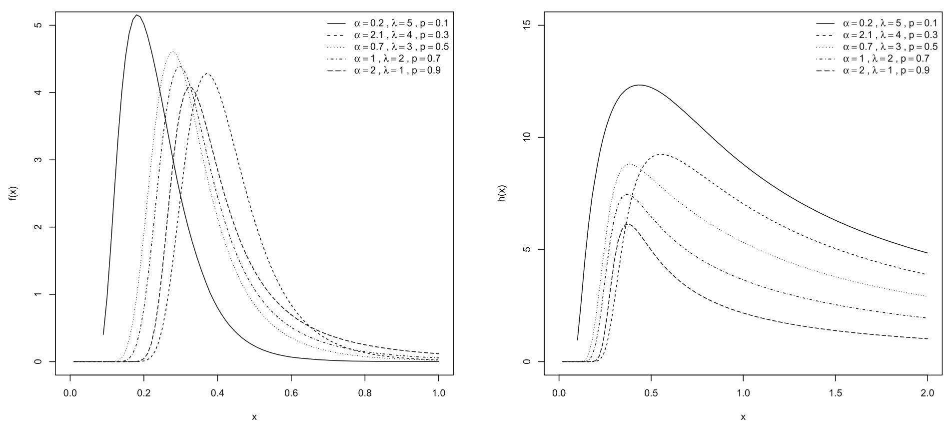
Proposition 1 *Let* $a = \inf\{x|\, G(x) > 0\}$. *The asymptotics of Eqs.* (2), (3) *and* (4) *when* $x \to a$ *are given by*

$$F(x) \sim [\lambda\, G(x)]^{\alpha}\,,$$
$$f(x) \sim \alpha\, \lambda^{\alpha}\, g(x)\, G(x)^{\alpha-1},$$
$$h(x) \sim \alpha\, \lambda^{\alpha}\, g(x)\, G(x)^{\alpha-1}.$$



(a) $\mu = 0.5$ and $\sigma = 0.5$ (b) $\mu = 0.5$ and $\sigma = 0.5$

Fig. 1 Plots of the EMON pdf (**a**) and EMON hrf (**b**) for some parameter values



(a) $\beta = 0.5$ and $\sigma = 1.5$ (b) $\beta = 0.5$ and $\sigma = 1.5$

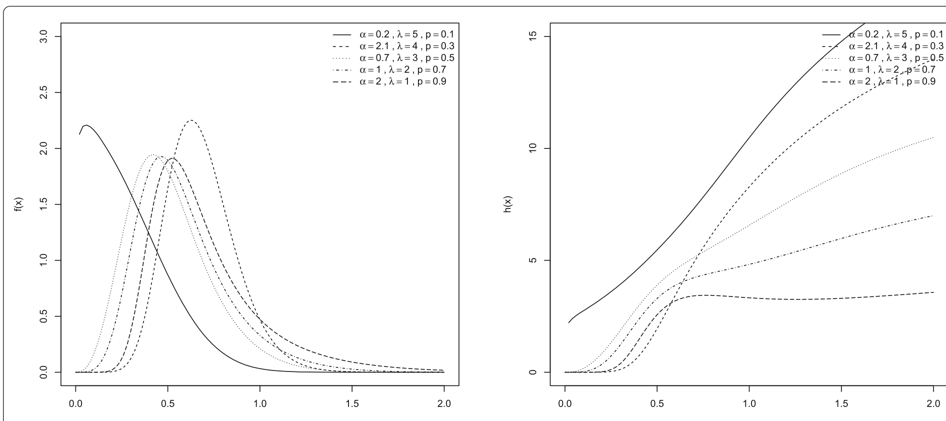
Fig. 2 Plots of the EMOFr pdf (**a**) and EMOFr hrf (**b**) for some parameter values

Proposition 2 *The asymptotics of Eqs.* (2), (3) *and* (4) *when* $x \to \infty$ *are given by*

$$\begin{aligned} 1 - F(x) &\sim \alpha\, \bar{G}(x)^{\lambda}, \\ f(x) &\sim \alpha\, \lambda\, g(x)\, \bar{G}(x)^{\lambda-1}, \\ h(x) &\sim \frac{\lambda\, g(x)}{\bar{G}(x)}. \end{aligned}$$

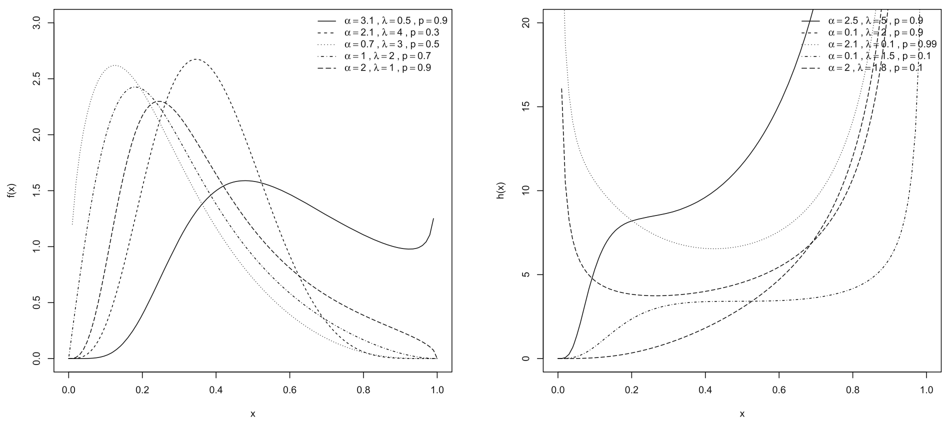
The shapes of the density and hazard rate functions can be described analytically. The critical points of the EMO-G density function are the roots of the equation

$$\begin{aligned} \frac{d\,\log[f(x)]}{dx} &= \frac{g'(x)}{g(x)} + (1-\lambda)\frac{g(x)}{1-G(x)} \\ &\quad - \lambda g(x)[1-G(x)]^{\lambda-1}\left\{\frac{1-\alpha}{1-[1-G(x)]^{\lambda}} + \frac{p(\alpha+1)}{1-p[1-G(x)]^{\lambda}}\right\} = 0 \quad (6) \end{aligned}$$



(a) $a = 5.5$ and $b = 5.5$ (b) $a = 5.5$ and $b = 5.5$

Fig. 3 Plots of the EMOGa pdf (**a**) and EMOGa hrf (**b**) for some parameter values



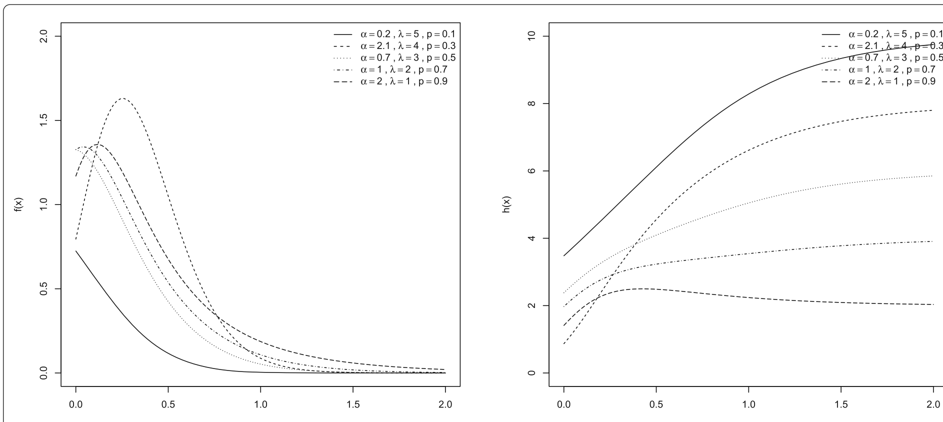
(a) $a = 2.0$ and $b = 1.5$ (b) $a = 2.0$ and $b = 1.5$

Fig. 4 Plots of the EMOB pdf (**a**) and EMOB hrf (**b**) for some parameter values

that corresponds to points where $f'(x) = 0$. There may be more than one root to (6). Let $\lambda(x) = d^2 \log[f(x)]/dx^2$. We have

$$\begin{aligned}\lambda(x) = {} & \frac{g''(x)g(x) - g'(x)^2}{g^2(x)} + (1-\lambda)\frac{g'(x)[1-G(x)] + g^2(x)}{[1-G(x)]^2} + \lambda(\alpha-1) \\ & \times \left\{ g'(x)\frac{[1-G(x)]^{\lambda-1}}{1-[1-G(x)]^{\lambda}} - (\lambda-1)g^2(x)\frac{[1-G(x)]^{\lambda-2}}{1-[1-G(x)]^{\lambda}} - \lambda g^2(x)\frac{[1-G(x)]^{2\lambda-2}}{\left\{1-[1-G(x)]^{\lambda}\right\}^2} \right\} \\ & - p\lambda(\alpha+1)\left\{ g'(x)\frac{[1-G(x)]^{\lambda-1}}{1-p[1-G(x)]^{\lambda}} - (\lambda-1)g^2(x)\frac{[1-G(x)]^{\lambda-2}}{1-p[1-G(x)]^{\lambda}} \right. \\ & \left. - p\lambda g^2(x)\frac{[1-G(x)]^{2\lambda-2}}{\left\{1-p[1-G(x)]^{\lambda}\right\}^2} \right\}.\end{aligned} \tag{7}$$

If $x = x_0$ is a root of (6) then it corresponds to a local maximum if $\lambda(x) > 0$ for all $x < x_0$ and $\lambda(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\lambda(x) < 0$ for all



(a) $\mu = 0.5$ and $\sigma = 1.5$ (b) $\mu = 0.5$ and $\sigma = 1.5$

Fig. 5 Plots of the EMOGu pdf (**a**) and EMOGu hrf (**b**) for some parameter values

$x < x_0$ and $\lambda(x) > 0$ for all $x > x_0$. It gives a point of inflexion if either $\lambda(x) > 0$ for all $x \neq x_0$ or $\lambda(x) < 0$ for all $x \neq x_0$.

The critical points of the (hrf) of X are obtained from the equation

$$\frac{d\,\log[h(x)]}{dx} = \frac{g'(x)}{g(x)} + (1-\lambda)\frac{g(x)}{1-G(x)} + \lambda(\alpha-1)\frac{g(x)[1-G(x)]^{\lambda-1}}{1-[1-G(x)]^{\lambda}} - \lambda p\frac{g(x)[1-G(x)]^{\lambda-1}}{1-p[1-G(x)]^{\lambda}}$$
$$-\frac{p\alpha\lambda g(x)[1-G(x)]^{\lambda-1}\left\{1-[1-G(x)]^{\lambda}\right\}^{\alpha-1} - \lambda\alpha g(x)[1-G(x)]^{\lambda-1}\left\{1-p[1-G(x)]^{\lambda}\right\}^{\alpha-1}}{\left\{1-p[1-G(x)]^{\lambda}\right\}^{\alpha} - \left\{1-[1-G(x)]^{\lambda}\right\}^{\alpha}} = 0. \tag{8}$$

There may be more than one root to (8). Let $\tau(x) = d^2 \log[h(x)]/dx^2$. If $x = x_0$ is a root of (8) then it refers to a local maximum if $\tau(x) > 0$ for all $x < x_0$ and $\tau(x) < 0$ for all $x > x_0$. It corresponds to a local minimum if $\tau(x) < 0$ for all $x < x_0$ and $\tau(x) > 0$ for all $x > x_0$. It gives an inflexion point if either $\tau(x) > 0$ for all $x \neq x_0$ or $\tau(x) < 0$ for all $x \neq x_0$.

3.2 Linear mixtures

We can demonstrate that the cdf (2) of X admits the expansion

$$F(x) = \sum_{k=0}^{\infty} b_k\, H_k(x;\boldsymbol{\xi}), \tag{9}$$

where $b_k = \sum_{i,j=0}^{\infty} w_{i,j,k}$,

$$w_{i,j,k} = w_{i,j,k}(\alpha,\lambda,p) = (-1)^{i+j+k}\binom{-\alpha}{i}\binom{\alpha}{j}\binom{(i+j)\lambda}{k}p^i,$$

and $H_k(x;\boldsymbol{\xi}) = G(x;\boldsymbol{\xi})^k$ denotes the exponentiated-G ("exp-G") cdf with power parameter k.

The density function of X can be expressed as an infinite linear mixture of exp-G density functions

$$f(x) = \sum_{k=0}^{\infty} b_{k+1}\, h_{k+1}(x;\boldsymbol{\xi}), \tag{10}$$

where (for $k \geq 0$) $h_{k+1}(x;\boldsymbol{\xi}) = (k+1)\, g(x;\boldsymbol{\xi})\, G(x;\boldsymbol{\xi})^k$ denotes the density function of the random variable $Y_{k+1} \sim$ exp-G$(k+1)$. Equation (10) reveals that the EMO-G density function is a linear mixture of exp-G density functions. Thus, some of its mathematical properties can be derived directly from those properties of the exp-G distribution. For example, the ordinary and incomplete moments and (mgf) of X can be obtained from those quantities of the exp-G distribution. Some structural properties of the exp-G distributions are well-defined by Mudholkar and Hutson (1996), Gupta and Kundu (2001) and Nadarajah and Kotz (2006), among others.

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as Maple, Mathematica and Matlab. These platforms have currently the ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration. The infinity limit in these sums can be substituted by a large positive integer such as 20 or 30 for most practical purposes.

3.3 Quantile power series

We obtain explicit expressions for the moments and generating function of the EMO family using a power series for the qf $x = Q(u) = F^{-1}(u)$ of X by expanding (5). If the G qf, say $Q_G(u)$, does not have a closed-form expression, this function can usually be expressed as a power series

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\, u^i, \tag{11}$$

where the coefficients a_i's are suitably chosen real numbers depending on the parameters of the parent distribution. For several important distributions such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in Eq. (11).

We use throughout the paper a result of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{12}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily determined from the recurrence equation, with $c_{n,0} = a_0^n$,

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [\, m(n+1) - i]\; a_m\, c_{n,i-m}. \tag{13}$$

Clearly, $c_{n,i}$ can be easily evaluated numerically from $c_{n,0}, \ldots, c_{n,i-1}$ and then from the quantities $a_0, \ldots, a_i$.

Next, we derive an expansion for the argument of $Q_G(\cdot)$ in Eq. (5)

$$A = 1 - \frac{(1 - u^{1/\alpha})^{1/\lambda}}{(1 - p\, u^{1/\alpha})^{1/\lambda}}.$$

Using the the generalized binomial expansion four times since $u \in (0, 1)$, we can write

$$A = \sum_{r,s,t=0}^{\infty} \sum_{m=0}^{t} (-1)^{r+s+t+m}\, p^r \binom{-\lambda^{-1}}{r} \binom{\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t} \binom{t}{m} u^m$$

Then, the qf of X can be expressed from (5) as

$$Q(u) = Q_G\left(\sum_{m=0}^{\infty} \delta_m\, u^m\right), \tag{14}$$

where

$$\delta_m = \begin{cases} 1 - \sum_{r,s,t=0}^{\infty} (-1)^{r+s+t}\, p^r \binom{-\lambda^{-1}}{r} \binom{-\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t}, & m = 0, \\ \sum_{r,s,t=0}^{\infty} (-1)^{r+s+t+m}\, p^r \binom{-\lambda^{-1}}{r} \binom{\lambda^{-1}}{s} \binom{(r+s)\alpha^{-1}}{t} \binom{t}{m}, & m > 0. \end{cases}$$

By combining (11) and (14), we have

$$Q(u) = \sum_{i=0}^{\infty} a_i \left(\sum_{m=0}^{\infty} \delta_m\, u^m\right)^i,$$

and then using (12) and (13),

$$Q(u) = \sum_{m=0}^{\infty} e_m\, u^m, \tag{15}$$

where $e_m = \sum_{i=0}^{\infty} a_i\, d_{i,m}$, $d_{i,0} = \delta_0^i$ and, for $m > 1$,

$$d_{i,m} = (m\,\delta_0)^{-1} \sum_{n=1}^{m} [\, n(i+1) - m]\; \delta_n\, d_{i,m-n}.$$

Equation (15) is the main result of this section. It allows to obtain various mathematical quantities for the EMO-G family as can be seen in the next sections. Note that

$$Q(u)^r = \left(\sum_{m=0}^{\infty} e_m u^m \right)^r = \sum_{m=0}^{\infty} f_{r,m}\, u^m, \tag{16}$$

where $f_{r,m}$ is obtained from the e_m's using (13).

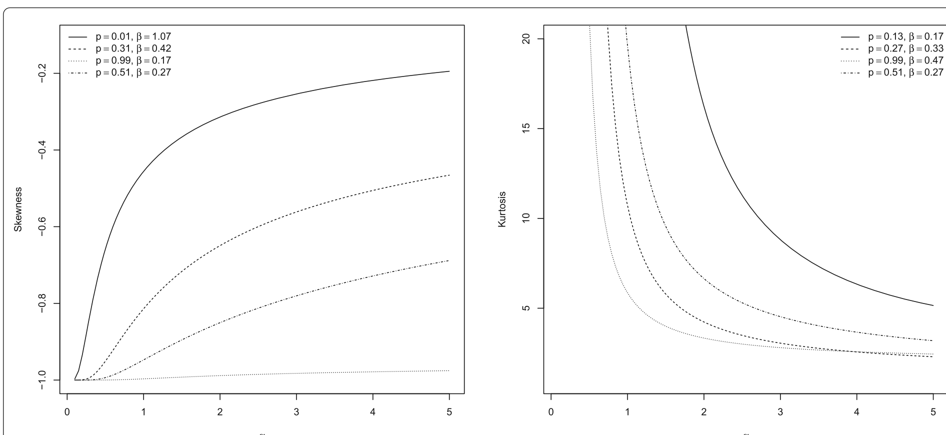
The effects of the shape parameters on the skewness and kurtosis can be determined from quantile measures. The shortcomings of the classical kurtosis measure are well-known. The Bowley skewness (Kenney and Keeping 1962) is one of the earliest skewness measures defined by the average of the quartiles minus the median divided by half the interquartile range, namely

$$B = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Since only the middle two quartiles are considered and the outer two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis (Moors 1998) is based on octiles

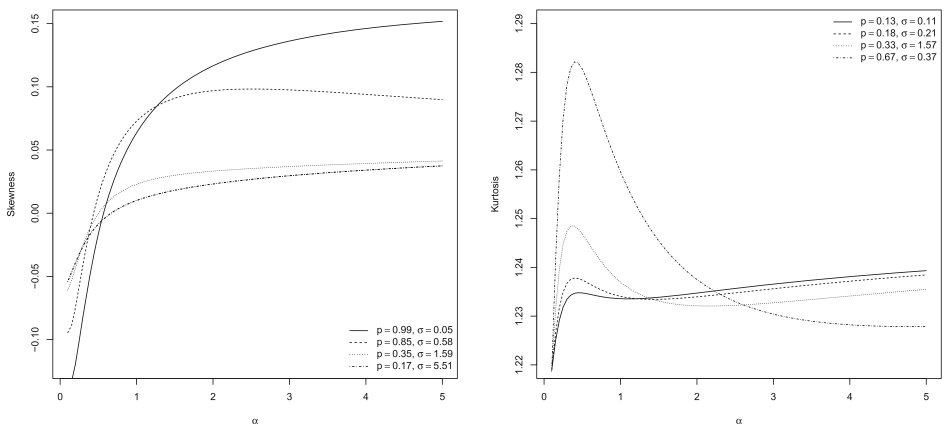
$$M = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}.$$

These measures are less sensitive to outliers and they exist even for distributions without moments. In Figs. 6 and 7, we plot the measures B and M for the EMOFr and EMON distributions (discussed in Section 2), respectively. These plots reveal how both measures B and M vary on the shape parameters.



(a) $\sigma = 1.0$ and $\lambda = 1.0$ (b) $\sigma = 1.0$ and $\lambda = 1.0$

Fig. 6 Skewness (**a**) and Kurtosis (**b**) of the EMOFr distribution



(a) $\mu = 3.3$ and $\lambda = 1.0$ (b) $\mu = 3.3$ and $\lambda = 1.0$

Fig. 7 Skewness (**a**) and Kurtosis (**b**) of the EMON distribution

3.4 Moments

Hereafter, we shall assume that G is the cdf of a random variable Z and that F is the cdf of a random variable X having density function (3). The rth ordinary moment of X can be obtained from the (r,k)th Probability Weighted Moment (PWM) of Z defined by

$$\tau_{r,k} = \mathrm{E}[\,Z^r\, G(Z)^k] = \int_{-\infty}^{\infty} z^r\, G(z)^k\, g(z) dz. \tag{17}$$

In fact, we have

$$\mathrm{E}(X^r) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, \tau_{r,k}. \tag{18}$$

Thus, the moments of any EMO-G distribution can be expressed as an infinite linear combination of the PWMs of G. A second formula for $\tau_{r,k}$ can be based on the parent qf $Q_G(u) = G^{-1}(u)$. Setting $G(x) = u$, we obtain

$$\tau_{r,k} = \int_0^1 Q_G(u)^r\, u^k du, \tag{19}$$

where the integral follows from (16) as

$$\mathrm{E}(X^r) = \int_0^1 Q(u)^r du = \sum_{m=0}^{\infty} \frac{f_{r,m}}{m+1}. \tag{20}$$

The PWMs for some well-known distributions will be determined in the following sections using alternatively Eqs. (17) and (19).

The central moments (μ_s) and cumulants (κ_s) of X can be obtained from Eqs. (18) and (20) as

$$\mu_s = \sum_{j=0}^{s} (-1)^j \binom{s}{j} \mu_1'^s \mu_{s-j}', \qquad \kappa_s = \mu_s' - \sum_{j=1}^{s-1} \binom{s-1}{j-1} \kappa_j \mu_{s-j}',$$

where $\kappa_1 = \mu'_1$. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The pth descending factorial moment of X is

$$\mu'_{(p)} = \mathrm{E}[X^{(p)}] = \mathrm{E}[X(X-1) \times \cdots \times (X-p+1)] = \sum_{k=0}^{p} s(p,k)\mu'_k,$$

where $s(r,k) = (k!)^{-1}[d^k x^{(r)}/dx^k]_{x=0}$ is the Stirling number of the first kind. So, we can obtain the factorial moments from the ordinary moments given before.

3.5 Incomplete moments

The nth incomplete moment of X is defined as $m_r(y) = \int_{-\infty}^{y} x^r f(x)dx$. For empirical purposes, the shape of many distributions can be usefully described by the incomplete moments. Here, we propose two methods to determine the incomplete moments of the new family. First, we can express $m_r(y)$ as

$$m_r(y) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1} \int_0^{G(y;\boldsymbol{\xi})} Q_G(u)^r\, u^k\, du. \tag{21}$$

The integral in (21) can be evaluated at least numerically for most baseline distributions.

A second method for the incomplete moments of X follows from (21) using Eqs. (12) and (13). We obtain

$$m_r(y) = \sum_{k,m=0}^{\infty} \frac{(k+1)\, b_{k+1}\, c_{r,m}}{m+k+1}\, G(y;\boldsymbol{\xi})^{m+k+1}. \tag{22}$$

Equations (21) and (22) are the main results of this section.

3.6 Generating function

Here, we provide three formulae for the mgf $M(s) = E(\mathrm{e}^{sX})$ of X. A first formula for $M(s)$ comes from Eq. (10) as

$$M(s) = \sum_{k=0}^{\infty} b_{k+1}\, M_{k+1}(s), \tag{23}$$

where $M_{k+1}(s)$ is the generating function of the exp-G$(k+1)$ distribution. Hence, $M(s)$ can be determined from an infinite linear combination of the exp-G generating functions.

A second formula for $M(s)$ can be derived from Eq. (10) as

$$M(s) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, \rho_k(s), \tag{24}$$

where

$$\rho_k(s) = \int_0^1 \exp\left[s\, Q_G(u)\right] u^k du. \tag{25}$$

We can derive the mgfs of several EMO distributions directly from Eqs. (24) and (25). For example, the mgfs of the exponentiated Marshall-Olkin exponential (EMOE) (such that $\lambda s < 1$) and EMO-standard logistic (for $s < 1$) distributions are given by

$$M(s) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, B(k+1, 1-\lambda s) \quad \text{and} \quad M(s) = \sum_{k=0}^{\infty} (k+1)\, b_{k+1}\, B(s+k+1, 1-s),$$

respectively.

3.7 Mean deviations

The mean deviations about the mean ($\delta_1 = E(|X - \mu'_1|)$) and about the median ($\delta_2 = E(|X - M|)$) of X can be expressed as

$$\delta_1 = 2\mu'_1 F\left(\mu'_1\right) - 2m_1\left(\mu'_1\right) \qquad \text{and} \qquad \delta_2 = \mu'_1 - 2m_1(M), \tag{26}$$

respectively, $F(\mu'_1)$ is easily evaluated from Eq. (2),

$$M = Q_G\left[1 - \left(\frac{1 - 2^{-1/\alpha}}{1 - p \times 2^{-1/\alpha}}\right)^{1/\lambda}\right]$$

is the median of X, $\mu'_1 = \mathrm{E}(X)$ comes from (18) and $m_1(y)$ is the first incomplete moment of X determined from (22) with $r = 1$.

Next, we provide three alternative ways to compute δ_1 and δ_2. A general equation for $m_1(z)$ is given by (21). A second general formula for $m_1(z)$ can be obtained from (10) as

$$m_1(z) = \sum_{k=0}^{\infty} b_{k+1}\, J_{k+1}(z), \tag{27}$$

where

$$J_{k+1}(z) = \int_{-\infty}^{z} x\, h_{k+1}(x) dx. \tag{28}$$

Equation (28) is the basic quantity to compute the mean deviations for the exp-G distributions. A simple application of (27) and (28) can be conducted to the exponentiated Marshall-Olkin Weibull (EMOW) distribution. The exponentiated Weibull density function (for $x > 0$) with power parameter $k + 1$, shape parameter c and scale parameter β is given by

$$h_{k+1}(x) = c\,(k+1)\,\beta^c\, x^{c-1}\, \exp\left\{-(\beta x)^c\right\}\left[1 - \exp\left\{-(\beta x)^c\right\}\right]^k,$$

and then

$$J_{k+1}(z) = c\,(k+1)\,\beta^c \sum_{r=0}^{\infty}(-1)^r \binom{k}{r} \int_0^z x^c\, \exp\left\{-(r+1)(\beta x)^c\right\} dx.$$

The last integral reduces to the incomplete gamma function

$$J_{k+1}(z) = c\,(k+1)\,\beta^c \sum_{r=0}^{\infty}(-1)^r \binom{k}{r}\, \gamma\left(c+1, (r+1)(\beta z)^c\right),$$

where $\gamma(a,x) = \int_0^x w^{a-1}\, \mathrm{e}^{-w} \mathrm{d}w$.

A third general formula for $m_1(z)$ can be derived by setting $u = G(x)$ in (10)

$$m_1(z) = \sum_{k=0}^{\infty}(k+1)\, b_{k+1}\; T_k(z), \tag{29}$$

where $T_k(z)$ is given by

$$T_k(z) = \int_0^{G(z)} Q_G(u)\, u^k du. \tag{30}$$

Applications of these equations are straightforward to obtain Bonferroni and Lorenz curves. These curves are defined (for a given probability π) by $B(\pi) = m_1(q)/(\pi\, \mu'_1)$ and $L(\pi) = m_1(q)/\mu'_1$, respectively, where $q = F^{-1}(\pi) = Q(\pi)$ comes from the qf of X for a given probability π.

4 Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals for the model parameters. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters for this family from complete samples only by maximum likelihood. Here, we determine the MLEs of the parameters of the new family of distributions from complete samples only.

Let $x_1, \ldots, x_n$ be the observed values from the EMOG distribution with parameters p, α, λ and $\boldsymbol{\xi}$. Let $\theta = (p, \alpha, \lambda, \boldsymbol{\xi})^\top$ be the $r \times 1$ parameter vector. The total log-likelihood function for θ is given by

$$\begin{aligned}\ell_n = \ell_n(\Theta) &= n \log\alpha + n \log\lambda + n \log(1-p) + \sum_{i=1}^{n} \log\left[g(x_i;\boldsymbol{\xi})\right] \\ &+(\lambda-1)\sum_{i=1}^{n}\log\left[1-G(x_i;\boldsymbol{\xi})\right] + (\alpha-1)\sum_{i=1}^{n}\log\left\{1-\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}\right\} \\ &-(\alpha+1)\sum_{i=1}^{n}\log\left\{1-p\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}\right\}. \qquad (31)\end{aligned}$$

The maximized log-likelihood can be either directly by using the NLMIXED procedure in SAS or the sub-routine MaxBFGS in the Ox program (see Doornik 2009) or by solving the nonlinear likelihood equations obtained by differentiating (31). The components of the score function

$$U_n(\theta) = (\partial\ell_n/\partial p, \partial\ell_n/\partial\alpha, \partial\ell_n/\partial\lambda, \partial\ell_n/\partial\boldsymbol{\xi})^\top$$

are given by

$$\begin{aligned}\frac{\partial\ell_n}{\partial p} &= (\alpha+1)\sum_{i=1}^{n}\frac{\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}}{1-p\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}} - \frac{n}{1-p}, \\ \frac{\partial\ell_n}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left\{1-\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}\right\} - \sum_{i=1}^{n}\log\left\{1-p\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}\right\}, \\ \frac{\partial\ell_n}{\partial\lambda} &= \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-G(x_i;\boldsymbol{\xi})\right] - (\alpha-1)\sum_{i=1}^{n}\frac{\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}\log\left[1-G(x_i;\boldsymbol{\xi})\right]}{1-\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}} \\ &+ p\,(\alpha+1)\sum_{i=1}^{n}\frac{\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}\log\left[1-G(x_i;\boldsymbol{\xi})\right]}{1-p\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell_n}{\partial\boldsymbol{\xi}} &= \sum_{i=1}^{n}\frac{g^{(\xi)}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})} - (\lambda-1)\sum_{i=1}^{n}\frac{G^{(\xi)}(x_i;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})} \\ &+ \lambda(\alpha-1)\sum_{i=1}^{n}\frac{g^{(\xi)}(x;\boldsymbol{\xi})\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda-1}}{1-\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}} - p\lambda(\alpha+1) \\ &\times\sum_{i=1}^{n}\frac{g^{(\xi)}(x_i;\boldsymbol{\xi})\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda-1}}{1-p\left[1-G(x_i;\boldsymbol{\xi})\right]^{\lambda}},\end{aligned}$$

where $h^{(\xi)}(\cdot)$ means the derivative of the function h with respect to ξ. For interval estimation of the model parameters, we can derive the observed information matrix $J_n(\theta)$, whose elements can be obtained from the authors upon request. Let $\widehat{\theta}$ be the MLE of θ. Under standard regularity conditions (Cox and Hinkley 1974), we can approximate the distribution of $\sqrt{n}(\widehat{\theta} - \theta)$ by the multivariate normal $N_r(0, K(\theta)^{-1})$, where $K(\theta) = \lim_{n\to\infty} n^{-1} J_n(\theta)$ is the unit information matrix and r is the number of parameters of the new distribution.

Often with lifetime data and reliability studies, one encounters censoring. In a very realistic random censoring mechanism, each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ and $\delta_i = I(X_i \leq C_i)$ is such that $\delta_i = 1$ if X_i is a time to event and $\delta_i = 0$ if it is right censored for $i = 1, \ldots, n$. The censored likelihood $L(\theta)$ for the model parameters is

$$L(\theta) \propto \prod_{i=1}^{n} [f(x_i; p, \alpha, \lambda, \boldsymbol{\xi})]^{\delta_i} \; [S(x_i; p, \alpha, \lambda, \boldsymbol{\xi})]^{1-\delta_i},$$

where $f(x; p, \alpha, \lambda, \boldsymbol{\xi})$ is given by (3) and $S(x; p, \alpha, \lambda, \boldsymbol{\xi})$ is the survival function evaluated from (2).

5 Applications

In this section, we use a real data set, collected by Prater (1956) and analyzed by Atkinson (1985), on the stress among women in Townsville, Queensland, Australia and the proportion of crude oil converted to gasoline after distillation and fractionation. This application aims to illustrate the potentiality of the EMO family. All the computations were done using the R software. For this application, we consider the following distributions: EMOB $(a, b, p, \lambda, \alpha)$ and Kw-WP$(a, b, c, \lambda, \beta)$ (Kwmaraswamy Weibull Poisson distribution) proposed by Ramos et al. (2015), both with five parameters. The density of the Kw-WP is given by

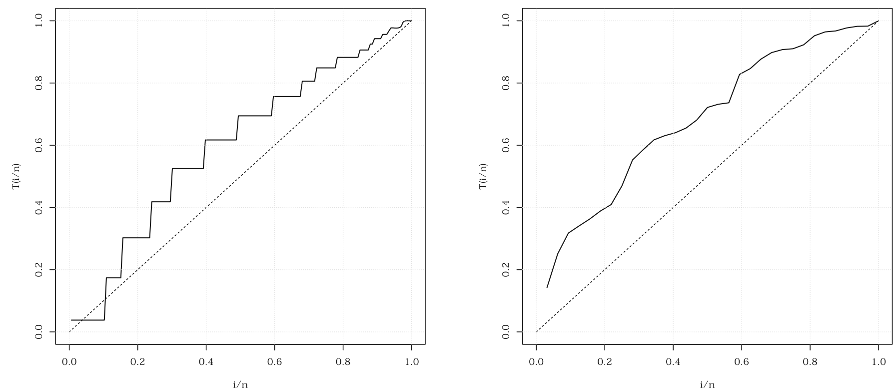
$$f(x) = \frac{\lambda abc\beta^c x^{c-1}[\,1 - \mathrm{e}^{-(\beta x)^c}]^{a-1} \exp[\,\lambda\{1 - [\,1 - \mathrm{e}^{-(\beta x)^c}]^a\,\}^b - (\beta x)^c]}{(\mathrm{e}^{\lambda} - 1)\{1 - [\,1 - \mathrm{e}^{-(\beta x)^c}]^a\,\}^{1-b}}.$$

A descriptive analysis of the data is presented in Table 3.

It is possible to obtain qualitative information about the hrf by means of plot analysis when we have the data censored or uncensored. We emphasize that the data sets here are uncensored. For this type of data, the total time in test (TTT) plot proposed

Table 3 Descriptive statistics

Statistics	Real data sets	
	Stress	Gasoline
Mean	0.2642	0.1966
Median	0.2500	0.1780
Mode	0.2500	0.1500
Variance	0.0376	0.0115
Skewness	0.9712	0.3867
Kurtosis	0.8272	–0.6561
Maximum	0.8500	0.4570
Minimum	0.0100	0.0280
n	166	32

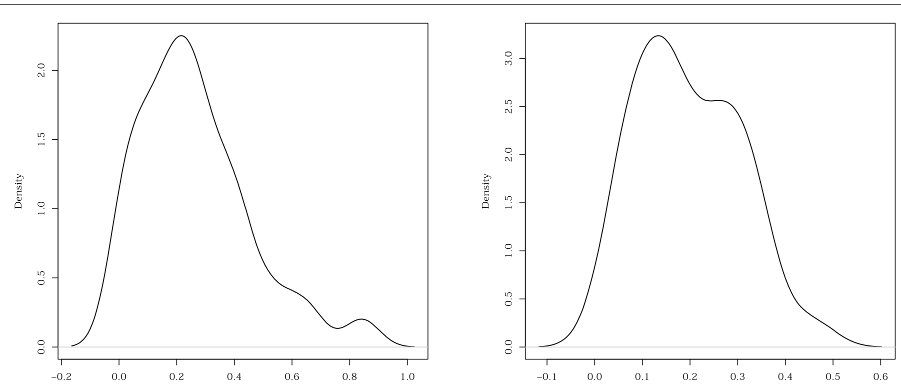


(a) (b)

Fig. 8 The TTT plot for: (**a**) stress among women in Townsville, Queensland, Australia (**b**) proportion of crude oil converted to gasoline after distillation and fractionation

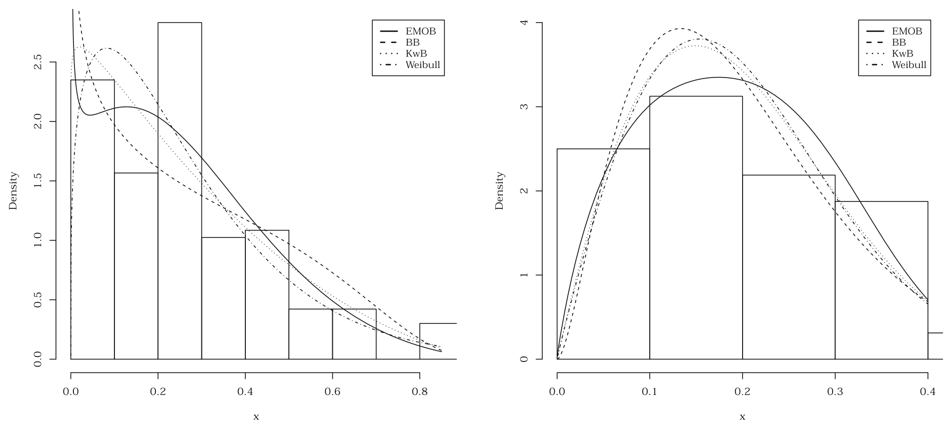
by Aarset (1987) may be used. Let T be a random variable with non-negative values that represents the survival time. The TTT curve is constructed by plotting the statistic $G(r/n) = \left[\sum_{i=1}^{r} T_{i:n} + (n+r)T_{r:n}\right] / \left(\sum_{i=1}^{n} T_{i:n}\right)$ versus r/n ($r = 1, \ldots, n$), where the values $T_{i:n}$ are the order statistics of the sample, for $i = 1, \ldots, n$. The plots can be easily obtained using the `TTT` function of the `AdequacyModel` package from the `R` software. More details about this package are available from `help(TTT)`. The TTT plots for the dataset in this application are shown in Fig. 8. For both plots, the TTT curve is concave, which, according to Aarset (1987), provides evidence that a monotonic increasing hrf is adequate.

The Fig. 9 displays the fitted densities to the current data obtained in a nonparametric manner using the gaussian kernel density estimation, defined as follows. Let $X_1, \ldots, X_n$ be a random vector of random variables independent and identically distributed where each variable follows an unknown distribution, denoted by f. The kernel density estimator is given by the following expression



(a) (b)

Fig. 9 Gaussian kernel density estimation for: (**a**) stress among women in Townsville, Queensland, Australia (**b**) proportion of crude oil converted to gasoline after distillation and fractionation



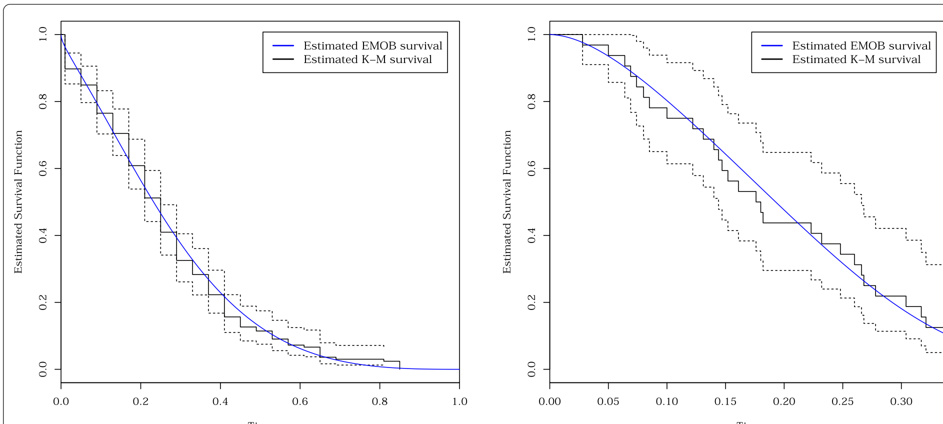
(a) (b)

Fig. 10 Estimates of the pdf for: (**a**) stress among women in Townsville, Queensland, Australia (**b**) proportion of crude oil converted to gasoline after distillation and fractionation

$$\hat{f}_h(x) = \frac{1}{n}\sum_{i=1}^{n} K_h(x - x_i) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{32}$$

where $K(\cdot)$ is the symmetrical kernel function and $\int_{-\infty}^{\infty} K(x)dx = 1$. Furthermore, $h > 0$ is known in literature as bandwith, which is a smoothing parameter. It is possible to find in literature numerous kernel function, as the normal standard distribution, for example. Silverman (1986) demonstrated that for the K standard normal, a reasonable bandwith is given by $h = \sqrt[5]{\left(4\hat{\sigma}^5/3n\right)} \approx 1.06\hat{\sigma}/\sqrt[5]{n}$, where $\hat{\sigma}$ is defined by the standard deviation of the sample.

The plots displayed in Fig. 10 indicates that the EMOB distribution provides the best fit compared with the other fitted distributions. We note the good adequacy of the fitted EMOB distribution in Fig. 11.



(a) (b)

Fig. 11 Estimated K-M survival (K-M estimates) compared with the EMOB survival estimates for: (**a**) tress among women in Townsville, Queensland, Australia (**b**) proportion of crude oil converted to gasoline after distillation and fractionation. The confidence intervals are 95 %

In this application, we use the package `AdequacyModel`. This package is intended to provide a computational support to work with probability distributions, mainly distributions aimed to survival analysis. This package was used to calculate some fitness statistics adjustment such as AIC (Akaike Information Criterion), CAIC (Consistent Akaikes Information Criterion), BIC (Bayesian Information Criterion), HQIC (Hannan-Quinn information criterion), KS (Test of Kolmogorov-Smirnov), A^* (statistic of Anderson-Darling) and W^* (statistic of Cramér-von Mises), which are described by Chen and Balakrishnan (1995), based on the results presented by Stephens (1986). When we want to test if one random sample, denoted by $x_1, x_2, \ldots, x_n$, with empirical distribuction function $F_n(x)$ comes from a specific distribution, we use these statistics. The Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics are given by the following expressions:

$$W^* = \left\{ n \int_{-\infty}^{+\infty} \{F_n(x) - F(x;\hat{\theta}_n)\}^2 \, dF(x;\hat{\theta}_n) \right\} \left(1 + \frac{0.5}{n}\right) \quad \text{and}$$

$$A^* = \left\{ n \int_{-\infty}^{+\infty} \frac{\{F_n(x) - F(x;\hat{\theta}_n)\}^2}{F(x;\hat{\theta}_n)(1 - F(x;\hat{\theta}_n))} dF(x;\hat{\theta}_n) \right\} \left(1 + \frac{0.75}{n} + \frac{2.25}{n^2}\right).$$

respectively. In these expressions, we have that $F_n(x)$ is the empirical distribution function, $F(x;\hat{\theta}_n)$ is the postulated distribution function evaluated at the MLE of θ, i.e. $\hat{\theta}_n$. Lower values of W^* and A^* provide evidence that $F(x;\hat{\theta}_n)$ generates the sample. More details about these statistics are given by Chen and Balakrishnan (1995). The `goodness.fit` function is used to calculate these statistics. More details can be obtained using the command `help(goodness.fit)`. The Table 4 shows the goodness-of-fit statistics (rounding to the fourth decimal place) for the dataset used in this application. The results showed that the EMOB $(a, b, p, \lambda, \alpha)$ distribution presented better results for the KS, A^* and W^* statistics when compared with the other distributions used in this application. In this study, the MLEs in Table 5 were obtained by global search heuristic method called Particle Swarm optmization - PSO proposed by Eberhart and Kennedy (1995). One of the advantages of using the PSO method in addition to being a robust optimization method is that there is no need to provide initial guesses. However, this is a computationally intensive method. The Appendix A shows the function `pso` implemented in `R`. At the end of the code there is a small example of how to use the function to minimize an objective function. The standard errors of the MLEs can be obtained by the bootstrap method. The standard errors were not obtained in these examples due to the use of the PSO method, which is computationally intensive.

Table 4 Goodness-of-fit statistics for the data: (I) stress among women in Townsville, Queensland, Australia (II) proportion of crude oil converted to gasoline after distillation and fractionation

Data set	Distribution	A*	W*
I	EMOB $(p, \alpha, \lambda, a, b)$	0.1554	0.0786
	Kw-WP$(a, b, c, \lambda, \beta)$	0.4974	0.1305
II	EMOB $(p, \alpha, \lambda, a, b)$	0.0348	0.0983
	Kw-WP$(a, b, c, \lambda, \beta)$	0.0489	0.0993

Table 5 MLEs for: (I) stress among women in Townsville, Queensland, Australia (II) proportion of crude oil converted to gasoline after distillation and fractionation

Data set	Distribution	Maximum Likelihood Estimates - MLE				
I	EMOB (p,α,λ,a,b)	−19.8703	1.4276	3.4036	0.2454	0.7793
	Kw-WP(a,b,c,λ,β)	12.3010	20.1431	0.1647	24.6569	2.3195
II	EMOB (p,α,λ,a,b)	−1.2077	0.3989	1.9915	4.6421	10.7513
	Kw-WP(a,b,c,λ,β)	15.2365	8.3171	0.2446	24.9571	10.5076

6 Conclusions

We derive general mathematical properties of a new continuous distributions with three extra shape parameters. We present some special models of the new EMO family of distributions. We investigate the shapes and derive explicit expressions for the ordinary and incomplete moments, quantile and generating functions and probability weighted moments, which hold for any baseline model. The estimation of the model parameters is by the method of maximum likelihood and the estimates were obtained by global search heuristic method called Particle Swarm Optmization-PSO. Ultimately, we fit some EMO-G distributions to two real data sets to demonstrate the potentiality of this family. We hope this generalization may attract wider applications in statistics.

Appendix

Code in R language for PSO method

```
pso <- function(func,S=150,lim_inf,lim_sup,e=0.0001,data=NULL,N=100){
b_lo = min(lim_inf)
b_up = max(lim_sup)
integer_max = .Machine$integer.max

if(length(lim_sup)!=length(lim_inf)){
stop("The␣vectors␣lim_inf␣and␣lim_sup␣must␣have␣the␣same␣dimension.")
}
dimension = length(lim_sup)
swarm_xi = swarm_pi = swarm_vi = matrix(NA,nrow=S,ncol=dimension)

# The best position of the particles.
g = runif(n=dimension,min=lim_inf,max=lim_sup)

# Objective function calculated in g.
f_g = func(par=as.vector(g),x=as.vector(data))

if(NaN%in%f_g==TRUE || Inf%in%abs(f_g)==TRUE{
while(NaN%in%f_g==TRUE || Inf%in%abs(f_g)==TRUE{
g = runif(n=dimension,min=lim_inf,max=lim_sup)
f_g = func(par=g,x=as.vector(data))
}
}

# Here begins initialization of the algorithm.
x_i = mapply(runif,n=S,min=lim_inf,max=lim_sup)

# Initializing the best position of particularities i to initial position.

swarm_pi = swarm_xi = x_i
f_pi = apply(X=x_i,MARGIN=1,FUN=func,x=as.vector(data))

is.integer0 <- function(x){
is.integer(x) && length(x)==0L
}

if(NaN%in%f_pi==TRUE || Inf%in%abs(f_pi)){
while(NaN%in%f_pi==TRUE || Inf%in%abs(f_pi)){
id_inf_fpi = which(abs(f_pi)==Inf)
if(is.integer0(id_inf_fpi)!=TRUE){
f_pi[id_inf_fpi] = integer_max
}
id_nan_fpi = which(f_pi==NaN)
if(is.integer0(id_nan_fpi)!=TRUE){
```

```
x_i[id_nan_fpi,] = mapply(runif,n=length(id_nan_fpi),min=lim_inf,
max=lim_sup)
swarm_pi = swarm_xi = x_i
f_pi = apply(X=x_i,MARGIN=1,FUN=func,x=as.vector(data))
}
}
}

minimo_fpi = min(f_pi)
if(minimo_fpi < f_g) g = x_i[which.min(f_pi),]

# Initializing the speeds of the particles.

swarm_vi = mapply(runif,n=S,min=-abs(rep(abs(b_up-b_lo),dimension)),
max=abs(rep(abs(b_up-b_lo),dimension)))

# Here ends the initialization of the algorithm

omega = 0.5
phi_p = 0.5
phi_g = 0.5

m=1
vector_f_g <- vector()

while(is.na(var(vector_f_g)) || m<50 ||
var(vector_f_g[length(vector_f_g):(length(vector_f_g)-10)])>e){
# r_p and r_g are randomized numbers in (0.1).
r_p = runif(n=dimension,min=0,max=1)
r_g = runif(n=dimension,min=0,max=1)

# Updating the vector speed.
swarm_vi = omega*swarm_vi+phi_p*r_p*(swarm_pi-swarm_xi)+
phi_g*r_g*(g-swarm_xi)

# Updating the position of each particle.
swarm_xi = swarm_xi+swarm_vi

myoptim = function(...) tryCatch(optim(...), error = function(e) NA)

f_xi = apply(X=swarm_xi,MARGIN=1,FUN=func,x=as.vector(data))
f_pi = apply(X=swarm_pi,MARGIN=1,FUN=func,x=as.vector(data))
f_g = func(par=g,x=as.vector(data))

if(NaN%in%f_xi==TRUE || NaN%in%f_pi==TRUE{
while(NaN%in%f_xi==TRUE){
id_comb = c(which(is.na(f_xi)==TRUE),which(is.na(f_pi)==TRUE))
if(is.integer0(id_comb)!=TRUE){
new_xi = mapply(runif,n=length(id_comb),min=lim_inf,
max=lim_sup)
swarm_pi[id_comb,]=swarm_xi[id_comb,] = new_xi
if(length(id_comb)>1){
if_xi[id_comb] = apply(X=swarm_xi[id_comb,],MARGIN=1,
FUN=func,x=as.vector(data))
f_pi[id_comb] = apply(X=swarm_pi[id_comb,],MARGIN=1,FUN=func,
x=as.vector(data))
}else{
f_xi[id_comb] = func(par=new_xi,x=as.vector(data))
}
}
}
}

if(Inf%in%abs(f_xi)==TRUE{
f_xi[which(is.infinite(f_xi))]=integer_max
}
if(InfInf%in%abs(f_pi)==TRUE{
f_pi[which(is.infinite(f_pi))]=integer_max
}

# There are values below the lower limit of restrictions?
id_test_inf=
which(apply(swarm_xi<t(matrix(rep(lim_inf,S),dimension,S)),1,sum)>=1)
id_test_sup=
which(apply(swarm_xi>t(matrix(rep(lim_sup,S),dimension,S)),1,sum)>=1)

if(is.integer0(id_test_inf)!=TRUE){
swarm_pi[id_test_inf,] = swarm_xi[id_test_inf,] =
mapply(runif,n=length(id_test_inf),
min=lim_inf,max=lim_sup)
```

```
}

if(is.integer0(id_test_sup)!=TRUE){
swarm_pi[id_test_sup,] = swarm_xi[id_test_sup,] =
mapply(runif,n=length(id_test_sup),
min=lim_inf,max=lim_sup)
}

if(is.integer0(which((f_xi<=f_pi)==TRUE))){
swarm_pi[which((f_xi<=f_pi)),] = swarm_pi[which((f_xi<=f_pi)),]
}

if(f_xi[which.min(f_xi)] <= f_pi[which.min(f_pi)]){
swarm_pi[which.min(f_pi),] = swarm_xi[which.min(f_xi),]
if(f_pi[which.min(f_pi)] < f_g) g = swarm_pi[which.min(f_pi),]
} # Here ends the block if.

vector_f_g[m] = f_g
m = m+1
if(m>N){
break
}

} # Here ends the block while.

f_x = apply(X=swarm_xi,MARGIN=1,FUN=func,x=as.vector(data))
list(par_pso=g,f_pso=vector_f_g)

} # Here ends the function.

# Example of using the PSO function. We are looking to minimize easom
# function in that -10<=x1<=10 and -10<=x2<=10.
easom <- function(par,x){
x1 = par[1]
x2 = par[2]
-cos(x1)*cos(x2)*exp(-((x1-pi)^2 + (x2-pi)^2))
}
set.seed(0)
# Using the PSO function
# S refers to the number of particles considered.
pso(func=easom,S=350,lim_inf=c(-10,-10),lim_sup=c(10,10))
```

Authors' contributions

The authors, CRBD, GMC, MA, PRDM and HFCC with the consultation of each other carried out this work and drafted the manuscript together. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Author details

[1]Department of Statistics, Federal University of Pernambuco, Recife, Brazil. [2]Department of Statistics, Federal Univeristy of Paraíba, João Pessoa, Brazil. [3]Department of Statistics, Persian Gulf University, Bushehr, Iran.

Received: 26 February 2016 Accepted: 8 September 2016
Published online: 05 November 2016

References

Aarset, MV: How to identify a bathtub hazard rate. IEEE Trans. Reliability. **36**, 106–8 (1987)
Adamidis, K, Loukas, S: A lifetime distribution with decreasing failure rate. Stat. Probab. Latt. **39**, 35–42 (1998)
Alzaatreh, A, Lee, C, Famoye, F: A new method for generating families of continuous distributions. METRON. **81**, 63–79 (2013)
Atkinson, AC: An introduction to graphical methods of diagnostic regression analysis. Clarendon Press, Oxford (1985)
Barreto-Souza, W, Morais, AL, Cordeiro, GM: The Weibull-geometric distribution. J. Stat. Comput. Simulation. **81**, 645–57 (2011)
Cordeiro, GM, de Castro, M: A new family of generalized distributions. J. Stat. Comput. Simulat. **81**, 883–98 (2011)
Cordeiro, GM, Ortega, EMM, Cunha, DCC: The exponentiated generalized class of distributions. J. Data Sci. **11**, 1–27 (2013)
Chen, G, Balakrishnan, N: A general purpose approximate goodness-of-fit test. J. Qual. Technol. **27**, 154–61 (1995)
Cox, DR, Hinkley, DV: Theoretical Statistics. Chapman and Hall, London (1974)
Doornik, JA: An Object-Oriented Matrix Language Ox 6. Timberlake Consultants Press, London (2009)
Eberhart, RC, Kennedy, J: A new optimizer using particle swarm theory. In: Proceedings of the sixth international symposium on micro machine and human science, vol. 1, pp. 39–43, New York, (1995)
Eugene, N, Lee, C, Famoye, F: Beta-normal distribution and its applications. Commun. Stat. Theory Methods. **31**, 497–512 (2002)
Gradshteyn, IS, Ryzhik, IM: Table of integrals, series, and products. Academic Press, San Diego (2000)
Gupta, RC, Gupta, RD: Proportional reversed hazard rate model and its applications. J. Stat. Planning Inference. **137**, 3525–36 (2007)

Gupta, RD, Kundu, D: Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions. Biometrical J. **43**, 117–30 (2001)

Jones, MC: Families of distributions arising from distributions of order statistics. Test. **13**, 1–43 (2004)

Kenney, JF, Keeping, ES: Mathematics of statistics, pp. 101–102, Part 1. 3rd ed, Princeton (1962)

Marshall, AW, Olkin, I: A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika. **84**, 641–52 (1997)

Mahmoudi, E, Shiran, M: Exponentiated weibull-geometric distribution and its applications (2012). arXiv:1206.4008vl [stat.ME]

Moors, JJA: A quantile alternative for kurtosis. J. R. Stat. Soc. Ser. D. Stat. **37**, 25–32 (1998)

Mudholkar, GS, Hutson, AD: The exponentiated Weibull family: some properties and a flood data application. Commun. Stat. Theory Methods. **25**, 3059–83 (1996)

Nadarajah, S, Kotz, S: The Exponentiated Type Distributions. Acta Appl. Math. **92**, 97–111 (2006)

Prater, NH: Estimate gasoline yields from crudes. Petroleum Refiner. **35**, 236–8 (1956)

Ramos, MWA, Marinho, PRD, Cordeiro, GM, Silva, RV, Hamedani, G: The kumaraswamy-G Poisson family of distributions. J. Stat. Theory Appl. **14**, 222–39 (2015)

Silva, RB, Barreto-Souza, W, Cordeiro, GM: A new distribution with decreasing, increasing and upside-down bathtub failure rate. Comput. Stat. Data Anal. **54**, 935–44 (2010)

Silverman, BW: Density estimation for statistics and data analysis, Vol. 26. CRC press (1986)

Stephens, MA: Tests based on EDF statistics. Goodness-of-fit Tech. **68**, 97–193 (1986)

Zografos, K, Balakrishnan, N: On families of beta and generalized gamma-generated distributions and associated inference. Stat. Methodol. **6**, 344–362 (2009)