International Journal of Statistics and Probability; Vol. 2, No. 3; 2013
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Exponentiated T-X Family of Distributions with Some Applications

Ahmad Alzaghal[1], Felix Famoye[1] & Carl Lee[1]

[1] Department of Mathematics, Central Michigan University, Mount Pleasant, Michigan, USA

Correspondence: Felix Famoye, Dept. of Mathematics, Central Michigan University, Mt. Pleasant, Michigan 48859, USA. Tel: 1-989-774-5497. E-mail: felix.famoye@cmich.edu

Received: March 24, 2013 Accepted: June 13, 2013 Online Published: July 9, 2013

doi:10.5539/ijsp.v2n3p31 URL: http://dx.doi.org/10.5539/ijsp.v2n3p31

Abstract

In this paper, a new family of distributions called exponentiated T-X distribution is defined. Some of its properties and special cases are discussed. A member of the family, namely, the three-parameter exponentiated Weibull-exponential distribution is defined and studied. Some of its properties including distribution shapes, limit behavior, hazard function, Shannon entropy, moments, skewness and kurtosis are discussed. The flexibility of the exponentiated Weibull-exponential distribution is assessed by applying it to three real data sets and comparing it with other distributions. The exponentiated Weibull-exponential distribution is found to adequately fit left-skewed and right-skewed data sets.

Keywords: hazard function, Shannon entropy, generalized distribution, quantile function, moments

1. Introduction

Statistical distributions are very useful in describing and predicting real world phenomena. Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. This has motivated researchers seeking and developing new and more flexible distributions. As a result, many new distributions have been developed and studied.

Mudholkar and Srivastava (1993) proposed the exponentiated Weibull distribution to analyze bathtub failure data. Gupta et al. (1998) introduced the general class of exponentiated distributions. Given a random variable X with the cumulative distribution function (CDF) $F(x)$, the class of exponentiated distributions is defined as

$$G_\alpha(x) = [F(x)]^\alpha. \tag{1}$$

Gupta et al. (1998) defined the exponentiated exponential distribution by taking $F(x)$ to be the CDF of an exponential distribution. The exponentiated Weibull distribution in Mudholkar and Srivastava (1993) is a member of the class of exponentiated distributions by taking $F(x)$ to be the CDF of a Weibull distribution. Many researchers utilized the class of exponentiated distributions to create new distributions. For example, Nadarajah and Kotz (2003) defined and studied the exponentiated Fréchet distribution, and Nadarajah (2005) defined and studied the exponentiated Gumbel distribution.

Eugene et al. (2002) introduced a new class of distributions generated from the beta distribution. Given a random variable X with the CDF $F(x)$, the class of beta-generated distributions is defined as

$$G_B(x) = \frac{1}{B(\alpha,\beta)} \int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}dt. \tag{2}$$

The corresponding probability density function (PDF) of the beta-generated distribution in (2) is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)F^{\alpha-1}(x)(1-F(x))^{\beta-1}.$$

Eugene et al. (2002) developed and studied the beta-normal distribution by taking $F(x)$ to be the CDF of a normal distribution. Many new distributions utilizing this technique have been defined and studied. Some examples include the beta Gumbel distribution by Nadarajah and Kotz (2004), the beta Fréchet distribution by Nadarajah and Gupta (2004), the beta-Weibull distribution by Famoye et al. (2005), the beta exponential distribution by Nadarajah and Kotz (2006), the beta-gamma distribution by Kong et al. (2007), the beta-Pareto distribution by Akinsete et al. (2008), the beta generalized exponential distribution by Barreto-Souza et al. (2010), the beta generalized Pareto distribution by Mahmoudi (2011), and the beta-Cauchy distribution by Alshawarbeh et al. (2012). For a review of beta-generated distributions and other generalizations, one may refer to Lee et al. (2013).

An extension of the beta-generated method was proposed in Jones (2009) and Cordeiro and de Castro (2011) by using the Kumaraswamy distribution (Kumaraswamy, 1980), as a generator instead of beta distribution. The PDF of the Kumaraswamy generalized distributions (KW-G) is given by

$$g(x) = \alpha\beta f(x)F^{\alpha-1}(x)(1 - F^{\alpha}(x))^{\beta-1}. \tag{3}$$

Recently, Alzaatreh, Lee and Famoye (2013) extended the beta-generated family of distributions by using any non-negative continuous random variable T as the generator, in place of the beta random variable. The new class of distributions is defined as

$$G(x) = \int_0^{-\log(1-F(x))} r(t)\,dt, \tag{4}$$

where $r(t)$ is the PDF of a non-negative continuous random variable T. The corresponding PDF to the CDF in (4) is given by

$$g(x) = \frac{f(x)}{1-F(x)} r\{-\log(1-F(x))\}. \tag{5}$$

In this new class, the distribution of the random variable T is the generator. The new family of distributions generated from (5) is called "T-X distribution". Alzaatreh, Famoye and Lee (2013) defined the Weibull-Pareto distribution from (5) by taking $r(t)$ to be the Weibull distribution and $F(x)$ to be the Pareto distribution.

Note that the upper limit for generating the T-X distribution is $-\log(1 - F(x))$. It is clear that one can define a different upper limit for generating different types of T-X distributions. In this article, we define the upper limit to be $-\log(1 - F^c(x))$, which leads to a new family of exponentiated T-X distributions. By including the additional parameter c, the exponentiated T-X family provides more flexible distributions for fitting real data.

The rest of this article is organized as follows. In section 2, we define the exponentiated T-X family and provide some of its properties. Some members of exponentiated T-X distributions are discussed in section 3. In section 4, we define the exponentiated Weibull-exponential distribution, provide some special cases of the distribution, and discuss some properties of the distribution, including distribution shapes, limit behavior, hazard function, quantile function, and Shannon entropy. Section 4 also contains expression for the moment generating function and the results of investigating the skewness of the exponentiated Weibull-exponential distribution. The applications of the exponentiated Weibull-exponential distribution to three real data sets are presented in section 5.

2. The Exponentiated T-X Family

Let $r(t)$ be the PDF of a non-negative continuous random variable T defined on $[0,\infty)$, and let $F(x)$ denote the CDF of a random variable X. We define the CDF for the exponentiated T-X class of distributions for a random variable X as

$$G(x) = \int_0^{-\log(1-F^c(x))} r(t)\,dt = R\{-\log(1-F^c(x))\}, \tag{6}$$

where $R(t)$ is the CDF of the random variable T. The corresponding PDF of the generalized distribution in (6) is given by

$$g(x) = \frac{c\,f(x)\,F^{c-1}(x)}{1-F^c(x)} r\{-\log(1-F^c(x))\},\ c>0. \tag{7}$$

By using a similar naming convention as "T-X distribution", we call each member of the new family of distributions generated from (7) as "exponentiated T-X distribution".

Some remarks on the exponentiated T-X distribution:

(a) The CDF and the PDF of exponentiated T-X distribution given in Equations (6) and (7), can be expressed as $G(x) = R\left(-\log\left(1-F^c(x)\right)\right) = R\left(H(x)\right)$ and $g(x) = h(x)\,r\left(H(x)\right)$, where $h(x)$ and $H(x)$ are the hazard and cumulative hazard functions of the random variable X with CDF $F^c(x)$. Hence, the exponentiated T-X distribution can be considered as a family of distributions arising from the hazard functions.

(b) The relationship between the random variable X that follows the family of distribution in (7) and the random variable T that follows the PDF $r(t)$ is given by $X = F^{-1}\{(1-e^{-T})^{1/c}\}$, which provides an easy way to simulate the random variable X by first simulating the random variable T and then computing $X = F^{-1}\{(1-e^{-T})^{1/c}\}$, which has the CDF $G(x)$. Therefore, $E(X)$ can be obtained by using

$$E(X) = E\left(F^{-1}\left\{(1-e^{-T})^{1/c}\right\}\right).$$

(c) We can generate new families of discrete distributions by taking the random variable X to be discrete. The probability mass function of the exponentiated T-X family of discrete distributions can be written as

$$g(x) = G(x) - G(x-1) = R\{-\log(1-F^c(x))\} - R\{-\log(1-F^c(x-1))\}.$$

Alzaatreh, Lee and Famoye (2012) defined and studied the family of discrete analogues of continuous distributions namely T-geometric family using (4) by taking X to be a geometric random variable. This article will discuss the case when X is a continuous random variable.

(d) When $c = 1$, an exponentiated T-X distribution reduces to T-X distribution. If in addition X follows the exponential distribution, an exponentiated T-X distribution reduces to T distribution.

The hazard function of exponentiated T-X family is given by

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{c\, f(x)\, F^{c-1}(x)\, r\{-\log(1-F^c(x))\}}{(1-F^c(x))\,(1-R\{-\log(1-F^c(x))\})}. \tag{8}$$

The quantile function for exponentiated T-X distribution, $Q(\lambda)$, $0 < \lambda < 1$, is obtained by solving $G(Q(\lambda)) = \lambda$, which is given by

$$Q(\lambda) = F^{-1}\left(1-e^{-R^{-1}(\lambda)}\right)^{1/c}. \tag{9}$$

The quantile function depends on the exponentiated parameter c. Since $0 < \left(1-e^{-R^{-1}(\lambda)}\right) < 1$ and F is monotonically non-decreasing, then $Q(\lambda|c<1) \le Q(\lambda|c=1)$ and $Q(\lambda|c>1) \ge Q(\lambda|c=1)$.

The entropy of a random variable X is a measure of variation of uncertainty. Entropy has several applications in physics, chemistry, engineering, and economics. The Shannon entropy of a continuous random variable with PDF $g(x)$ is defined as $E[-\log g(X)]$ (Shannon, 1948). The relationship between the Shannon entropy for a random variable X that has the PDF $g(x)$ and the Shannon entropy of a random variable T with PDF $r(t)$ is given by the following theorem.

Theorem 1 *If T has a PDF $r(t)$ and X follows the exponentiated T-X distribution in (7), then the Shannon entropy of X, η_x, is given by*

$$\eta_x = -\log c - E\left(\log f\left(F^{-1}\left\{1-e^{-T}\right\}^{1/c}\right)\right) + \frac{1-c}{c}E\left(\log\left\{1-e^{-T}\right\}\right) - \mu_T + \eta_T, \tag{10}$$

where μ_T and η_T are the mean and the Shannon entropy for the random variable T.

Proof. See the Appendix. □

3. Some Members of Exponentiated T-X Family with Different T-Distributions

There are two sub-families in the exponentiated T-X family. In the first sub-family, the X distribution is the same but the T distributions are different. In the other sub-family, the T distribution is the same but the X distributions are different. Table 1 lists the exponentiated T-X families for different T distributions.

Table 1. Some members of exponentiated T-X distributions for different T distributions

Name	The density $r(t)$	The density $g(x)$ of the T-X random variable
Exponential	$\beta e^{-\beta t}$	$c\beta f(x)F^{c-1}(x)(1-F^c(x))^{\beta-1}$
Beta exponential	$\frac{\lambda e^{-\lambda\beta x}(1-e^{-\lambda x})^{\alpha-1}}{B(\alpha,\beta)}$	$\frac{c\lambda}{B(\alpha,\beta)} f(x)F^{c-1}(x)(1-F^c(x))^{\lambda\beta-1}[1-(1-F^c(x))^{\lambda}]^{\alpha-1}$
Exponentiated exponential	$\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}$	$c\alpha\lambda f(x)F^{c-1}(x)(1-(1-F^c(x))^{\lambda})^{\alpha-1}(1-F^c(x))^{\lambda-1}$
Gamma	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1}e^{-t/\beta}$	$\frac{c}{\Gamma(\alpha)\beta^{\alpha}} f(x)F^{c-1}(x)(1-F^c(x))^{\frac{1}{\beta}-1}(-\log(1-F^c(x))^{\alpha-1}$
Half normal	$\frac{1}{\sigma}\left(\frac{2}{\pi}\right)^{1/2} e^{-t^2/2\sigma^2}$	$\frac{c}{\sigma}\left(\frac{2}{\pi}\right)^{1/2} \frac{f(x)F^{c-1}(x)}{1-F^c(x)} \exp\left(-\{\log(1-F^c(x)\}^2/2\sigma^2\right)$
Levy	$\left(\frac{\gamma}{2\pi}\right)^{1/2}\frac{e^{-\gamma/2t}}{t^{3/2}}$	$c\left(\frac{\gamma}{2\pi}\right)^{1/2}\frac{f(x)F^{c-1}(x)}{1-F^c(x)}\frac{\exp\left(-\gamma/2(-\log(1-F^c(x))\right)}{\left(-\log(1-F^c(x))\right)^{3/2}}$
Log logistic	$\frac{\beta(t/\alpha)^{\beta-1}}{\alpha\left(1+(t/\alpha)^{\beta}\right)^2}$	$\frac{c\beta}{\alpha^{\beta}}\frac{f(x)F^{c-1}(x)}{1-F^c(x)}\frac{\{-\log(1-F^c(x))\}^{\beta-1}}{\left(1+\{-\log(1-F^c(x))/\alpha\}^{\beta}\right)^2}$
Rayleigh	$\frac{t}{\sigma^2}e^{-t^2/2\sigma^2}$	$\frac{-c}{\sigma^2}\frac{f(x)F^{c-1}(x)}{1-F^c(x)}\log(1-F^c(x))\exp\left\{-(\log(1-F^c(x)))^2/2\sigma^2\right\}$
Type-2 Gumbel	$\alpha\beta\frac{e^{-\beta t^{-\alpha}}}{t^{\alpha+1}}$	$c\alpha\beta\frac{f(x)F^{c-1}(x)}{1-F^c(x)}\frac{\exp\left(-\beta\{-\log(1-F^c(x))\}^{-\alpha}\right)}{\{-\log(1-F^c(x))\}^{\alpha+1}}$
Lomax	$\frac{\lambda k}{(1+\lambda t)^{k+1}}$	$ck\lambda\frac{f(x)F^{c-1}(x)}{1-F^c(x)}(1-\lambda\log(1-F^c(x)))^{-k-1}$
Inverted beta	$\frac{t^{\beta-1}(1+t)^{-\beta-\gamma}}{B(\beta,\gamma)}$	$\frac{c}{B(\beta,\gamma)}\frac{f(x)F^{c-1}(x)}{1-F^c(x)}\frac{\left(-\log(1-F^c(x))\right)^{\beta-1}}{\{1-\log(1-F^c(x))\}^{\beta+\gamma}}$
Burr	$\frac{\alpha k\, t^{\alpha-1}}{(1+t^{\alpha})^{k+1}}$	$c\alpha k\,\frac{f(x)F^{c-1}(x)}{1-F^c(x)}\frac{\left(-\log(1-F^c(x))\right)^{\alpha-1}}{\left(1+\{-\log(1-F^c(x))\}^{\alpha}\right)^{k+1}}$
Weibull	$\frac{\alpha}{\gamma}\left(\frac{t}{\gamma}\right)^{\alpha-1}e^{-(t/\gamma)^{\alpha}}$	$\frac{c\alpha}{\gamma}\frac{f(x)F^{c-1}(x)\{-\log(1-F^c(x))^{1/\gamma}\}^{\alpha-1}\exp\left(-\{-\log(1-F^c(x))^{1/\gamma}\}^{\alpha}\right)}{1-F^c(x)}$

In the rest of this section, we will discuss some properties of the exponentiated T-X family for different T distributions.

3.1 Exponentiated Gamma-X Family

If the random variable T follows the gamma distribution with parameters α and β, then $r(t) = (\Gamma(\alpha)\beta^{\alpha})^{-1}t^{\alpha-1}e^{-t/\beta}$, $t > 0$. The PDF of exponentiated gamma-X family using (7) is defined as

$$g(x) = \frac{c}{\Gamma(\alpha)\beta^{\alpha}} f(x)F^{c-1}(x)(-\log(1-F^c(x)))^{\alpha-1}(1-F^c(x))^{\frac{1}{\beta}-1}. \tag{11}$$

Using the incomplete gamma function $\delta(\alpha, t) = \int_0^t u^{\alpha-1}e^{-u}du$, the CDF of T is $R(t) = \delta(\alpha,\ t/\beta)/\Gamma(\alpha)$. Hence, the CDF of exponentiated gamma-X family from (11) is given by

$$G(x) = \delta\{\alpha, -\log(1-F^c(x))/\beta\}/\Gamma(\alpha).$$

The Shannon entropy of the exponentiated gamma-X family of distributions is given by

$$\eta_x = -E\left(\log f\left(F^{-1}\left\{1-e^{-T}\right\}^{1/c}\right)\right) + \frac{1-c}{c}E\left(\log\left\{1-e^{-T}\right\}\right) + \alpha(1-\beta) + \log\left(\frac{\beta\,\Gamma(\alpha)}{c}\right) + (1-\alpha)\psi(\alpha),$$

where $\psi(\cdot)$ is the digamma function. The result follows from Theorem 1 by using the mean $\mu_T = \alpha\beta$ and the Shannon entropy $\eta_T = \alpha + \log\beta\,\Gamma(\alpha) + (1-\alpha)\psi(\alpha)$ for the gamma distribution, which is given by Song (2001).

Some special cases of exponentiated gamma-X family:

(1) When $\alpha = 1$, the exponentiated gamma-X family reduces to

$$g(x) = (c/\beta)f(x)F^{c-1}(x)(1-F^c(x))^{\frac{1}{\beta}-1}. \tag{12}$$

By using $\gamma = 1/\beta$ in (12), the exponentiated gamma-X family reduces to the *KW-G* family in (3).

(2) When $\beta = 1$, the exponentiated gamma-X family reduces to

$$g(x) = \frac{c}{\Gamma(\alpha)} f(x)F^{c-1}(x)(-\log(1 - F^c(x)))^{\alpha-1},$$

which is named exponentiated standard gamma-X family.

(3) When $\beta = \alpha = 1$, the exponentiated gamma-X family reduces to the exponentiated family in (1). Hence, all distributions that belong to the exponentiated family in (1) can be generated by using the family of distributions in (11).

3.2 Exponentiated Weibull-X Family

If the random variable T follows the Weibull distribution with parameters α and γ, then $r(t) = (\alpha/\gamma)(t/\gamma)^{\alpha-1}e^{-(t/\gamma)^{\alpha}}$, $t > 0$. The PDF of exponentiated Weibull-X family using (7) is defined as

$$g(x) = \frac{c\alpha}{\gamma}\frac{f(x)F^{c-1}(x)}{1 - F^c(x)}\{-\log(1 - F^c(x))/\gamma\}^{\alpha-1}\exp\{-(-\log(1 - F^c(x))/\gamma)^{\alpha}\}.$$

By using the CDF of Weibull distribution and (6), the CDF of exponentiated Weibull-X family is given by

$$G(x) = 1 - \exp\{-(-\log(1 - F^c(x))/\gamma)^{\alpha}\}. \quad (13)$$

The Shannon entropy of the exponentiated Weibull-X family of distributions is given by

$$\eta_x = -E\left(\log f\left(F^{-1}\left\{1 - e^{-T}\right\}^{1/c}\right)\right) + \frac{1-c}{c}E\left(\log\left\{1 - e^{-T}\right\}\right) - \gamma\Gamma\left(1 + \frac{1}{\alpha}\right) + \upsilon\left(1 - \frac{1}{\alpha}\right) - \log\left(\frac{c\alpha}{\gamma}\right) + 1,$$

where υ is the Euler's constant. The result follows from Theorem 1 by using the mean $\mu_T = \gamma\Gamma(1 + 1/\alpha)$ and the Shannon entropy $\eta_T = \upsilon(1 - 1/\alpha) - \log(\alpha/\gamma) + 1$ for the Weibull distribution (Song, 2001).

Some special cases of exponentiated Weibull-X family:

(1) When $\alpha = 1$, the exponentiated Weibull-X family reduces to

$$g(x) = \frac{c}{\gamma} f(x)F^{c-1}(x)(1 - F^c(x))^{\frac{1}{\gamma}-1}. \quad (14)$$

By using $\lambda = 1/\gamma$ in (14), the exponentiated Weibull-X family reduces to the KW-G family in (3).

(2) When $\alpha = \gamma = 1$, the exponentiated Weibull-X family reduces to the exponentiated family in (1).

In the remaining sections, we will study the properties of a new distribution named the exponentiated Weibull-exponential distribution by taking the random variable X to be the standard exponential distribution in the exponentiated Weibull-X family.

4. The Exponentiated Weibull-Exponential Distribution

The CDF of the exponentiated Weibull-exponential distribution (EWED) when X follows the standard exponential distribution in Equation (13) is given by

$$G(x) = 1 - \exp\left[-\left(-\log\left\{1 - (1 - e^{-x})^c\right\}/\gamma\right)^{\alpha}\right], \ x \geq 0, \ \alpha, \gamma \text{ and } c > 0, \quad (15)$$

and the corresponding PDF of the exponentiated Weibull-exponential distribution is given by

$$g(x) = \frac{c\alpha}{\gamma}\frac{e^{-x}(1 - e^{-x})^{c-1}}{1 - (1 - e^{-x})^c}\left(-\log\left\{1 - (1 - e^{-x})^c\right\}/\gamma\right)^{\alpha-1}\exp\left[-\left(-\log\left\{1 - (1 - e^{-x})^c\right\}/\gamma\right)^{\alpha}\right]. \quad (16)$$

Some special cases of the EWED:

(1) When $c = 1$, the EWED reduces to the Weibull distribution with parameters α and γ.

(2) When $c = \alpha = 1$, the EWED reduces to the exponential distribution with parameter γ.

(3) When $\alpha = 1$, the EWED reduces to KW-standard exponential distribution defined by Cordeiro and de Castro (2011).

(4) When $\alpha = \gamma = 1$, the EWED reduces to the standard exponentiated exponential distribution.

(5) When $c = 1$ and $\alpha = -k$, the EWED reduces to type 2 extreme value distribution defined by Johnson et al. (1995).

4.1 Some Properties of EWED

Transformation: The relationship between the exponentiated Weibull-exponential distribution and the uniform, Weibull and exponential distributions is given by the following theorem.

Theorem 2

(a) If a random variable Y follows the uniform distribution, then the random variable

$$X = -\log\left\{1 - \left(1 - e^{-\gamma\left(-\log(Y)\right)^{1/\alpha}}\right)^{1/c}\right\} \; follows\ EWED.$$

(b) If a random variable Y follows the Weibull distribution with parameters α and γ, then the random variable

$$X = -\log\left\{1 - \left(1 - e^{-Y}\right)^{1/c}\right\} \; follows\ EWED.$$

(c) If a random variable Y follows the standard exponential distribution, then the random variable

$$X = -\log\left\{1 - \left(1 - e^{-\gamma Y^{1/\alpha}}\right)^{1/c}\right\} \; follows\ EWED.$$

Proof. Using transformation technique, it is easy to show that the random variable X has exponentiated Weibull-exponential density function as given in Equation (16). □

Limit behavior: The following lemma is on the limit behavior of the PDF in Equation (16).

Lemma 1 *The limit of exponentiated Weibull-exponential density as $x \to \infty$ is 0, and the limit as $x \to 0$ is given by*

$$\lim_{n\to\infty} g(x) = \begin{cases} 0, & c\alpha > 1 \\ \gamma^{-\alpha}, & c\alpha = 1 \\ \infty, & c\alpha < 1. \end{cases}$$

Proof. When $x \to \infty$, the exponentiated Weibull-exponential density in (16) goes to 0. The limit of $g(x)$ as $x \to 0$ may be written as

$$\lim_{x\to 0} g(x) = \frac{c\alpha}{\gamma^{\alpha}} \lim_{x\to 0} \left(-\log\left\{1 - (1 - e^{-x})^{c}\right\}\right)^{\alpha-1} (1 - e^{-x})^{c-1}. \tag{17}$$

By using the series representation of the logarithm function when $|z| \leq 1$

$$\log(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{n}, \tag{18}$$

the limit in Equation (17) can be expressed as

$$\begin{aligned} \lim_{x\to 0} g(x) &= \frac{c\alpha}{\gamma^{\alpha}} \lim_{x\to 0}\left[(1 - e^{-x})^{c} + \frac{(1 - e^{-x})^{2c}}{2} + \frac{(1 - e^{-x})^{3c}}{3} + \frac{(1 - e^{-x})^{4c}}{4} + \cdots\right]^{\alpha-1} (1 - e^{-x})^{c-1} \\ &= \frac{c\alpha}{\gamma^{\alpha}} \lim_{x\to 0}\left[1 + \frac{(1 - e^{-x})^{c}}{2} + \frac{(1 - e^{-x})^{2c}}{3} + \frac{(1 - e^{-x})^{4c}}{4} + \cdots\right]^{\alpha-1} (1 - e^{-x})^{c\alpha-1}. \end{aligned} \tag{19}$$

The limit of the square brackets in Equation (19) as x goes to 0 is 1. Hence, we have

$$\lim_{x\to 0} g(x) = \frac{c\alpha}{\gamma^{\alpha}} \lim_{x\to 0} (1 - e^{-x})^{c\alpha-1}. \tag{20}$$

Therefore, as $x \to 0$ Equation (20) goes to zero when $\alpha c > 1$. It goes to ∞ when $\alpha c < 1$ and when $c\alpha = 1$ the limit reduces to the constant $\gamma^{-\alpha}$. This completes the proof. □

The graphs of exponentiated Weibull-exponential distribution for various values of α, γ and c are given in Figure 1. The figure shows that the density function can take different shapes such as left-skewed, right-skewed, symmetric or reversed J-shape.

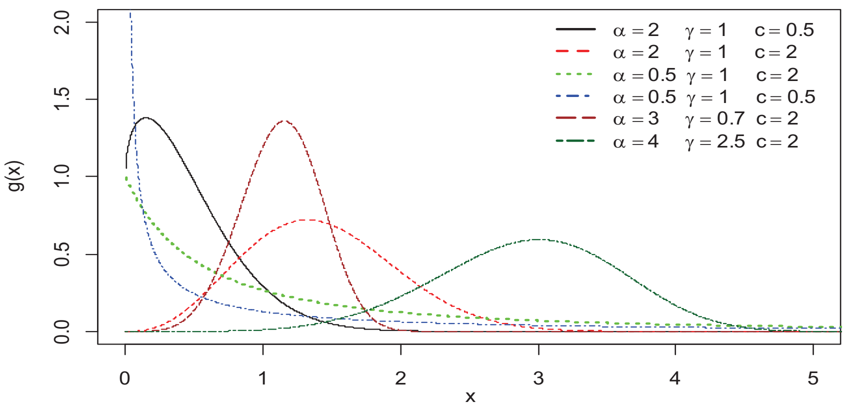

Figure 1. Density functions for various values of α, γ, and c

Hazard function: The hazard function of EWED using Equations (15), (16) and (8) is given by

$$h(x) = \frac{c\alpha}{\gamma} \frac{e^{-x}(1-e^{-x})^{c-1}}{1-(1-e^{-x})^c} \left(-\log\left\{1-(1-e^{-x})^c\right\}/\gamma\right)^{\alpha-1}. \tag{21}$$

By setting $c = 1$, the hazard function in (21) reduces to the hazard function of the Weibull distribution. The following lemma addresses the limit behaviors of the hazard function in (21).

Lemma 2 *The limit of exponentiated Weibull-exponential hazard function as $x \to 0$ is given by*

$$\lim_{x\to 0} h(x) = \begin{cases} 0, & c\alpha > 1 \\ \gamma^{-\alpha}, & c\alpha = 1 \\ \infty, & c\alpha < 1, \end{cases}$$

and the limit as $x \to \infty$ is given by

$$\lim_{x\to \infty} h(x) = \begin{cases} 0, & \alpha < 1 \\ \gamma^{-1}, & \alpha = 1 \\ \infty, & \alpha > 1. \end{cases}$$

Proof. The proof of the limit as $x \to 0$ follows from Lemma 1 and it is straight forward to show the result as $x \to \infty$ by taking the limit of exponentiated Weibull-exponential hazard function. □

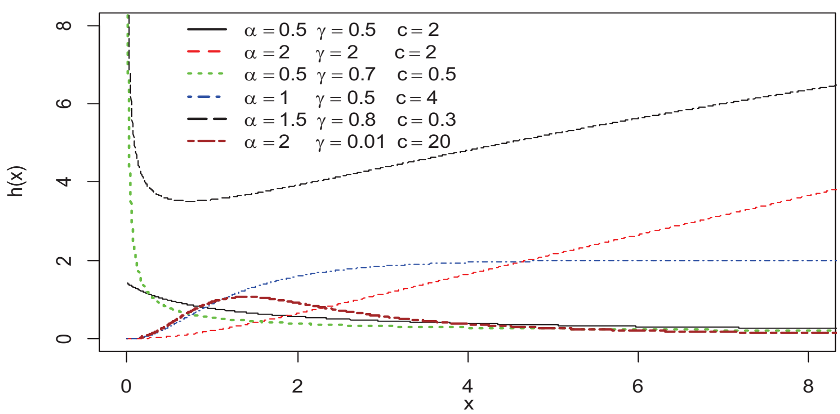

Figure 2. Hazard functions for various values of α, γ, and c

Figure 2 displays the graph of the hazard function of the EWED for various values of α, γ and c. When $\alpha = c = 1$, the EWED has a constant failure rate ($= \gamma^{-1}$). For $c\alpha < 1$ and $\alpha < 1$, the EWED has a decreasing failure rate and when $c\alpha < 1$ and $\alpha > 1$, the EWED has a bathtub failure rate. When $c\alpha > 1$ and $\alpha > 1$, the EWED has an increasing failure rate. When $c\alpha > 1$ and $\alpha < 1$, the EWED has an upside down bathtub (or unimodal) failure rate. The exponent c gives the hazard function of the EWED more shapes than the hazard function of the Weibull distribution.

Quantile function: The quantile function for EWED is given by

$$Q(\lambda) = -\log\left\{1 - \left(1 - \exp\left\{-\gamma(-\log(1-\lambda))^{1/\alpha}\right\}\right)^{1/c}\right\}. \tag{22}$$

The result follows by using (9) with $R(t)$ and $F(x)$ being the CDF of Weibull distribution and standard exponential distribution, respectively.

Shannon entropy: The Shannon entropy for the exponentiated Weibull-exponential variable X with density $g(x)$ is given by the following lemma.

Lemma 3 *If a random variable X follows EWED, then the Shannon entropy of X, η_x, is given by*

$$\eta_x = \mu_x + \frac{(1-c)\alpha}{c}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\gamma^{\alpha(k+1)}k!}\left(\sum_{n=1}^{\infty}\frac{\Gamma(\alpha(k+1))}{n^{\alpha(k+1)+1}}\right) + \gamma\left(\Gamma\left(1+\frac{1}{\alpha}\right) + 1 - \frac{1}{\alpha}\right) + \log\left(\frac{\gamma}{c\alpha}\right) + 1,$$

where μ_x is the mean of EWED.

Proof. By using Theorem 1 and the fact that T follows a Weibull distribution with parameters α and γ, and $f(x) = e^{-x}$, Equation (10) can be written as

$$\eta_x = \mu_x + \frac{1-c}{c}E\left(\log\left\{1-e^{-T}\right\}\right) + \gamma\left(\Gamma\left(1+\frac{1}{\alpha}\right) + 1 - \frac{1}{\alpha}\right) + \log\left(\frac{\gamma}{c\alpha}\right) + 1. \tag{23}$$

So, to complete the proof we need to evaluate

$$E\left(\log\{1-e^{-T}\}\right) = (\alpha/\gamma)\int_0^{\infty}\log(1-e^{-t})\,(t/\gamma)^{\alpha-1}e^{-(t/\gamma)^{\alpha}}dt. \tag{24}$$

By using the series representation for the exponential function

$$e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}, \tag{25}$$

the integral in (24) can be simplified to

$$E\left(\log\left\{1-e^{-T}\right\}\right) = \alpha\sum_{k=0}^{\infty}\frac{(-1)^k}{\gamma^{\alpha(k+1)}k!}\int_0^{\infty}\log(1-e^{-t})t^{\alpha(k+1)-1}\,dt. \tag{26}$$

By using the series representation of the logarithm function in Equation (18), Equation (26) can be expressed as

$$E\left(\log\left\{1-e^{-T}\right\}\right) = \alpha\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\gamma^{\alpha(k+1)}k!}\left(\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}t^{\alpha(k+1)-1}e^{-nt}dt\right) = \alpha\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\gamma^{\alpha(k+1)}k!}\left(\sum_{n=1}^{\infty}\frac{\Gamma(\alpha(k+1))}{n^{\alpha(k+1)+1}}\right).$$

Therefore, Equation (23) reduces to

$$\eta_x = \mu_x + \frac{(1-c)\alpha}{c}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{\gamma^{\alpha(k+1)}k!}\left(\sum_{n=1}^{\infty}\frac{\Gamma(\alpha(k+1))}{n^{\alpha(k+1)+1}}\right) + \gamma\left(\Gamma\left(1+\frac{1}{\alpha}\right) + 1 - \frac{1}{\alpha}\right) + \log\left(\frac{\gamma}{c\alpha}\right) + 1,$$

which completes the proof. □

4.2 Moment Generating Function

The moment generating function of EWED is given by

$$M(t) = E(e^{tX}) = \int_0^{\infty}e^{tx}g(x)\,dx = \frac{c\alpha}{\gamma}\int_0^{\infty}e^{tx}\frac{e^{-x}(1-e^{-x})^{c-1}}{1-(1-e^{-x})^c}u_0^{\alpha-1}\exp(-u_0^{\alpha})dx, \tag{27}$$

where $u_0 = -\log\{1-(1-e^{-x})^c\}/\gamma$. On using the substitution $u = u_0^{\alpha}$, the integral in (27) can be simplified as

$$M(t) = \int_0^{\infty} e^{tx} g(x) dx = \int_0^{\infty} e^{-u}\left(1-\left(1-e^{-\gamma u^{1/\alpha}}\right)^{1/c}\right)^{-t} du. \tag{28}$$

By using the series expansion

$$\left(1-\left(1-e^{-\gamma u^{1/\alpha}}\right)^{1/c}\right)^{-t} = \sum_{k=0}^{\infty}\binom{t+k-1}{k}\left(1-e^{-\gamma u^{1/\alpha}}\right)^{k/c},$$

Equation (28) reduces to

$$M(t) = 1 + \sum_{k=1}^{\infty}\frac{(t)_k}{k!}\int_0^{\infty} e^{-u}\left(1-e^{-\gamma u^{1/\alpha}}\right)^{k/c} du, \tag{29}$$

where $(t)_k = t(t+1)\cdots(t+k-1)$ is the ascending factorial. By using the series expansion

$$\left(1-e^{-\gamma u^{1/\alpha}}\right)^{k/c} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j e^{-j\gamma u^{1/\alpha}},$$

Equation (29) reduces to

$$M(t) = 1 + \sum_{k=1}^{\infty}\frac{(t)_k}{k!}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j \int_0^{\infty} e^{-u}\, e^{-j\gamma u^{1/\alpha}} du\right). \tag{30}$$

By using the series representation for the exponential function given in Equation (25), the integral in (30) can be simplified to

$$M(t) = 1 + \sum_{k=1}^{\infty}\frac{(t)_k}{k!}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j\left\{\sum_{n=0}^{\infty}\frac{(-1)^n j^n}{n!}\gamma^n\int_0^{\infty} e^{-u}\, u^{n/\alpha} du\right\}\right).$$

Thus, the moment generating function of EWED is given by

$$M(t) = 1 + \sum_{k=1}^{\infty}\frac{(t)_k}{k!}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j\left\{\sum_{n=0}^{\infty}\frac{(-1)^n j^n}{n!}\gamma^n\Gamma(1+n/\alpha)\right\}\right).$$

Therefore, by taking the n^{th} derivative of the moment generating function and evaluating it at $t = 0$, the n^{th} moment of the EWED can be obtained as

$$E(X^n) = \sum_{k=1}^{\infty}\frac{d^n(t)_k}{dt^n}\frac{1}{k!}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j\left\{\sum_{n=0}^{\infty}\frac{(-1)^n j^n}{n!}\gamma^n\Gamma(1+n/\alpha)\right\}\right),$$

where the n^{th} derivative of the ascending factorial is defined recursively as

$$\left.\frac{d^n(t)_k}{dt^n}\right|_{t=0} = \sum_{r=0}^{n-1}\binom{n-1}{r}\left[\psi^{(n-1-r)}(t+k)-\psi^{(n-1-r)}(t)\right]\frac{d^r(t)_k}{dt^r}.$$

Therefore, the mean and the variance of EWED are respectively given by

$$\mu = E(X) = \sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j\left\{\sum_{n=0}^{\infty}\frac{(-1)^n j^n}{n!}\gamma^n\,\Gamma(1+n/\alpha)\right\}\right),$$

and

$$\sigma^2 = E(X^2)-\mu^2 = \sum_{k=1}^{\infty}\frac{2\left(-\psi(1)+\psi(k)\right)}{k}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}(k/c)_j\left\{\sum_{n=0}^{\infty}\frac{(-1)^n j^n}{n!}\gamma^n\,\Gamma(1+n/\alpha)\right\}\right)-\mu^2.$$

The mean and the variance of EWED are reported in Table 2 for some values of α, γ and c. For fixed values of c and α, the mean and the variance of EWED increase as γ increases. When the values of α and γ are fixed, the mean and the variance of EWED increase as c increases. For fixed values of γ and c, the variance of EWED decreases as α increases while the mean of EWED first decreases and then increases as α increases.

Table 2. Mean and variance of EWED for some values of α, γ and c

		$\gamma = 0.5$		$\gamma = 1.0$		$\gamma = 2.0$	
c	α	Mean	Variance	Mean	Variance	Mean	Variance
0.5	0.9	0.2550	0.1848	0.6728	1.0244	1.6233	4.8705
	1	0.2274	0.1218	0.6137	0.7101	1.5071	3.4837
	2	0.1552	0.0165	0.4586	0.1199	1.2148	0.6867
	4	0.1481	0.0044	0.4497	0.0345	1.2189	0.2068
	6	0.1510	0.0022	0.4601	0.0175	1.2503	0.1049
0.7	0.9	0.3691	0.2559	0.8420	1.1944	1.8458	5.1864
	1	0.3414	0.1787	0.7858	0.8517	1.7353	3.7498
	2	0.2731	0.0321	0.6491	0.1660	1.4758	0.7774
	4	0.2732	0.0094	0.6545	0.0494	1.4993	0.2348
	6	0.2797	0.0047	0.6706	0.0249	1.5369	0.1182
0.9	0.9	0.4757	0.3163	0.9868	1.3202	2.0256	5.4010
	1	0.4490	0.2281	0.9334	0.9570	1.9195	3.9300
	2	0.3882	0.0469	0.8127	0.2006	1.6830	0.8362
	4	0.3952	0.0141	0.8287	0.0603	1.7187	0.2522
	6	0.4046	0.0070	0.8486	0.0302	1.7600	0.1263
1	0.9	0.5261	0.3429	1.0522	1.3715	2.1044	5.4860
	1	0.5000	0.2500	1.0000	1.0000	2.0000	4.0000
	2	0.4431	0.0537	0.8862	0.2146	1.7724	0.8584
	4	0.4532	0.0162	0.9064	0.0647	1.8128	0.2587
	6	0.4639	0.0081	0.9277	0.0323	1.8554	0.1293
2	0.9	0.9372	0.5183	1.5439	1.6703	2.6650	5.9351
	1	0.9167	0.3958	1.5000	1.2500	2.5708	4.3741
	2	0.8899	0.0990	1.4279	0.2937	2.3915	0.9710
	4	0.9190	0.0298	1.4705	0.0882	2.4543	0.2899
	6	0.9365	0.0146	1.4992	0.0434	2.5032	0.1435
4	0.9	1.4640	0.6610	2.1209	1.8794	3.2829	6.2201
	1	1.4488	0.5146	2.0833	1.4236	3.1955	4.6079
	2	1.4473	0.1347	2.0401	0.3455	3.0458	1.0360
	4	1.4895	0.0401	2.0964	0.1028	3.1211	0.3070
	6	1.5114	0.0194	2.1292	0.0502	3.1734	0.1511
6	0.9	1.8097	0.7212	2.4850	1.9616	3.6618	6.3267
	1	1.7968	0.5644	2.4500	1.4914	3.5769	4.6948
	2	1.8055	0.1492	2.4177	0.3650	3.4381	1.0592
	4	1.8527	0.0441	2.4790	0.1081	3.5177	0.3130
	6	1.8762	0.0213	2.5133	0.0527	3.5711	0.1537

4.3 Skewness and Kurtosis Based on Moments

The skewness of Weibull distribution is approximately equal to zero, when the shape parameter α is approximately equal to 3.60. Also, the skewness of the Weibull distribution is a decreasing function of the shape parameter (e.g., see Johnson et al., 1994). In this sub-section, we will investigate the skewness of the exponentiated Weibull-exponential distribution. The skewness of the EWED is computed by using

$$Sk = \frac{\mu_3}{\sigma^3} = E\left[(X-\mu)^3\right]\left(E\left[(X-\mu)^2\right]\right)^{-3/2}. \tag{31}$$

The skewness in (31) depends on the three parameters α, γ and c. In order to examine when the skewness is equal to zero, the expression in (31) is set to zero and numerical solutions are obtained for α and c for fixed values of γ.

Note that equating (31) to zero is equivalent to equating μ_3 to zero. In this analysis, we first select a fixed value of γ and consider values of c from 1.0 to 5.0 at an increment of 0.01. For each value of c, we solve for α for which μ_3 is equal to zero. Thus, we obtain a set of (c, α) for fixed γ. Figure 3 shows the curve where the EWED is symmetric for two different values of γ. Regression lines are drawn to estimate each curve.

For example, when $\gamma = 1$ and c is in the interval $[1, 5]$, the equation of the curve is estimated by

$$\hat{\alpha} = -0.1092\ln^3(c) + 0.5943\ln^2(c) - 1.364\ln(c) + 3.598. \quad (32)$$

Thus, if the ordered pair $(\ln(c), \alpha)$ is on the curve (32), then the EWED is symmetric. If the ordered pair $(\ln(c), \alpha)$ lies above (or below) the curve (32), then the distribution is skewed to the left (or right). When $\gamma = 3$ and c is in the interval [1, 5], we obtain the equation

$$\hat{\alpha} = -0.03184\ln^3(c) + 0.1892\ln^2(c) - 0.4722\ln(c) + 3.601.$$

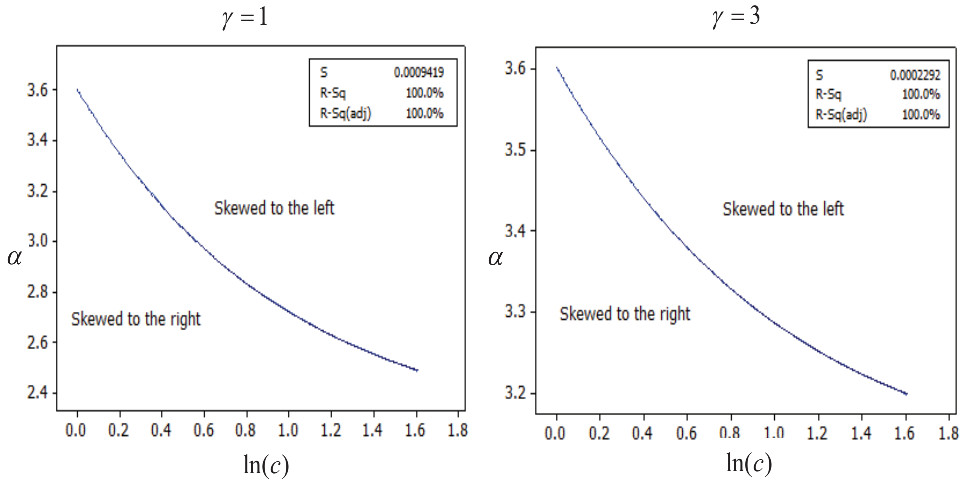

Figure 3. Fitted line plots of skewness for α against $\ln(c)$ when $\gamma = 1$ and 3

The skewness and kurtosis of EWED are given in Table 3 for some values of α, γ and c. When the values of γ and c are fixed, the skewness of EWED decreases as α increases, and the distribution changes from right skewed to left skewed; while the kurtosis of EWED first decreases and then increases as α increases. For fixed values of γ and α, the skewness of EWED decreases as c increases. When the values of α and $c < 1$ are fixed, the skewness of EWED decreases as γ increases. Also, when the values of α and $c > 1$ are fixed, the skewness of EWED increases as γ increases. From Table 3, EWED can be left or right skewed and it can be leptokurtic (cone headed) or platykurtic (flat headed). From Table 3, all the three parameters seem to affect the shape of EWED. This behavior is observed from the graphs of EWED PDFs (not included to save space) for various values of the parameters.

Table 3. Skewness and kurtosis of EWED for some values of α, γ and c

		$\gamma = 0.5$		$\gamma = 1.0$		$\gamma = 2.0$	
c	α	Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
0.5	0.9	3.5175	21.8311	2.9792	16.2641	2.6335	13.3543
	1	3.1264	17.6229	2.6324	13.0832	2.2982	10.6486
	2	1.3316	5.1845	1.0998	4.3023	0.8858	3.6744
	4	0.3487	2.7969	0.2188	2.6920	0.0842	2.6510
	6	−0.0388	2.6975	−0.1391	2.7419	−0.2445	2.8320
0.7	0.9	2.8659	15.5859	2.6392	13.5795	2.4798	12.3324
	1	2.4933	12.2897	2.2901	10.7260	2.1396	9.7371
	2	0.9322	3.8863	0.8382	3.6268	0.7458	3.4121
	4	0.1016	2.6898	0.0463	2.6835	−0.0119	2.6918
	6	−0.2293	2.8402	−0.2722	2.8837	−0.3173	2.9392
0.9	0.9	2.4833	12.5254	2.4252	12.0635	2.3798	11.7002
	1	2.1299	9.7929	2.0785	9.4425	2.0382	9.1952
	2	0.7096	3.3843	0.6859	3.3327	0.6617	3.2854
	4	−0.0381	2.7209	−0.0523	2.7250	−0.0675	2.7312
	6	−0.3361	2.9781	−0.3471	2.9929	−0.3588	3.0098
1	0.9	2.3450	11.5300	2.3450	11.5300	2.3450	11.5300
	1	2.0000	9.0000	2.0000	9.0000	2.0000	9.0000
	2	0.6311	3.2451	0.6311	3.2451	0.6311	3.2451
	4	−0.0873	2.7481	−0.0873	2.7481	−0.0873	2.7481
	6	−0.3735	3.0365	−0.3735	3.0365	−0.3735	3.0365
2	0.9	1.6855	7.6029	1.9398	9.1325	2.1453	10.3432
	1	1.3896	5.9861	1.6100	7.0800	1.8047	8.0716
	2	0.2700	2.8647	0.3730	2.9647	0.4839	3.0966
	4	−0.3105	2.9816	−0.2480	2.9118	−0.1795	2.8469
	6	−0.5403	3.3649	−0.4927	3.2687	−0.4406	3.1703
4	0.9	1.3262	6.0419	1.7026	7.9638	2.0265	9.7223
	1	1.0627	4.8606	1.3866	6.1936	1.6891	7.5821
	2	0.0837	2.8403	0.2357	2.9073	0.4039	3.0482
	4	−0.4219	3.1689	−0.3301	3.0335	−0.2273	2.9117
	6	−0.6220	3.5696	−0.5526	3.4081	−0.4748	3.2465
6	0.9	1.2011	5.5961	1.6171	7.5864	1.9829	9.5049
	1	0.9497	4.5525	1.3070	5.9163	1.6472	7.4161
	2	0.0207	2.8594	0.1887	2.9028	0.3761	3.0369
	4	−0.4588	3.2420	−0.3576	3.0805	−0.2435	2.9359
	6	−0.6488	3.6444	−0.5725	3.4591	−0.4863	3.2739

4.4 Skewness and Kurtosis Based on Quantiles

Another way to study the relationships of the shape parameters α, γ and c and the skewness and kurtosis is by using Galton's skewness (Galton, 1883) and Moors' kurtosis (Moors, 1988), both of which are based on the quantile function. By using the quantile function $Q(\cdot)$ in Equation (22), Galton's skewness and Moors' kurtosis, respectively, are given by

$$S = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)},$$

and

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Figure 4 depicts Galton's skewness and Moors' kurtosis for the EWED using the parameters α and c when $\gamma = 1$.

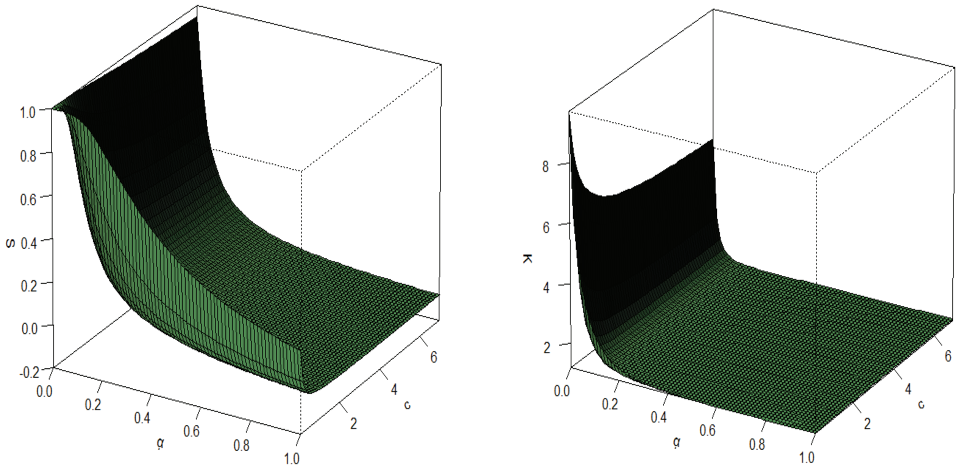

Figure 4. Galton's skewness and Moors' kurtosis for the EWED

We created a table for Galton's skewness and Moors' kurtosis for the same combination of parameters in Table 3 (not included to save space). The observations made from that table are identical to the observations made from Table 3.

5. Applications of EWED

This section presents three applications of the exponentiated Weibull-exponential distribution using real data sets. The data sets are chosen to illustrate the ability of the EWED to fit skewed (left or right) and heavy-tailed data. In these applications the maximum likelihood method is applied to estimate the parameters of fitted distributions. The maximized log-likelihood, the Kolmogorov-Smirnov test (K-S) along with the corresponding p-value, the Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are reported in order to compare the EWED with the other distributions. For graphical illustration of the goodness of fit, the plot of fitted density functions along with the histogram of the data is presented.

5.1 Strengths of 1.5 cm Glass Fibers

The data set ($n = 63$) is on the strengths of 1.5 cm glass fibers and it is obtained from Smith and Naylor (1987). Barreto-Souza et al. (2010) applied the beta generalized exponential distribution to fit the data and Barreto-Souza et al. (2011) fitted the beta Fréchet distribution to the data. The MLEs of EWED parameters and the goodness of fit statistics are reported in Table 4. The MLEs of the beta exponential distribution and beta generalized exponential distribution (BGED) are taken from Barreto-Souza et al. (2010), and the MLEs of beta Fréchet distribution (BFD) are from Barreto-Souza et al. (2011).

Table 4. Parameters estimates (standard error in parentheses) for the glass fibers data

Distribution	Beta exponential	Beta Fréchet	BGED	EWED
Parameter estimates	$\hat{a} = 17.779\,(3.289)$	$\hat{a} = 0.396\,(0.174)$	$\hat{a} = 0.413\,(0.302)$	$\hat{\alpha} = 23.614\,(3.954)$
	$\hat{b} = 22.722\,(33.338)$	$\hat{b} = 225.727\,(164.476)$	$\hat{b} = 93.457\,(120.085)$	$\hat{\gamma} = 7.249\,(0.994)$
	$\hat{\lambda} = 0.390\,(0.455)$	$\hat{\lambda} = 1.302\,(0.270)$	$\hat{\lambda} = 0.923\,(0.501)$	$\hat{c} = 0.0033\,(0.003)$
		$\hat{\sigma} = 6.863\,(1.992)$	$\hat{\alpha} = 22.612\,(21.925)$	
K-S	0.216	0.214	0.167	0.137
p-value	0.005	0.006	0.059	0.195
Log likelihood	−24.13	−19.59	−15.60	−14.33
AIC	54.1	47.2	39.2	34.7
BIC	60.7	55.8	47.8	41.1

From Table 4, Both the beta generalized exponential distribution and the EWED provide adequate fit to the data with EWED providing the best fit based on every criterion. The distribution of the data is skewed to the left (skewness = −0.92). By using the Wald statistic to test the null hypothesis $c = 1$ against the alternative $c \neq 1$, we obtain a p-value that is less than 0.0001, which shows that the parameter c is significant for fitting the data. Hence, the EWED is superior to using the Weibull distribution to fit the data. This application suggests that the EWED has the ability to fit left-skewed data sets. The exponentiated parameter plays an important role in capturing the left skewness. Figure 5 displays the estimated densities of the EWED, beta Fréchet and beta generalized exponential distributions. The plots show that the EWED fits better than the beta generalized exponential distribution.

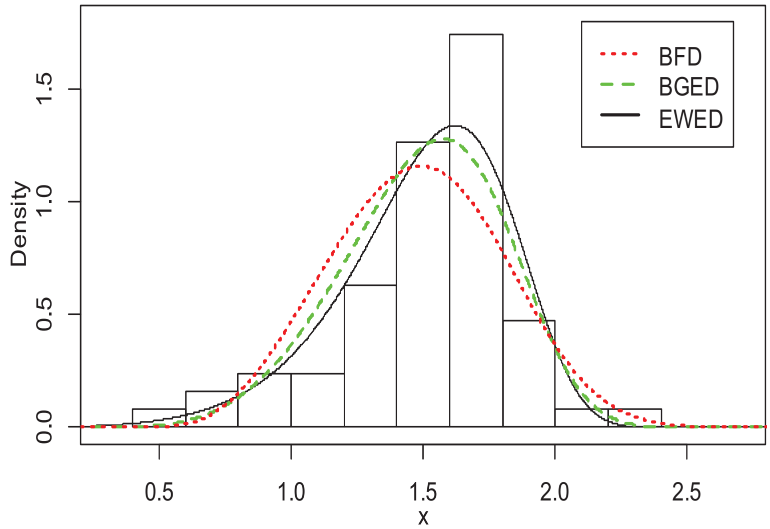

Figure 5. PDFs for glass fibers data

5.2 Depressive Condition Data

The depressive condition data is obtained from a study conducted by Leiva et al. (2010) in a city located at the south part of Chile. The data corresponds to a scale that measures the behavioral and emotional problems of children. The score was based on a study of 19 items from a random sample of 134 children. Each item was scored on a scale of 1 to 3, with 3 being a higher tendency towards depressiveness.

The depressive condition data was analyzed by Leiva et al. (2010), using a mixture of two skewed distributions. Balakrishnan et al. (2011) analyzed the depressive condition data using three mixture models based on the Birnbaum Saunders distribution. We apply the EWED to fit the depressive condition data and compare the results with the three mixture models in Table 5. The MLEs and the goodness of fit for the three mixture models are from Balakrishnan et al. (2011).

Table 5. Parameters estimates (standard error in parentheses) for the glass fibers data

Distribution	MTBS Distribution (model 1)	MBSLBS Distribution (model 2)	RMBSLBS Distribution (model 3)	EWED
Parameter Estimates	$\hat{\alpha}_1 = 0.579\,(0.041)$ $\hat{\beta}_1 = 7.300\,(0.440)$ $\hat{\alpha}_2 = 0.135\,(0.095)$ $\hat{\beta}_2 = 20.256\,(2.674)$ $\hat{p} = 0.9631\,(0.041)$	$\hat{\alpha}_1 = 0.603\,(0.46)$ $\hat{\beta}_1 = 7.579\,(0.478)$ $\hat{\alpha}_2 = 61.406\,(50.859)$ $\hat{\beta}_2 = 0.001\,(0.006)$ $\hat{p} = 0.997\,(0.238)$	$\hat{\alpha}_1 = 0.598\,(0.044)$ $\hat{\beta}_1 = 5.665\,(1.393)$ $\hat{\alpha}_2 = 0.598\,(0.044)$ $\hat{\beta}_2 = 5.665\,(1.393)$ $\hat{p} = 0.176\,(0.653)$	$\hat{\alpha} = 1.045\,(0.124)$ $\hat{\gamma} = 5.920\,(0.748)$ $\hat{c} = 25.744\,(9.291)$
K-S	0.091	0.092	0.085	0.091
p-value	0.218	0.206	0.283	0.216
Log likelihood	−387.52	−388.09	−388.73	−386.25
AIC	785.0	786.2	787.5	778.2
BIC	799.5	800.7	801.9	787.2

Balakrishnan et al. (2011) observed that model 3 provides the best fit out of the three mixture models based on its lowest K-S statistic and highest corresponding p-value as shown in Table 5. From our analysis, the EWED provides a good fit; it has the highest log likelihood and lowest AIC among the four models in Table 5. The EWED with three parameters and model 3 with five parameters both provide adequate fit to the data.

The estimated PDFs of the EWED, Model 2, and Model 3 are given in Figure 6, which show that the EWED fits the depressive condition data very well. Based on the plots in Figure 6, one can see that the depressive condition data is skewed to the right (skewness = 1.13) indicating that the EWED has the ability to fit right-skewed data and capture long tails very well. The Wald statistic for testing the null hypothesis $c = 1$ against the alternative $c \neq 1$ has a p-value of 0.008. Hence, we reject the null hypothesis in favor of the EWED. This indicates that the exponentiated parameter plays a critical role in capturing the right skewness.

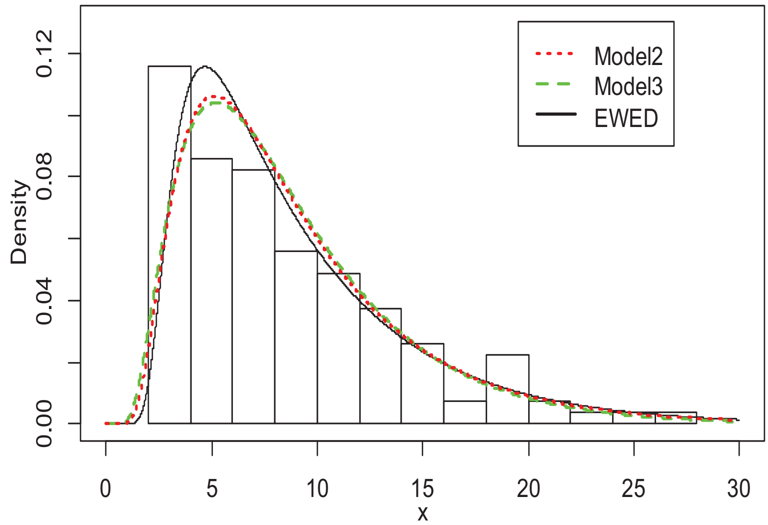

Figure 6. PDFs for depressive condition data

5.3 Airborne Data

The data set in this application represents the repair times (in hours) for an airborne communication transceiver and was originally analyzed by Von Alven (1964) by fitting a two-parameter log-normal distribution. Recently, Cordeiro et al. (2013) analyzed the data by using the beta generalized Raleigh distribution. We also apply the EWED to fit the airborne data and the results are given in Table 6. The parameter estimates, the K-S, AIC, and BIC values (in Table 6) for the beta generalized Raleigh (BGR), exponentiated generalized Rayleigh (EGR), and generalized Rayleigh distributions are taken from Cordeiro et al. (2013).

The results from Table 6 indicate that the BGR distribution provides a better fit than the EGR distribution, while the EWED provides the best fit with the lowest K-S statistic and highest corresponding p-value. The distribution of the data is highly skewed to the right (skewness = 2.99).

Table 6. Estimates of the model parameters (standard error in parentheses) for airborne data

Distribution	Generalized Rayleigh	EGR	BGR	EWED
Parameter Estimates	$\hat{\alpha} = -0.703\,(0.049)$ $\hat{\theta} = 0.0079\,(0.003)$	$\hat{a} = 5.712\,(2.400)$ $\hat{\alpha} = -0.946\,(0.024)$ $\hat{\theta} = 0.0073\,(0.002)$	$\hat{a} = 10.482\,(0.476)$ $\hat{b} = 20.761\,(0.228)$ $\hat{\alpha} = -0.893\,(0.022)$ $\hat{\theta} = 4.7 \times 10^{-6}\,(10^{-5})$	$\hat{\alpha} = 0.498\,(0.133)$ $\hat{\gamma} = 1.279\,(0.702)$ $\hat{c} = 4.512\,(1.992)$
K-S	0.176	0.179	0.122	0.091
p-value	0.116	0.105	0.500	0.838
Log likelihood	−217.10	−108.15	−99.55	−100.00
AIC	221.1	222.3	207.1	206.0
BIC	224.7	227.7	214.4	211.5

The Wald statistic for testing the null hypothesis $c = 1$ against the alternative $c \neq 1$ has a p-value of 0.078 and this is significant at 10% level. This suggests that the parameter c is critical for the EWED to be flexible for fitting long-tailed and highly skewed data. Figure 7 displays the estimated densities of the EWED, BGR distribution, and EGR distribution fitted to the airborne data.

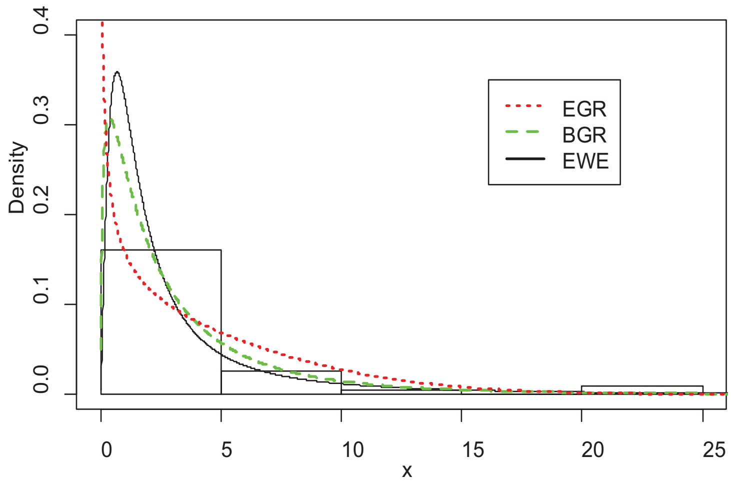

Figure 7. PDFs for Airborne data

6. Summary and Conclusion

In this article, we introduce the exponentiated T-X family, which is an extension of the T-X family proposed by Alzaatreh, Lee and Famoye (2013). The exponentiated parameter c provides additional flexibility for fitting diverse shapes of data. Some of its properties are derived and some members of the family are defined. A member of the exponentiated Weibull-X family, namely, the three-parameter exponentiated Weibull-exponential distribution is defined and studied. Various properties of the exponentiated Weibull-exponential distribution including, limiting behavior, hazard function, moments, and Shannon entropy are derived. The EWED is applied to fit three real data sets. These applications show that the EWED has the ability to fit skewed (left or right) and heavy-tailed data due to its flexibility. The need for the exponentiated parameter c can be tested using the Wald statistic. Among the three data analyzed, the parameter c is very critical for two data sets, and the third one is significant at 10% level.

Figure 8 provides the various families that can be obtained from the exponentiated T-X distributions.

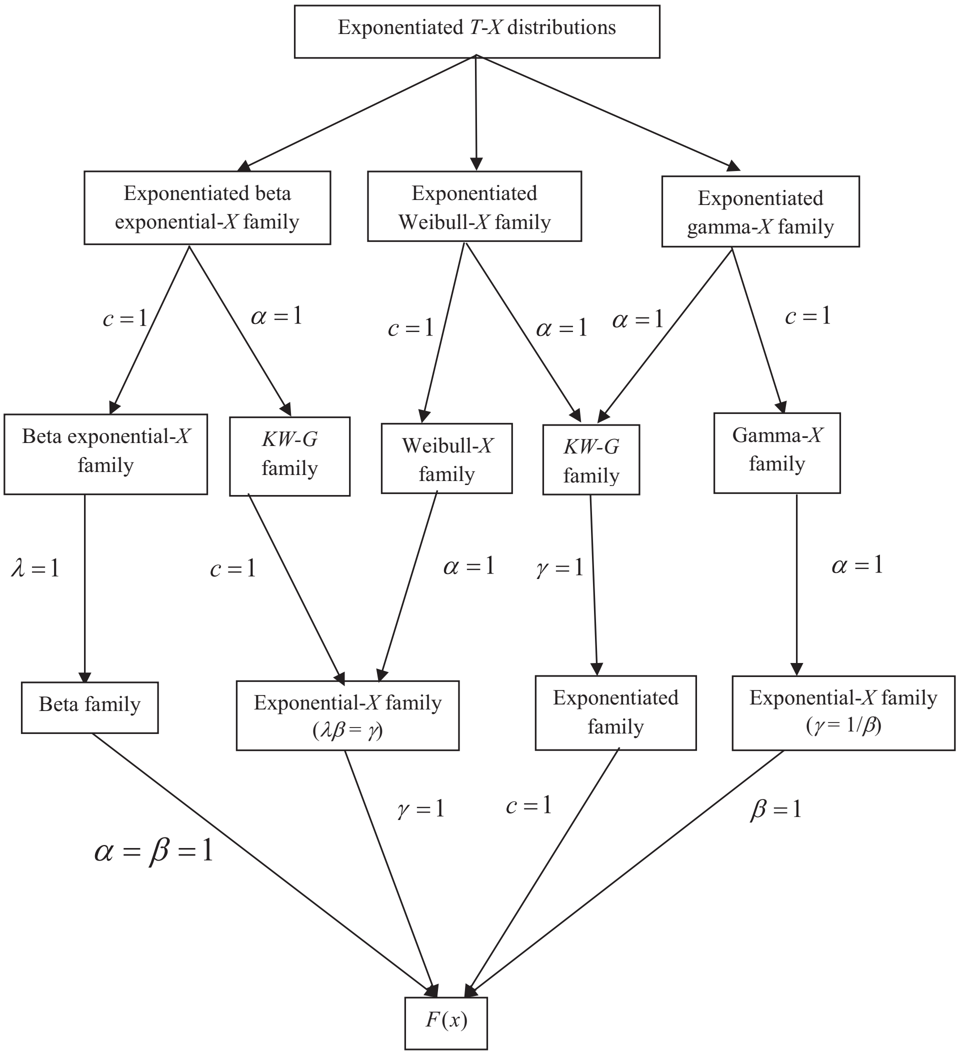

Figure 8. Families of exponentiated T-X distributions

Acknowledgements

The authors are grateful for the comments and suggestions by the referees and the Editor-in-Chief. Their comments and suggestions have greatly improved the paper.

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Appendix

Proof of Theorem 1. By the definition of Shannon entropy,

$$\begin{aligned} \eta_x &= E(-\log g(X)) = -\log c - E(\log f(X)) - E\left(\log F^{c-1}(X) - E(-\log(1-F^c(X)))\right) \\ &\quad + E(-\log[r\{-\log(1-F^c(X))\}]). \end{aligned} \tag{A.1}$$

The random variable T has a PDF $r(t)$ and using $T = -\log(1 - F^c(X))$, we have the following $E(\log f(X)) = E\left(\log f\left(F^{-1}\left\{1 - e^{-T}\right\}^{1/c}\right)\right)$, $E\left(\log F^{c-1}(X)\right) = \frac{c-1}{c} E\left(\log\left(1 - e^{-T}\right)\right)$, $E(-\log(1 - F^c(X))) = E(T) = \mu_T$, and $E(-\log[r\{-\log(1 - F^c(X))\}]) = E(-\log[r(t)]) = \eta_T$.

Applying these results in (A.1), we obtain

$$\eta_x = -\log c - E\left(\log f\left(F^{-1}\left\{1 - e^{-T}\right\}^{1/c}\right)\right) + \frac{1-c}{c} E\left(\log\left\{1 - e^{-T}\right\}\right) - \mu_T + \eta_T,$$

which completes the proof. □

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