# Expressions of content-parametrized Schur multiple zeta-functions via the Giambelli formula 

Kohji Matsumoto and Maki Nakasuji *


#### Abstract

In this article, we consider the expressions for content-parametrized Schur multiple zeta-functions in terms of multiple zeta-functions of Euler-Zagier type and their star-variants, or in terms of modified zeta-functions of root systems. First of all, we focus on the Schur multiple zeta-function of hook type. And then, applying the Giambelli formula and induction argument, we obtain the expressions for general contentparametrized Schur multiple zeta-functions.


Keywords: Schur multiple zeta-functions, zeta-functions of root systems, Euler-Zagier multiple zeta-functions, Giambelli formula

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## 1 Introduction

The multiple zeta-function of Euler-Zagier type and its star-variant are defined by the series

$$
\zeta\left(s_{1}, \ldots, s_{r}\right)=\sum_{0<m_{1}<\cdots<m_{r}} \frac{1}{m_{1}^{s_{1}} \cdots m_{r}^{s_{r}}}, \quad \zeta^{\star}\left(s_{1}, \ldots, s_{r}\right)=\sum_{0<m_{1} \leq \cdots \leq m_{r}} \frac{1}{m_{1}^{s_{1} \cdots m_{r}^{s_{r}}}}
$$

respectively, where $\left(s_{1}, \ldots, s_{r}\right) \in \mathbb{C}^{r}$ with $\operatorname{Re}\left(s_{1}\right), \ldots, \operatorname{Re}\left(s_{r-1}\right) \geq 1$ and $\operatorname{Re}\left(s_{r}\right)>1$ for convergence. More precisely, both $\zeta\left(s_{1}, \ldots, s_{r}\right)$ and $\zeta^{\star}\left(s_{1}, \ldots, s_{r}\right)$ converge in $\sum_{j=r-i+1}^{r} \operatorname{Re}\left(s_{j}\right)>i$ for $1 \leq i \leq r$ (see [M]). Recently, some representation-theoretic generalizations of them have been studied. One of them is the multiple zeta-functions attached to semisimple Lie algebras, called the zeta-functions of root systems introduced by K. Matsumoto, Y. Komori and H. Tsumura ([KMT1]). It is well known that semisimple Lie algebras over $\mathbb{C}$ are classified by root systems coming from their Dynkin

[^0]diagrams, and the exact formula for this kind of multiple zeta-function is different for each root system. For example, the zeta-function of the root system of type $A_{r}$ is of the following simple form;
$$
\zeta_{r}\left(\underline{\mathbf{s}}, A_{r}\right)=\sum_{m_{1}}^{\infty} \cdots \sum_{m_{r}=1}^{\infty} \prod_{1 \leq i<j \leq r+1}\left(m_{i}+\cdots+m_{j-1}\right)^{-s(i, j)},
$$
where $s(i, j)$ is the variable corresponding to the root parametrized by $(i, j)(1 \leq i, j \leq r+1, i \neq j)$ and
\[

$$
\begin{aligned}
\underline{\mathbf{s}}= & (s(1,2), s(2,3), \ldots, s(r, r+1), s(1,3), s(2,4), \ldots, \\
& s(r-1, r+1), \ldots, s(1, r), s(2, r+1), s(1, r+1)) .
\end{aligned}
$$
\]

In their papers ( KMT1] and [KMT2]), we can see that various relations among multiple zeta values can be regarded as special cases of functional relations among zeta-functions of root systems.

The other generalization of multiple zeta-functions of Euler-Zagier type is that associated with combinatorial objects called semi-standard Young tableaux. It is called the Schur multiple zetafunctions, which was introduced by M. Nakasuji, O. Phuksuwan and Y. Yamasaki ([NPY). As details are given in the next section, this function has the form of

$$
\zeta_{\lambda}(s)=\sum_{M \in \operatorname{SSYT}(\lambda)} \frac{1}{M^{s}},
$$

where $\lambda$ is a partition and $\operatorname{SSYT}(\lambda)$ is a set of semi-standard Young tableaux of shape $\lambda$. In their paper ([ $\mathbb{N P Y}]$ ), after they studied basic properties of this function, they obtained some determinant formulas such as Jacobi-Trudi, Giambelli and dual Cauchy formulas under the assumption that $\zeta_{\lambda}(\boldsymbol{s})$ is content-parametrized (in the sense defined in Section (2). Furthermore, they investigated skew Schur multiple zeta-functions which is associated with the set difference $\lambda / \mu$ of two partitions $\lambda$ and $\mu$, and quasi-symmetric functions as extensions.

In our previous research $([M N)$, the relation between zeta-functions of root systems and Schur multiple zeta-functions was discussed. There, we obtained various expressions of Schur multiple zeta-functions of anti-hook type which are special cases of skew Schur multiple zeta-functions. One of them is as follows.

Theorem 1.1 ([[MN, Theorem 3.2]) For $k, \ell \in \mathbb{N}$, if $\lambda=(\underbrace{k+1, \cdots, k+1}_{\ell+1 \text { times }}), \mu=(\underbrace{k, \cdots, k}_{\ell \text { times }})$ and

$\boldsymbol{s}=$|  |  |
| :--- | :--- |
|  |  |

(note that the way of indexing the variables here is not standard), then

$$
\begin{equation*}
\zeta_{\lambda / \mu}(\boldsymbol{s})=\sum_{i=0}^{k}(-1)^{k-i} \zeta^{\star}\left(s_{00}, s_{10}, \ldots, s_{i-1,0}\right) \zeta\left(s_{k \ell}, s_{k, \ell-1}, \ldots, s_{k 0}, s_{k-1,0}, \ldots, s_{i 0}\right) \tag{1.1}
\end{equation*}
$$

holds in the whole space $\mathbb{C}^{k+\ell+1}$, where $\zeta^{\star}=1$ for $i=0$.
Other results in [MN] (Theorem 4.1 in [MN], for example) are expressions in terms of modified zeta-functions of root systems of type $A$ defined by (3.5), (3.7) and (3.8) in Section 3.

Remark 1 One expression among them gives us an analogue of Weyl group multiple Dirichlet series in the sense of Bump, Goldfeld and others (see $\left[\begin{array}{l}B\end{array}\right)$ and so, this may mean a first link for some undiscovered connections between the theory of Weyl group multiple Dirichlet series and the theory of zeta-functions of root systems which Bump questioned in [B, p.19].

Our aim in this article is to obtain the expressions, analogous to the results proved in [MN], for Schur multiple zeta-functions of shape $\lambda$. First of all, we will focus on the Schur multiple zetafunction of hook type. In Section 3, we will prove the expressions similar to (1.1) in Theorem 1.1 and those in terms of modified zeta-functions of root systems of type $A$ in [MN] for hook types. The aforementioned Giambelli formula for the content-parametrized Schur multiple zeta-functions is a determinant formula, in which the elements in the matrix in the determinant expression are Schur multiple zeta-functions of hook type. Therefore, by using this formula, the results in Section 3 can be used to obtain the expressions of more general content-parametrized Schur multiple zetafunctions. In Section [4 applying the Giambelli formula with more discussions, we will obtain the expressions for the content-parametrized Schur multiple zeta-function of shape $\lambda$.

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## 2 Schur multiple zeta-functions and their Giambelli formula

Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ be a partition that is a non-increasing sequence of a positive integer $n$, i.e. $\lambda_{1} \geq \lambda_{2} \geq \cdots \lambda_{m}>0$ with $\sum_{i} \lambda_{i}=n$. Then a Young diagram of shape $\lambda$ is obtained by drawing $\lambda_{i}$ boxes in the $i$-th row. The conjugate $\lambda^{\prime}=\left(\lambda_{1}^{\prime}, \ldots, \lambda_{s}^{\prime}\right)$ of $\lambda$ is defined by $\lambda_{i}^{\prime}=\#\left\{j \mid \lambda_{j} \geq i\right\}$. In other words, $\lambda^{\prime}$ is the partition whose Young diagram is the transpose of that of $\lambda$. Let $T(\lambda, X)$ be the set of all Young tableaux of shape $\lambda$ over a set $X$ and, in particular, $\operatorname{SSYT}(\lambda) \subset T(\lambda, \mathbb{N})$ the set of all semi-standard Young tableaux of shape $\lambda$. Recall that $M=\left(m_{i j}\right) \in \operatorname{SSYT}(\lambda)$ if and only if $m_{i 1} \leq m_{i 2} \leq \cdots$ for all $i$ and $m_{1 j}<m_{2 j}<\cdots$ for all $j$. For $s=\left(s_{i j}\right) \in T(\lambda, \mathbb{C})$, the Schur
multiple zeta-function associated with $\lambda$ is defined as in [NPY] by the series

$$
\zeta_{\lambda}(\boldsymbol{s})=\sum_{M \in \operatorname{SSYT}(\lambda)} \frac{1}{M^{s}},
$$

where $M^{s}=\prod_{(i, j) \in \lambda} m_{i j}^{s_{i j}}$ for $M=\left(m_{i j}\right) \in \operatorname{SSYT}(\lambda)$ as in Section 亿. This series converges absolutely if $\boldsymbol{s} \in W_{\lambda}$ where

$$
W_{\lambda}=\left\{\begin{array}{l|l}
s=\left(s_{i j}\right) \in T(\lambda, \mathbb{C}) & \begin{array}{l}
\operatorname{Re}\left(s_{i j}\right) \geq 1 \text { for all }(i, j) \in \lambda \backslash C(\lambda) \\
\operatorname{Re}\left(s_{i j}\right)>1 \text { for all }(i, j) \in C(\lambda)
\end{array}
\end{array}\right\}
$$

with $C(\lambda)$ being the set of all corners of $\lambda$.
For a partition $\lambda$, we define two sequences of indices $p_{1}, \ldots, p_{N}$ and $q_{1}, \ldots, q_{N}$ by $p_{i}=\lambda_{i}-i$ and $q_{i}=\lambda_{i}^{\prime}-i$ for $1 \leq i \leq N$ where $N$ is the number of the main diagonal entries of the Young diagram of $\lambda$. We sometimes write $\lambda=\left(p_{1}, \ldots, p_{N} \mid q_{1}, \ldots, q_{N}\right)$, which is called the Frobenius notation of $\lambda$ (see [Mac, Section 1.1]).

Let

$$
W_{\lambda}^{\text {diag }}=\left\{\boldsymbol{s}=\left(s_{i j}\right) \in W_{\lambda} \mid s_{i j}=s_{l m} \text { if } j-i=m-l\right\} .
$$

When $\boldsymbol{s} \in W_{\lambda}^{\text {diag }}$, we can introduce new variables $\left\{z_{k}\right\}_{k \in \mathbb{Z}}$ by the condition $s_{i j}=z_{j-i}$ (for all $i, j)$, and we may regard $\zeta_{\lambda}(\boldsymbol{s})$ as a function in variables $\left\{z_{k}\right\}_{k \in \mathbb{Z}}$. We call the Schur multiple zeta-function associated with $\left\{z_{k}\right\}$ content-parametrized Schur multiple zeta-function, since $j-i$ is named "content" (cf. Hamel ( $[\underline{\mathrm{H}}])$ ).

The following theorem is the Giambelli formula for Schur multiple zeta-functions.

Theorem 2.1 ([NPY, Theorem 4.5]) Let $\lambda$ be a partition such that $\lambda=\left(p_{1}, \cdots, p_{N} \mid q_{1}, \cdots, q_{N}\right)$ in the Frobenius notation. Assume $\boldsymbol{s} \in W_{\lambda}^{\text {diag }}$. Then we have

$$
\begin{equation*}
\zeta_{\lambda}(\mathbf{s})=\operatorname{det}\left(\zeta_{i, j}\right)_{1 \leq i, j \leq N}, \tag{2.1}
\end{equation*}
$$

where $\zeta_{i, j}=\zeta_{\left(p_{i}+1,1^{q_{j}}\right)}\left(\mathbf{s}_{i j}^{F}\right)$ with $\mathbf{s}_{i j}^{F}=$| $z_{0}$ | $z_{1}$ | $z_{2}$ | $\cdots$ | $z_{p_{i}}$ |
| :---: | :---: | :---: | :---: | :---: |
| $z_{-1}$ |  |  |  |  |
| $\vdots$ |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
| $\left.q_{j}, q_{j} q_{j}\right)$ |  |  |  |  |
|  |  |  |  |  |

Remark 2 The original Giambelli formula is for Schur functions, and the above is its zetaanalogue. In [NPY, they use the notation $\left\{a_{k}\right\}$ as a variable in $W_{\lambda}^{\text {diag }}$, but we here use the notation $\left\{z_{k}\right\}$, instead.

Example 2.2 When $\lambda=(6,4,4,2,2)$ then $\lambda=(5,2,1 \mid 4,3,0)$ is the Frobenius notation, and in
this example we write $T$ shortly for $\zeta_{\lambda}(T)(T \in T(\lambda, \mathbb{C}))$. Then for $\boldsymbol{s} \in W_{\lambda}^{\text {diag }}$, Theorem 2.1 gives


## 3 Expressions of Schur multiple zeta-functions of hook type

For $\lambda=\left(p+1,1^{q}\right)$, let

using the notation in the previous section. The Schur multiple zeta-function of this type is called the Schur multiple zeta-function of hook type, and we obtain the following expression for it in terms of multiple zeta-functions of Euler-Zagier type and their star-variants.

Theorem 3.1 For $\lambda=\left(p+1,1^{q}\right)$, we have

$$
\begin{equation*}
\zeta_{\lambda}(\boldsymbol{s})=\sum_{j=0}^{q}(-1)^{j} \zeta^{\star}\left(z_{-j}, \ldots, z_{-1}, z_{0}, z_{1}, \ldots, z_{p}\right) \zeta\left(z_{-j-1}, \ldots, z_{-q}\right), \tag{3.2}
\end{equation*}
$$

where we put $\zeta\left(z_{-j-1}, \ldots, z_{-q}\right)=1$ when $j=q$, and also

$$
\begin{equation*}
\zeta_{\lambda}(\boldsymbol{s})=\sum_{j=0}^{p}(-1)^{j} \zeta\left(z_{j}, \ldots, z_{1}, z_{0}, z_{-1}, \ldots, z_{-q}\right) \zeta^{\star}\left(z_{j+1}, \ldots, z_{p}\right) \tag{3.3}
\end{equation*}
$$

where we put $\zeta^{\star}\left(z_{j+1}, \ldots, z_{p}\right)=1$ when $j=p$.
Proof. For the first assertion (3.2), we carry out the induction for $q$. In the case $q=0$, it is trivial that the assertion holds. Assume that the assertion is true for $q-1$, and consider the case of $q$ :

$$
\begin{equation*}
\zeta_{\lambda}(\boldsymbol{s})=\sum_{\substack{m_{11} \leq m_{12} \leq \ldots \leq m_{1, p+1} \\ m_{11}<m_{21}<\ldots<m_{q+1,1}}} m_{11}^{-z_{0}} m_{12}^{-z_{1}} \ldots m_{1, p+1}^{-z_{p}} m_{21}^{-z_{-1}} m_{31}^{-z_{-2}} \ldots m_{q+1,1}^{-z_{-q}} . \tag{3.4}
\end{equation*}
$$

The summation on the right-hand side can be divided into two parts:

$$
\sum_{\substack{m_{11} \leq m_{12} \leq . . \leq m_{1, p+1}+1 \\ m_{21}<\ldots<m_{q+1,1}}}-\sum_{\substack{m_{11} \leq m_{12} \leq \ldots \leq m_{1, p+1} \\ m_{11} \leq m_{21}<\ldots<m_{q+1,1}}}=\sum_{1}-\sum_{2},
$$

say. Then obviously

$$
\sum_{1}=\zeta^{\star}\left(z_{0}, \ldots, z_{p}\right) \zeta\left(z_{-1}, \ldots, z_{-q}\right),
$$

while

$$
\sum_{2}=\sum_{\substack{m_{21} \leq m_{11} \leq m_{12} \leq \ldots \leq m_{1, p+1} \\ m_{21}<m_{31}<\ldots<m_{1, q+1,1}}}=\zeta_{\lambda_{-}}\left(s^{b}\right)
$$



By the assumption for induction, we obtain

$$
\begin{aligned}
\sum_{2} & =\sum_{i=0}^{q-1}(-1)^{i} \zeta^{\star}\left(z_{-(i+1)}, \ldots, z_{-2}, z_{-1}, z_{0}, \ldots, z_{p}\right) \zeta\left(z_{-(i+2)}, \ldots, z_{-q}\right) \\
& =\sum_{j=1}^{q}(-1)^{j-1} \zeta^{\star}\left(z_{-j}, \ldots, z_{-2}, z_{-1}, z_{0}, \ldots, z_{p}\right) \zeta\left(z_{-(j+1)}, \ldots, z_{-q}\right) .
\end{aligned}
$$

This leads to

$$
\begin{aligned}
& \sum_{1}-\sum_{2} \\
& =\zeta^{\star}\left(z_{0}, \ldots, z_{p}\right) \zeta\left(z_{-1}, \ldots, z_{-q}\right)-\sum_{j=1}^{q}(-1)^{j-1} \zeta^{\star}\left(z_{-j}, \ldots, z_{-2}, z_{-1}, z_{0}, \ldots, z_{p}\right) \zeta\left(z_{-(j+1)}, \ldots, z_{-q}\right) \\
& =\sum_{j=0}^{q}(-1)^{j} \zeta^{\star}\left(z_{-j}, \ldots, z_{-2}, z_{-1}, z_{0}, \ldots, z_{p}\right) \zeta\left(z_{-(j+1)}, \ldots, z_{-q}\right),
\end{aligned}
$$

which completes the proof of (3.2). The second assertion (3.3) can be similarly proved by using the induction for $p$.

Theorem 3.1 is a kind of analogue of Theorem 1.1 in the case of hook-type. As mentioned in Section 1, in MN, another expression of $\zeta_{\lambda}(\boldsymbol{s})$ in terms of modified zeta-functions of root systems has been shown. An analogue of such an expression for the case of hook-type also exists.

First we have to define the modified zeta-function of root system. For $r>0$ and $0 \leq d \leq r$, we define the modified zeta-function of the root system of type $A_{r}$ by

$$
\begin{equation*}
\zeta_{r, d}^{\bullet}\left(\mathbf{s}, A_{r}\right)=\underbrace{\left(\sum_{m_{1}=0}^{\infty} \cdots \sum_{m_{d}=0}^{\infty}\right)^{\prime}}_{d \text { times }} \underbrace{\sum_{m_{d+1}=1}^{\infty} \cdots \sum_{m_{r}=1}^{\infty}}_{r-d \text { times }} \prod_{1 \leq i<j \leq r+1}\left(m_{i}+\cdots+m_{j-1}\right)^{-s(i, j)}, \tag{3.5}
\end{equation*}
$$

where

$$
\begin{array}{r}
\underline{\mathbf{s}}=(s(1,2), s(2,3), \ldots, s(r, r+1), s(1,3), s(2,4), \ldots, s(r-1, r+1), \ldots \\
 \tag{3.6}\\
s(1, r), s(2, r+1), s(1, r+1)) .
\end{array}
$$

with $s(i, j)$ being the variable corresponding to the root parametrized by $(i, j)$, and the prime means that the terms $\left(m_{i}+\cdots+m_{j-1}\right)^{-s(i, j)}$, where $1 \leq i<j \leq d+1$ and $m_{i}=\cdots=m_{j-1}=0$, are
omitted. We also introduce

$$
\begin{equation*}
\zeta_{r}^{H}\left(\underline{\mathbf{s}}, x, A_{r}\right)=\sum_{m_{1}=1}^{\infty} \cdots \sum_{m_{r}=1}^{\infty} \prod_{1 \leq i<j \leq r+1}\left(x+m_{i}+\cdots+m_{j-1}\right)^{-s(i, j)}, \tag{3.7}
\end{equation*}
$$

and
where $x>0$.
From the definition of semi-standard Young tableaux, the runnning indices for the Schur multiple zeta-function of hook type (3.4) satisfy $1 \leq m_{11} \leq m_{12} \leq \cdots \leq m_{1, p+1}$ and $1 \leq m_{11}<m_{21}<$ $\cdots<m_{1, q+1,1}$. Therefore, setting $m_{12}=m_{11}+a_{1}\left(a_{1} \geq 0\right), m_{13}=m_{11}+a_{1}+a_{2}\left(a_{1}, a_{2} \geq 0\right), \cdots$,

$$
\begin{equation*}
m_{1, p+1}=m_{11}+a_{1}+a_{2}+\cdots+a_{p} \quad\left(a_{i} \geq 0\right), \tag{3.9}
\end{equation*}
$$

and $m_{21}=m_{11}+b_{1}\left(b_{1} \geq 1\right), m_{31}=m_{11}+b_{1}+b_{2}\left(b_{1}, b_{2} \geq 1\right), \cdots$,

$$
m_{1, q+1,1}=m_{11}+b_{1}+b_{2}+\cdots+b_{q} \quad\left(b_{j} \geq 1\right)
$$

we can write the Schur multiple zeta-function of hook type as

$$
\begin{align*}
\zeta_{\lambda}(\boldsymbol{s})= & \sum_{\substack{m_{11} \geq 1 \\
a_{1} \geq 0(1 \leq i \leq p) \\
b_{j} \geq 1(0 \leq j \leq q)}} m_{11}^{-z_{0}}\left(m_{11}+a_{1}\right)^{-z_{1}} \cdots\left(m_{11}+a_{1}+\cdots+a_{p}\right)^{-z_{p}} \\
& \times\left(m_{11}+b_{1}\right)^{-z_{-1}}\left(m_{11}+b_{1}+b_{2}\right)^{-z_{-2}} \cdots\left(m_{11}+b_{1}+b_{2}+\cdots+b_{q}\right)^{-z_{-q}} \\
= & \sum_{m_{11} \geq 1} m_{11}^{-z_{0}} \zeta_{p, p}^{\bullet, H}\left(\boldsymbol{s}_{+}, m_{11}, A_{p}\right) \zeta_{q}^{H}\left(\boldsymbol{s}_{-}, m_{11}, A_{q}\right), \tag{3.10}
\end{align*}
$$

where $\boldsymbol{s}_{+}=\left(z_{1}, z_{2}, \cdots, z_{p}\right)$ and $\boldsymbol{s}_{-}=\left(z_{-1}, z_{-2}, \cdots, z_{-q}\right)$. We note that this equation is the first step in Theorem 4.2 in the next section.

## 4 Expressions of content-parametrized Schur multiple zeta-functions

Now we are ready to prove the main results in the present paper. Our aim is to generalize the results in Section 3 to more general content-parametrized Schur multiple zeta-functions $\zeta_{\lambda}(\boldsymbol{s})$ which can be written in terms of $\left\{z_{k}\right\}$ with content $k$. The results which we will prove in this section are

Theorem 4.1 and Theorem 4.2, which are analogues of [MN, Theorem 3.2] and of [MN, Theorem 4.1], respectively.

First, as a generalization of Theorem 3.1, we prove the following theorem.
Theorem 4.1 For the symmetric group $\mathfrak{S}_{N}$, we have

$$
\begin{aligned}
\zeta_{\lambda}(\boldsymbol{s})= & \sum_{\sigma \in \mathfrak{S}_{N}} \operatorname{sgn}(\sigma) \sum_{j_{1}=0}^{q_{1}} \cdots \sum_{j_{N}=0}^{q_{N}}(-1)^{j_{1}+\cdots+j_{N}} \\
& \times \zeta^{\star}\left(z_{-j_{1}}, \ldots, z_{0}, \ldots, z_{p_{\sigma(1)}}\right) \zeta^{\star}\left(z_{-j_{2}}, \ldots, z_{0}, \ldots, z_{p_{\sigma(2)}}\right) \ldots \zeta^{\star}\left(z_{-j_{N}}, \ldots, z_{0}, \ldots, z_{p_{\sigma(N)}}\right) \\
& \times \zeta\left(z_{-j_{1}-1}, \ldots, z_{-q_{1}}\right) \zeta\left(z_{-j_{2}-1}, \ldots, z_{-q_{2}}\right) \ldots \zeta\left(z_{-j_{N}-1}, \ldots, z_{-q_{N}}\right)
\end{aligned}
$$

Proof. We use the induction for $N$. When $N=1$, it holds from (3.2) in Theorem 3.1, Assume that the assertion is true for $N-1$, and consider the case of $N$. The Giambelli formula for the Schur multiple zeta-function (2.1) can be written as

$$
\begin{equation*}
\zeta_{\lambda}(\boldsymbol{s})=\sum_{h=1}^{N}(-1)^{h+N} \zeta_{h, N} \cdot \Delta_{h N} \tag{4.1}
\end{equation*}
$$

where $\zeta_{i, j}$ is the same notation as in (2.1) and

$$
\Delta_{h N}=\operatorname{det}\left(\begin{array}{cccc}
\zeta_{1,1} & \zeta_{1,2} & \cdots & \zeta_{1, N-1} \\
\vdots & & & \vdots \\
\zeta_{h-1,1} & \zeta_{h-1,2} & \cdots & \zeta_{h-1, N-1} \\
\zeta_{h+1,1} & \zeta_{h+1,2} & \cdots & \zeta_{h+1, N-1} \\
\vdots & & & \vdots \\
\zeta_{N, 1} & \zeta_{N, 2} & \cdots & \zeta_{N, N-1}
\end{array}\right)
$$

is of size $(N-1) \times(N-1)$. Introducing the notation

$$
\zeta_{i, j}^{\circ}= \begin{cases}\zeta_{i, j} & i \leq h-1 \\ \zeta_{i+1, j} & i \geq h\end{cases}
$$

we have

$$
\Delta_{h N}=\operatorname{det}\left(\begin{array}{cccc}
\zeta_{1,1}^{\circ} & \zeta_{1,2}^{\circ} & \cdots & \zeta_{1, N-1}^{\circ}  \tag{4.2}\\
\vdots & & & \vdots \\
\zeta_{N-1,1}^{\circ} & \zeta_{N-1,2}^{\circ} & \cdots & \zeta_{N-1, N-1}^{\circ}
\end{array}\right)
$$

Note that $\zeta_{i, j}^{\circ}=\zeta_{i, j}^{\circ}\left(s_{i j}^{\circ}\right)$ where
for $h \leq i \leq N-1$. We can apply the assumption for the induction to $\Delta_{h N}$ (compare (2.1) and (4.2)). We have

$$
\begin{align*}
\Delta_{h N} & =\sum_{\tau \in \mathfrak{G}_{N-1}} \operatorname{sgn}(\tau) \sum_{j_{1}=0}^{q_{1}} \cdots \sum_{j_{N-1}=0}^{q_{N-1}}(-1)^{j_{1}+\cdots+j_{N-1}} \\
& \times \zeta^{\star}\left(z_{-j_{1}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(1)}}\right) \zeta^{\star}\left(z_{-j_{2}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(2)}}\right) \ldots \zeta^{\star}\left(z_{-j_{N-1}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(N-1)}}\right) \\
& \times \zeta\left(z_{-j_{1}-1}, \ldots, z_{-q_{1}}\right) \zeta\left(z_{-j_{2}-1}, \ldots, z_{-q_{2}}\right) \ldots \zeta\left(z_{-j_{N-1}-1}, \ldots, z_{-q_{N-1}}\right) \tag{4.3}
\end{align*}
$$

where

$$
\tau^{\prime}=\left(\begin{array}{cccccccc}
1 & \cdots & h-1 & h & h+1 & \cdots & N-1 & N \\
1 & \cdots & h-1 & h+1 & h+2 & \cdots & N & h
\end{array}\right) \circ \tau \in \mathfrak{S}_{N} .
$$

On the other hand, Theorem 3.1 implies

$$
\begin{equation*}
\zeta_{h, N}=\sum_{j_{N}=0}^{q_{N}}(-1)^{j_{N}} \zeta^{\star}\left(z_{-j_{N}}, \ldots, z_{-1}, z_{0}, z_{1}, \ldots, z_{p_{h}}\right) \zeta\left(z_{-j_{N}-1}, \ldots, z_{-q_{N}}\right) . \tag{4.4}
\end{equation*}
$$

Substituting (4.3) and (4.4) into (4.1), we have

$$
\begin{aligned}
\zeta_{\lambda}(\boldsymbol{s})= & \sum_{h=1}^{N}(-1)^{h+N} \sum_{j_{N}=0}^{q_{N}}(-1)^{j_{N}} \zeta^{\star}\left(z_{-j_{N}}, \ldots, z_{-1}, z_{0}, z_{1}, \ldots, z_{p_{h}}\right) \zeta\left(z_{-j_{N}-1}, \ldots, z_{-q_{N}}\right) \\
& \times \sum_{\tau \in \mathfrak{G}_{N-1}} \operatorname{sgn}(\tau) \sum_{j_{1}=0}^{q_{1}} \cdots \sum_{j_{N-1}=0}^{q_{N-1}}(-1)^{j_{1}+\cdots+j_{N-1}} \\
& \times \zeta^{\star}\left(z_{-j_{1}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(1)}}\right) \zeta^{\star}\left(z_{-j_{2}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(2)}}\right) \ldots \zeta^{\star}\left(z_{-j_{N-1}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(N-1)}}\right) \\
& \times \zeta\left(z_{-j_{1}-1}, \ldots, z_{-q_{1}}\right) \zeta\left(z_{-j_{2}-1}, \ldots, z_{-q_{2}}\right) \ldots \zeta\left(z_{-j_{N-1}-1}, \ldots, z_{-q_{N-1}}\right) \\
= & \sum_{h=1}^{N}(-1)^{h+N} \sum_{\tau \in \mathfrak{G}_{N-1}} \operatorname{sgn}(\tau) \sum_{j_{1}=0}^{q_{1}} \cdots \sum_{j_{N}=0}^{q_{N}}(-1)^{j_{1}+\cdots+j_{N}} \\
& \times \zeta^{\star}\left(z_{-j_{1}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(1)}}\right) \zeta^{\star}\left(z_{-j_{2}}, \ldots, z_{0}, \ldots, z_{p_{\tau^{\prime}(2)}}\right) \ldots \zeta^{\star}\left(z_{-j_{N-1}}, \ldots, z_{0}, \ldots, z_{\left.p_{\tau^{\prime}(N-1)}\right)}\right) \\
& \times \zeta^{\star}\left(z_{-j_{N}}, \ldots, z_{0}, \ldots, z_{p_{h}}\right) \\
& \times \zeta\left(z_{-j_{1}-1}, \ldots, z_{-q_{1}}\right) \zeta\left(z_{-j_{2}-1}, \ldots, z_{-q_{2}}\right) \ldots \zeta\left(z_{-j_{N}-1}, \ldots, z_{-q_{N}}\right) .
\end{aligned}
$$

Since $\left\{\tau^{\prime} \in \mathfrak{S}_{N} \mid \tau \in \mathfrak{S}_{N-1}\right\}=\left\{\sigma \in \mathfrak{S}_{N} \mid \sigma(N)=h\right\}$, summing over $h$ we have

$$
\begin{equation*}
\sum_{h=1}^{N}(-1)^{h+N} \sum_{\tau \in \mathfrak{S}_{N-1}} \operatorname{sgn}(\tau)=\sum_{\sigma \in \mathfrak{S}_{N}} \operatorname{sgn}(\sigma) . \tag{4.5}
\end{equation*}
$$

This leads to the theorem.
Remark 3 If we use (3.3) in Theorem 3.1 instead of (3.2), we obtain an alternative expression of $\zeta_{\lambda}(\boldsymbol{s})$, where the roles of $\zeta$ and of $\zeta^{\star}$ are reversed.

Next, as a generalization of (3.10), we prove the following theorem, which is an expression in terms of modified zeta-functions of root systems.

Theorem 4.2 For the symmetric group $\mathfrak{S}_{N}$, we have

$$
\begin{aligned}
\zeta_{\lambda}(\boldsymbol{s})= & \sum_{m_{11}, m_{22}, \ldots, m_{N N} \geq 1}\left(m_{11} \ldots m_{N N}\right)^{-z_{0}} \\
& \times \sum_{\sigma \in \mathfrak{S}_{N}} \operatorname{sgn}(\sigma) \prod_{k=1}^{N} \zeta_{p_{k}, p_{k}}^{\bullet H}\left(\boldsymbol{s}_{+(k)}, m_{\sigma(k) \sigma(k)}, A_{p_{k}}\right) \prod_{j=1}^{N} \zeta_{q_{j}}^{H}\left(\boldsymbol{s}_{-(j)}, m_{j j}, A_{q_{j}}\right),
\end{aligned}
$$

where $\boldsymbol{s}_{+(k)}=\left(z_{1}, z_{2}, \cdots, z_{p_{k}}\right)$ and $\boldsymbol{s}_{-(j)}=\left(z_{-1}, z_{-2}, \cdots, z_{-q_{j}}\right)$.
Proof. We use the induction for $N$. For $N=1$, the assertion is true from (3.10). Assume that it is true for $N-1$ and apply it to (4.21). Then

$$
\begin{align*}
\Delta_{h N} & =\sum_{m_{11}, m_{22}, \ldots, m_{(N-1)(N-1)} \geq 1}\left(m_{11} \ldots m_{(N-1)(N-1)}\right)^{-z_{0}} \\
& \times \sum_{\tau \in \mathfrak{G}_{N-1}} \operatorname{sgn}(\tau) \prod_{k=1}^{N-1} \zeta_{p_{k}^{\prime}, p_{k}^{\prime}}^{\bullet}\left(s_{+\left(k^{\prime}\right)}, m_{\tau(k) \tau(k)}, A_{p_{k}^{\prime}}\right) \prod_{j=1}^{N-1} \zeta_{q_{j}}^{H}\left(s_{-(j)}, m_{j j}, A_{q_{j}}\right), \tag{4.6}
\end{align*}
$$

where

$$
k^{\prime}=\left\{\begin{array}{ll}
k & k \leq h-1 \\
k+1 & k \geq h,
\end{array} \quad p_{k}^{\prime}= \begin{cases}p_{k} & k \leq h-1 \\
p_{k+1} & k \geq h\end{cases}\right.
$$

Dividing the first product in the right-hand side into two parts, we have

$$
\begin{align*}
\Delta_{h N} & =\sum_{m_{11}, m_{22}, \ldots, m_{(N-1)(N-1)} \geq 1}\left(m_{11} \ldots m_{(N-1)(N-1)}\right)^{-z_{0}} \\
& \times \sum_{\tau \in \mathfrak{S}_{N-1}} \operatorname{sgn}(\tau) \prod_{k=1}^{h-1} \zeta_{p_{k}, p_{k}}^{\bullet, H}\left(\boldsymbol{s}_{+(k)}, m_{\tau(k) \tau(k)}, A_{p_{k}}\right) \prod_{k=h+1}^{N} \zeta_{p_{k}, p_{k}}^{\bullet, H}\left(\boldsymbol{s}_{+(k)}, m_{\tau(k-1) \tau(k-1)}, A_{p_{k}}\right) \\
& \times \prod_{j=1}^{N-1} \zeta_{q_{j}}^{H}\left(\boldsymbol{s}_{-(j)}, m_{j j}, A_{q_{j}}\right) \tag{4.7}
\end{align*}
$$

Substituting (4.7) and

$$
\zeta_{h, N}=\sum_{m_{N N} \geq 1} m_{N N}^{-z_{0}} \zeta_{p_{h}, p_{h}}^{\bullet, H}\left(\boldsymbol{s}_{+(h)}, m_{N N}, A_{p_{h}}\right) \zeta_{q_{N}}^{H}\left(\boldsymbol{s}_{-(N)}, m_{N N}, A_{q_{N}}\right)
$$

(which follows from (3.10)) into (4.1), we have

$$
\begin{aligned}
\zeta_{\lambda}(\boldsymbol{s})= & \sum_{h=1}^{N}(-1)^{h+N} \sum_{m_{11}, m_{22}, \ldots, m_{N N} \geq 1}\left(m_{11} \ldots m_{N N}\right)^{-z_{0}} \prod_{j=1}^{N} \zeta_{q_{j}}^{H}\left(\boldsymbol{s}_{-(j)}, m_{j j}, A_{q_{j}}\right) \\
& \times \sum_{\tau \in \mathfrak{G}_{N-1}} \operatorname{sgn}(\tau) \prod_{k=1}^{h-1} \zeta_{p_{k}, p_{k}}^{\bullet, H}\left(\boldsymbol{s}_{+(k)}, m_{\tau(k) \tau(k)}, A_{p_{k}}\right) \zeta_{p_{h}, p_{h}}^{\bullet, \boldsymbol{p}_{+(h)}}\left(\boldsymbol{s}_{+( }, m_{N N}, A_{p_{h}}\right) \\
& \times \prod_{k=h+1}^{N} \zeta_{p_{k}, p_{k}}^{\bullet, H}\left(\boldsymbol{s}_{+(k)}, m_{\tau(k-1) \tau(k-1)}, A_{p_{k}}\right)
\end{aligned}
$$

Again noting (4.5), we obtain the theorem.

Remark 4 The right-hand side of (4.6) also gives an analogue of Weyl group multiple Dirichlet series in the sense of Bump ([B]); compare with Remark 1 ,

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## Kohji Matsumoto

Graduate School of Mathematics,
Nagoya University,
Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan
kohjimat@math.nagoya-u.ac.jp
Maki Nakasuji
Department of Information and Communication Science, Faculty of Science, Sophia University,
7-1 Kio-cho, Chiyoda-ku, Tokyo, 102-8554, Japan
nakasuji@sophia.ac.jp
and
Mathematical Institute,
Tohoku University,
Sendai 980-8578, Japan


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