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EXPRESSIONS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS BY STANDARD FUNCTIONS

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Abstract. The concepts associated with conditions of power series expressions by finite sum of standard functions are introduced in this paper. This method of calculation can be used for solving ordinary differential equations when solutions are expressed by power series. The practical formulas of calculation and some examples are presented in this paper.

Key words: differential equations, standard functions

1. Introduction

Various dynamical systems can be described by ordinary differential equations. The solutions of those equations can be expressed by series with coefficients–functions of initial conditions and parameters using the operator method [2].

These series are changed to polynomials with optional order in the case of a computer realization. It is possible to approximate these polynomials by sum of exponential functions if solutions are periodical or aperiodical functions [3].

The conditions of expression by sum of trigonometric functions and the algorithm of unessential terms subject elimination to precision are formulated in this paper. Few new concepts dedicated to theory of expression of series by finite sum of standard functions are proposed. The given methodology is applied for a system of ordinary differential equations describing transformation of self-vibrations to the rotational motion.

2. The General Part

The new concepts and theorems used to define conditions of series expression by finite sum of standard functions and applied in practical calculation are presented in this part.

DEFINITION 1. A set of numbers $\lambda_1, \lambda_2, ..., \lambda_n \in C$, when $n \in N$ is called the *Van-der-Mond set* (V-set), if it satisfies a condition $\lambda_k \neq \lambda_r$ when $k \neq r$.

With every V-set it is possible to construct a determinant of Van-der-Mond not equal to zero:

$$V_n(\lambda_1, \lambda_2, ..., \lambda_n) = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \cdot ... \cdot (\lambda_n - \lambda_1) \\ \times (\lambda_n - \lambda_2) \cdot ... \cdot (\lambda_n - \lambda_{n-1}).$$

It follows that that $V_1(\lambda_1) = 1$.

Let $(p_j; j \in Z)$ be a sequence of complex numbers. Then for every $m \in N$ and fixed $j_0 \in Z_0$ it is possible to construct the following matrices:

$$H_{j_0}^{(m)} = \left\| \begin{matrix} p_{j_0} & p_{j_0+1} & \cdots & p_{j_0+m-1} \\ p_{j_0+1} & p_{j_0+2} & \cdots & p_{j_0+m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{j_0+m-1} & p_{j_0+m} & \cdots & p_{j_0+2m-2} \end{matrix} . \right\|,$$

$$\bar{H}_{j_0}^{(m)}(\rho) = \left\| \begin{matrix} p_{j_0} & p_{j_0+1} & \cdots & p_{j_0+m} \\ p_{j_0+1} & p_{j_0+2} & \cdots & p_{j_0+m+1} \\ \cdots & \cdots & \cdots & \cdots \\ p_{j_0+m-1} & p_{j_0+m} & \cdots & p_{j_0+2m-1} \\ 1 & \rho & \cdots & \rho^m \end{matrix} \right\|.$$

DEFINITION 2. If for a sequence of complex numbers $(p_j; j \in Z)$ there exists a number r_0 such that condition $r_0 = \max_{\substack{m \in N \\ j_0 \in Z_0}} \operatorname{rang} H_{j_0}^{(m)}$ is satisfied, then sequence $(p_j; j \in Z)$ has H-rank r_0. This rank is denoted $H - rang\,(p_j; j \in Z) = r_0$.

Example 1. It is easy to see that:

$$H - rang\,(a_0 + jd; j \in Z) = 2, \quad \text{if } d \neq 0,$$
$$H - rang\,\left(j^2; j \in Z\right) = 3.$$

It is possible to demonstrate that

$$H - rang\,\left(a_l j^l + a_{l-1} j^{l-1} + ... + a_0; j \in Z\right) = l + 1, \quad a_l \neq 0, l \in N.$$

It follows directly that $H - rang\,(a_0, a_1, ...a_n, 0, 0, ...) = n + 1$, when $a_n \neq 0$. Also, by convention $H - rang\,(0, 0, ...) = 0$. The sequence $(j!; j \in Z_0)$ does not have H-rang, because $\det H_m^{(j)} \neq 0$ for all $m \in N$ and $j \in Z_0$.

The following corollaries are obtained directly from the definition of H-rang:

Corollary 1. Let $q_j = p_{j+n}$, when $n \in N$ is fixed, besides, $m_1 < m_2$. Then we get:

$$H - rang\,(p_j; j \in Z_0) \geq H - rang\,(q_j; j \in Z_0)\,, rangH_{j_0}^{(m_1)} \leq rangH_{j_0}^{(m_2)}.$$

Corollary 2. If $H - rang\,(p_j; j \in Z_0) = r_0$ then

$$\det H_0^{(r_0)} \neq 0, \;\; \det H_j^{(r_0+1+n)} \equiv 0, \; j, n \in Z_0.$$

So, $H - rang\,(p_j; j \in Z_0) = \max\limits_{m \in N} rangH_0^{(m)}$.

Corollary 3. The given sequence $(p_j; j \in Z_0)$ has H-rang, satisfying coincidence $H - rang\,(p_j; j \in Z_0) = m, m \in Z_0$, if and only if, when there exist constants $A_0, A_1, ..., A_{m-1} \in C$, which are independent on j and satisfy condition

$$A_0 p_j + A_1 p_{j+1} + ... + A_{m-1} p_{j+m-1} = p_{j+m} \tag{2.1}$$

for all values of $j \in Z_0$. Equality (2.1) is not possible for m', when $m' < m$.

Lemma 1. *Let a sequence $(p_j; j \in Z_0)$ be given and*

$$p_j = \sum_{r=1}^{m} \mu_r \lambda_r^j, \tag{2.2}$$

where $\lambda_1, ..., \lambda_m$ is a V-set. The the following equalities are valid:

$$\det H_j^{(m)} = (\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_m)\,(\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_m)^j\, V_m^2\,(\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_m)\,,$$
$$\det \bar{H}_j^{(m)} = \det H_j^{(m)}\,(\rho - \lambda_1)\,(\rho - \lambda_2) \cdot ... \cdot (\rho - \lambda_m)\,.$$

Corollary 4. Note that $\det H_j^{(m)} \equiv 0$, when p_j is specified by (2.2) for all $j \in Z_0$ and $n = m+1, m+2$,.... Therefore, $H - rang\,(p_j; j \in Z) = m$.

Let

$$\binom{j}{k} := \begin{cases} 0, 0 \leq j < k; \\ \frac{j!}{k!(j-k)!}, k \leq j; \end{cases} \binom{j}{k} \cdot 0^{j-k} := \begin{cases} 0, j \neq k; \\ 1, k = j, \end{cases}, j, k \in Z_0. \tag{2.3}$$

Then using V-set $\lambda_1, \lambda_2 ..., \lambda_m$ it is possible to compose a sequence of numbers $(p_j; j \in Z_0)$ described by relationships

$$p_j := \sum_{r=1}^{n} \sum_{k_r=0}^{m_r-1} a_{rk_r} \binom{j}{k_r} \lambda_r^{j-k_r} \tag{2.4}$$

for all $m_1, m_2, ..., m_n \in N$ and $a_{rk_r} \in C$.

DEFINITION 3. The sequence of numbers specified by (2.4) set of Van-der-Mond ($V-set$) relationships is called *algebraic progression* and the coefficients $\lambda_1, \lambda_2 ..., \lambda_m$ are called *denominators*.

Let $\hat{\lambda}_1, \hat{\lambda}_2 ..., \hat{\lambda}_m \in C / \{0\}$ be a $V - set$. Then it is possible to compose a set of expressions: $\Delta_k^{(0)}(j) := \hat{\lambda}_k^j$,

$$\Delta_k^{(r+1)}(j) := \frac{\Delta_k^{(r)}(j) - \Delta_{k+1}^{(r)}(j)}{\hat{\lambda}_k - \hat{\lambda}_{k+r+1}}, \ r = 0, 1, ...; \ k = 1, 2, ..., n - r - 1.$$

Because $\Delta_k^{(r)}(j)$ can be expressed in the following way:

$$\Delta_k^{(r)}(j) = \mu_{k_0} \hat{\lambda}_k^{(j)} + \mu_{k_1} \hat{\lambda}_{k+1}^{(j)} + ... + \mu_{k_r} \hat{\lambda}_{k_r}^{(j)}$$

with coefficients $\mu_{k_0}, \mu_{k_1}, ..., \mu_{k_r}$ independent on j, the sequence

$$\left(\Delta_k^{(r)}(j); j \in Z_0\right), \ \ r = 1, 2, ..., m - k$$

is an algebraic progression. Besides, H-rang $\left(\Delta_k^{(r)}(j); j \in Z_0\right) = r + 1$. It can be observed that limits $\lim\limits_{\hat{\lambda}_k, \hat{\lambda}_{k+1}, ..., \hat{\lambda}_{k+r} \to \lambda_k} \Delta_k^{(r)}(j) = \binom{j}{r} \lambda_k^{j-r}$ exist when λ_k is any fixed complex number.

Using Lemma 1 and relationships (2.3) we get that H-rang of the sequence of numbers $\left(\binom{j}{r} \lambda_k^{j-r}; j \in Z_0\right)$ satisfies the following relationship: $H - rang \left(\binom{j}{r} \lambda_k^{j-r}; j \in Z_0\right) = r + 1$. From qualities of $\Delta_k^{(r)}(j)$ it is obtained that expression (2.4) is calculated using appropriate limit conversion from (2.2), i.e. after suitable choice of coefficients $\mu_1, \mu_2, ..., \mu_r$ which depend on denominators $\lambda_1, \lambda_2, ..., \lambda_n$:

$$p_j := \lim_{\substack{\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_{m_1} \to \lambda_1 \\ \cdots \\ \hat{\lambda}_{m_1 + ... + m_{n-1} + 1}, ..., \hat{\lambda}_{m_1 + ... + m_n} \to \lambda_n}} \sum_{r=1}^{m} \mu_r \hat{\lambda}_r^j = \sum_{r=1}^{n} \sum_{k_r=0}^{m_r - 1} a_{r k_r} \binom{j}{k_r} \lambda_r^{j - k_r}, j \in Z_0,$$

when $m_1 + \ldots + m_{n-1} = m$, besides, $\lambda_1, \lambda_2, ..., \lambda_n \in C$ is V-set. Then, after the use of limit conversation we get $H - rang\,(p_j; j \in Z_0) = m$.

Corollary 5. If term p_j of the algebraic progression $(p_j; j \in Z_0)$ is described by (2.4), then its Hankel's matrices $H_j^{(m)}$ and $\bar{H}_j^{(m)}$ satisfy the following transitions:

$$\det H_j^{(m)} = \sigma_m \left(\lambda_1^{m_1} \cdot \lambda_2^{m_2} \cdot ... \cdot \lambda_n^{m_n}\right)^j,$$
$$\det \bar{H}_j^{(m)} = \det H_j^{(m)} \cdot (\rho - \lambda_1)^{m_1} \cdot (\rho - \lambda_2)^{m_2} \cdot ... \cdot (\rho - \lambda_n)^{m_n},$$

where coefficient $\sigma_m = \det H_0^{(m)} \neq 0$.

Let note, that the equality $(\lambda_1^{m_1} \cdot \lambda_2^{m_2} \cdot ... \cdot \lambda_n^{m_n})^0 = 1$ is valid for all V-sets $\lambda_1, \lambda_2, ..., \lambda_n$.

Theorem 1. *If $a_{rm_{r-1}} \neq 0$ then H-rang of algebraic progression $(p_j; j \in Z_0)$ described by* (2.4) *satisfies the relationship*

$$H - rang\,(p_j; j \in Z_0) = m_1 + m_2 + ... + m_n.$$

Let a power series of real variable x

$$y\,(x) = \sum_{j=0}^{+\infty} p_j \frac{(x - x_0)^j}{j!}, |p_j| \leq M^j, 0 \leq M < +\infty \tag{2.5}$$

be given and it converges for all $x, x_0 \in R$, when x_0 is fixed.

Definition 4. A function specified by relationship (2.5) is called *algebraic function with finite rank*, if it is possible to express it by a finite sum

$$y\,(x) = \sum_{r=1}^{n} Q_r\,(x)\, e^{\lambda_r x}, \tag{2.6}$$

where $Q_r\,(x) = \sum\limits_{k_r=0}^{m_r-1} a_{rk_r} x^{k_r}, m_r = 1, 2, ...; a_{rk_r} \in C$ and $a_{rm_r-1} \neq 0$.

Theorem 2. *The power series, described by* (2.5)*, can be expressed by* (2.6) *if and only if the coefficients of series* (2.5) *compose an algebraic progression* $(p_j; j \in Z_0)$ *and* $H - rang\,(p_j; j \in Z_0) = m_1 + m_2 + ... + m_n$.

Corollary 6. It is possible to express algebraic function with finite rank $y\,(x)$ described by (2.2) as:

$$y\,(x) = \mu_0 + \sum_{r=1}^{m} \big(\mu_r \cos\left(\lambda_r\,(x - x_0)\right) + \gamma_r \sin\left(\nu_r\,(x - x_0)\right)\big),$$

if and only if the sequence $(p_j; j \in Z_0)$ of coefficients of power series (2.5) is algebraic progression satisfying relationships (2.2) and the real parts of its denominators are equal to zero.

3. Applications

The formulas for realization of the method and one applied example are presented in this section. Let the solution of some differential equation be given:

$$f\,(t) = \sum_{k=0}^{+\infty} p_k \frac{t^k}{k!}, \quad k = 0, 1, 2, ..., t \in R.$$

Our purpose is to express this series by a finite sum of exponential functions

$$f(t)=\sum_{r=1}^{m}\mu_r\exp(\lambda_r t), \tag{3.1}$$

where m is computed from the following relation

$$max\left(l:l\in N, detH_0^{(m)}\neq 0\right)=m. \tag{3.2}$$

Having m and using some methods of linear algebra we compute coefficients μ_r and λ_r of (3.1) from

$$det\bar{H}_0^{(m)}(\rho)=0,\quad p_k=\sum_{r=0}^{m}\lambda_r^k\mu_r,\quad k=0,1,2,\dots,m-1. \tag{3.3}$$

Suppose $\rho_1,\rho_2,\dots,\rho_m$ is V-set. Then coefficients $\lambda_k=\rho_k,\ k=1,2,\dots,m.$

Example 2. Let we have a solution of some differential equation

$$y(x)=2\sum_{k=0}^{+\infty}3^{k-1}\frac{x^{2k}}{(2k)!}+\sum_{k=0}^{+\infty}3^k\frac{x^{2k+1}}{(2k+1)!}.$$

Then $H-rang\,(p_j;j\in Z_0)=2$. After some calculations we get the expression

$$y(x)=\frac{2+\sqrt{3}}{6}\exp\left(\sqrt{3}x\right)+\frac{2-\sqrt{3}}{6}\exp\left(-\sqrt{3}x\right).$$

Example 3. Let a system of ordinary differential equations

$$\begin{cases}\varphi''-x''\sin\varphi+H\varphi'=0,\\ x''+p^2x-q_1x'+q_3x'^3-\mu\left(\varphi''\sin\varphi+\varphi'^2\cos\varphi\right)=0\end{cases} \tag{3.4}$$

be given. There H,p,q_1,q_3,μ are numerical parameters with values, providing autonomous vibration $x(t)$ and stimulating rotational motion $\varphi=\varphi(t)$ to be found. Here H is dissipative coefficient, p is self frequency of system, q_1 and q_{31} are parameters of self-vibration, μ is mass of rotor.

One of the main steady modes is when the rotor is rotating with uniform speed. For instance, the mode with a fixed speed of rotor rotation is obtained if use the following parameters:

$$H=0.05,\quad \mu=0.1,\quad p=1,\quad q_1=0.5,\quad q_3=1.75$$

and its initial conditions are

$$x(0)=x'(0)=0.1,\quad \varphi(0)=0.1,\varphi'(0)=1.$$

Using operator method [2] we obtain the solution in steady regime written in series form

$$\varphi'(t) = 0.7678 + 0.0531t + 0.2877t^2 - 0.0309t^3 - 0.0839t^4 + 0.0054t^5 + 0.0098t^6 - ...$$

After using formulas for expression of the solution by trigonometric functions described in [1] and similar to (3.3) we get the expression

$$\tilde{\varphi}'(t) = 0.9324 + 0.1646\cos 1.87t - 0.0283\sin 1.87t. \qquad (3.5)$$

There values of coefficients are given up to four exact numbers after comma.

It is possible to prove that the function described by relationship (3.1) is a solution of linear 2 or 3 order differential equation with constant coefficients. Then expression (3.5) can be interpreted as a linear part of system (3.4). This method allows us to pick the influence of higher harmonics on the solution and on that ground to prove the existence of domains of steady state regimes.

4. Conclusions

The new concepts describing functions and sequences of coefficients of series are proposed. The conditions defining a possibility to reduce a power series to a finite sum of standard functions are formulated. Formulas of calculation are described and a model of dynamical system is investigated as an example. This method can be applied for analysis and correctness of domains of steady regimes of differential equations as well as for the influence of the highest harmonics to the solution.

References

[1] L. Bikulčienė, R. Marcinkevičius and Z. Navickas. Adapted operator method for solving of ordinary differential equations. In: Vesna Lužar-Stiffler and Vesna Hljuz Dobrić(Eds.), *Proc. of 27th Intern. Conference Information Technology Interfaces ITI2005, Cavtat/Dubrovnik, Croatia, 2005*, Information Technology Interfaces, SRCE University Computing Centre, University of Zagreb, Zagreb, Croatia, 611 – 616, 2005.

[2] L. Bikulčienė, R. Marcinkevičius and Z. Navickas. Computer realization of the operator method for solving of differential equations. In: Lubin Vulkov Zhilin Li and Jerzy Waśnewski(Eds.), *Numerical Analysis and Its Applications. Third International Conference, NAA 2004*, Lecture Notes in Computer Science, Springer-Verlag, Berlin, 182 – 189, 2005.

[3] L. Bikulčienė and Z. Navickas. Solution of differential equations using sum of exponential functions. *Lithuanian Mathematical Journal*, **44**(1), 138 – 143, 2004.