EXTENDABILITY OF SIMPLICIAL MAPS IS UNDECIDABLE

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ABSTRACT. We present a short proof of the Čadek-Krčál-Matoušek-Vokřínek-Wagner result from the title (in the following form due to Filakovský-Wagner-Zhechev).

For any fixed even l there is no algorithm recognizing the extendability of the identity map of S^l to a PL map $X \to S^l$ of given $2l$-dimensional simplicial complex X containing a subdivision of S^l as a given subcomplex.

We also exhibit a gap in the Filakovský-Wagner-Zhechev proof that embeddability of complexes is undecidable in codimension > 1.

CONTENTS

1. EXTENDABILITY OF SIMPLICIAL MAPS IS UNDECIDABLE

We present short proofs of recent topological undecidability results for hypergraphs (complexes): Theorems 1.1 and 1.2 [CKM+, FWZ].

A **complex** $K = (V, F)$ is a finite set V together with a collection F of subsets of V such that if a subset σ is in F, then every subset of σ is in F.[1] (Hence $F \ni \emptyset$.) In an equivalent geometric language, a complex is a collection of closed faces (=subsimplices) of some simplex. A **k-complex** is a complex containing at most $(k+1)$-element subsets, i.e., at most k-dimensional simplices. Elements of V and of F are called **vertices** and **faces**.

The complete k-complex on n vertices (or the k-skeleton of the $(n-1)$-simplex) is the collection of all at most $(k+1)$-element subsets of an n-element set. For $k = 0$ we denote this complex by $[n]$, for $n = k+1$ by D^k (k-simplex or k-disk), and for $n = k+2$ by S^k (k-sphere).

The *subdivision of an edge* operation is shown in fig. 1. Exercise: represent the *subdivision of a face* operation shown in fig. 1 as composition of several subdivisions of an edge and inverse operations). A **subdivision** of a complex K is any complex obtained from K by several subdivisions of edges.

A **simplicial map** $f : (V, F) \to (V', F')$ between complexes is a map $f : V \to V'$ (not necessarily injective) such that $f(\sigma) \in F'$ for each $\sigma \in F$. A **piecewise-linear (PL) map** $K \to K'$ between complexes is a simplicial map between certain their subdivisions.

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[1]We do not use longer name 'abstract finite simplicial complex'. A *k-hypergraph* (more precisely, a $(k+1)$-uniform hypergraph) (V, F) is a finite set V together with a collection F of $(k+1)$-element subsets of V. In topology it is more traditional (because often more convenient) to work with complexes not hypergraphs. The following results are stated for complexes, although some of them are correct for hypergraphs.

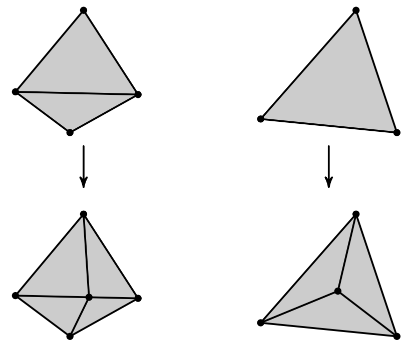

Figure 1. Subdivision of an edge (left) and of a face (right)

The **body** (or geometric realization) $|K|$ of a complex K is the union of simplices of K. Below we often abbreviate $|K|$ to K; no confusion should arise. A simplicial or PL map between complexes induces a map between their bodies, which is called *simplicial* or *PL*, respectively.[2]

The **wedge** $K_1 \vee \ldots \vee K_m$ of complexes $K_1 = (V_1, F_1), \ldots, K_m = (V_m, F_m)$ with disjoint vertices is the complex whose vertex set is obtained by choosing one vertex from each V_j and identifying chosen vertices, and whose of faces is obtained from $F_1 \sqcup \ldots \sqcup F_m$ by such identification. The choice of vertices is important in general, but is immaterial in the examples below. Let $K \vee K$ be the wedge of two copies of K.

Let $Y_l = S^l$ for l even and $Y_l = S^l \vee S^l$ for l odd.

Theorem 1.1 (retractability is undecidable)**.** *For any fixed integer $l > 1$ there is no algorithm recognizing the extendability of the identity map of Y_l to a PL map $X \to Y_l$ of given $2l$-complex X containing a subdivision of Y_l as a given subcomplex.*

This is implied [FWZ] by the following theorem and Proposition 1.8.b.

Let $V^d_m = S^d_1 \vee \ldots \vee S^d_m$ be the wedge of m copies of S^d.

Theorem 1.2 (extendability is undecidable)**.** *For some fixed integer m and any fixed integer $l > 1$ there is no algorithm recognizing extendability of given simplicial map $V^{2l-1}_m \to Y_l$ to a PL map $X \to Y_l$ of given $2l$-complex X containing a subdivision of V^{2l-1}_m as a given subcomplex.*

This is a 'concrete' version of [CKM+, Theorem 1.1.a].

Remarks and examples below are formally not used later.

Remark 1.3. (a) *Relation to earlier known results.* For $l > 1$ any PL map $S^1 \to Y_l$ extends to D^2. The analogues of Theorems 1.1 and 1.2 for Y_l replaced by a complex without this property (called *simply-connectedness*) were well-known by mid 20th century. See more in [CKM+, §1].

(b) *Why this text might be interesting.* Exposition of the proofs of Theorems 1.1 and 1.2 here is shorter and simpler than in [CKM+]. I structure the proof by explicitly stating the Brower-Hopf-Whitehead Theorems 1.5, 1.6, and Propositions 1.7, 1.8. Theorems 1.5 and 1.6 relate homotopy classification to quadratic functions on integers. Thus they allow to prove the equivalence of extendability / retractability to homotopy of certain maps, and to solvability of certain Diophantine equations, see Propositions 1.7 and 1.8. These results are essentially known before [CKM+] and are essentially deduced in [CKM+] from other known results. (As far as I know, they were not explicitly stated earlier, not even in [CKM+]; cf. [CKM+, §4.2] and [Sk21d, Remarks 2.1.b and 2.2.e].)

Also I present definitions in an economic way accessible to non-specialists (including computer scientists). In particular, I do not use cell complexes and simplicial sets.

[2]The related different notion of a *continuous* map between bodies of complexes is not required to state and prove the results of this text. In theorems below the existence of a continuous extension is equivalent to the existence of a PL extension (by the PL Approximation Theorem).

A reader might want to consider the proof below first for l even. Then he/she can omit parts (b,b') of Lemma 1.4, part (c) of Theorem 1.5, and parts (b1,b2) of Theorem 1.6.

Lemma 1.4. *(a) For some (fixed) integers m, s there is no algorithm that for given arrays $a = ((a^{i,j})_1, \dots, (a^{i,j})_m)$, $1 \le i < j \le s$, and $b = (b_1, \dots, b_m)$ of integers decides whether*

(SYM) there are integers $x_1, \dots, x_s$ such that

$$\sum_{1 \le i < j \le s} a_q^{i,j} x_i x_j = b_q \quad \text{for any} \quad 1 \le q \le m.$$

(b) Same as (a) for

(SKEW) there are integers $x_1, \dots, x_s, y_1, \dots, y_s$ such that

$$\sum_{1 \le i < j \le s} a_q^{i,j} (x_i y_j - x_j y_i) = b_q \quad \text{for any} \quad 1 \le q \le m.$$

(b') Same as (a) for the property (SKEW') obtained from (SKEW) by replacing $a_q^{i,j}$ with $2a_q^{i,j}$.

See [CKM+, §2] for deduction of (a,b) from insolvability of general Diophantine equations. Part (b') follows by (b) because either all b_q in (SKEW') are even or the system (SKEW') is unsolvable.

Denote by $\simeq$ homotopy between maps. For $n > 1$ we use *abelian group structure* on the set $\pi_n(X)$ of homotopy classes of PL maps $S^n \to X$. Let $u, v : S^l \vee S^l \to S^l$ be the contractions of the second and the first sphere of $S^l \vee S^l$.

Theorem 1.5. *For any integer l and simplicial map $\varphi : P \to Q$ between subdivisions of S^l there is an effectively constructible integer* $\deg \varphi$ *(called the* degree *of φ) such that*

(a) for any integer k there is an effectively constructible PL map $\widehat{k} : S^l \to S^l$ of degree k;

(b) for maps $\varphi, \psi : S^l \to S^l$ if $\deg \varphi = \deg \psi$, then $\varphi \simeq \psi$.

(c) for $l > 1$ and maps $\varphi, \psi : S^l \to S^l \vee S^l$ if $\deg(u \circ \varphi) = \deg(u \circ \psi)$ and $\deg(v \circ \varphi) = \deg(v \circ \psi)$, then $\varphi \simeq \psi$.

Sketch of a proof. Define $\deg \varphi$ by to be the sum of signs of a finite number of points from $\varphi^{-1} y$, where $y \in S^l$ is a 'random' (i.e. *regular*) value of φ. More precisely, take y outside the image of any $(l-1)$-simplex of P. For the definition of sign and the proof of (b) see e.g. [Ma03] or [Sk20, §8]. Part (c) is a simple case of the Hilton Theorem.

Clearly, deg defines a homomorphism $\pi_l(S^l) \to \mathbb{Z}$. Let $\widehat{1} := \mathrm{id}\, S^l$, let $\widehat{0}$ be the constant map, and let $\widehat{-1}$ be the reflection w.r.t. the equator $S^{l-1} \subset S^l$. Then for $k \ne 0$ let $\widehat{k}$ be a representative of the sum of $|k|$ summands $\widehat{\operatorname{sgn} k}$. $\square$

For a set $x = (x_1, \dots, x_s)$ of integers let $\widehat{x} : V_s^l \to S^l$ be the map whose restriction to S_j^l is $\widehat{x_j}$. Let $\lambda, \mu : S^l \to S^l \vee S^l$ be the inclusions into the first and the second sphere of the wedge.

Theorem 1.6 (proved in §2)**.** *For any integer a there exists an effectively constructible PL map $W_2(a) : S^{2l-1} \to S_1^l \vee S_2^l$ such that for any $l > 1$*

(a) for the composition $W(a) : S^{2l-1} \overset{W_2(a)}{\to} S_1^l \vee S_2^l \overset{\mathrm{id} \vee \mathrm{id}}{\to} S^l$ we have

(a1) for l even $W(a) \simeq W(a')$ only when $a = a'$;

(a2) $\widehat{(x_1, x_2)} \circ W_2(a) \simeq W(a x_1 x_2)$.

(b1) $W_2(a) \simeq W_2(a')$ only when $a = a'$;

(b2) for l odd $(\lambda \circ \widehat{(x_1, x_2)} + \mu \circ \widehat{(y_1, y_2)}) \circ W_2(2a) \simeq W_2(2a(x_1 y_2 - x_2 y_1))$, where the map $\lambda \circ \widehat{x} + \mu \circ \widehat{y} : S_1^l \vee S_2^l \to S^l \vee S^l$ is defined to be $\lambda \circ \widehat{x_j} + \mu \circ \widehat{y_j}$ on S_j^l.

Proposition 1.7. *Let $a = ((a^{i,j})_1, \dots, (a^{i,j})_m)$, $1 \le i < j \le s$, and $b = (b_1, \dots, b_m)$ be arrays of integers. There are effectively constructible PL maps*

$$W_s(a) : V^{2l-1}_m \to V^l_s \quad \text{and} \quad W(b) : V^{2l-1}_m \to S^l$$

such that the property (SYM) for even l, and the property (SKEW) for odd $l > 1$ and all $a^{i,j}_q$ even, is equivalent to

(LD) there is a PL map $\varkappa : V^l_s \to Y_l$ such that $\varkappa \circ W_s(a) \simeq \beta_l(b)$.

Here $\beta_l = W$ for l even, and $\beta_l = W_2$ for l odd.

Proposition 1.7 and Lemma 1.4.ab' imply that *homotopy left divisibility* is undecidable. See the right triangle of the left diagram below for $P = V^{2l-1}_m$, $Q = V^l_s$, $g = W_s(a)$, and $\beta = \beta_l(b)$.

$$\begin{tikzcd} & P \arrow[dl, "\subset"'] \arrow[d, "g"] \arrow[dr, "\beta"] & \\ \mathrm{Cyl}\, g \arrow[rr, dashed, bend right] & Q \arrow[l, "\subset"'] \arrow[r, dashed, "\varkappa"] & Y \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\subset"] \arrow[d, "g"'] & X \arrow[r, "\subset"] \arrow[dl, dashed] & X \cup_P \mathrm{Cyl}\, g \arrow[dll, dashed] \\ Q \arrow[rr, equal] & & Q \arrow[u, "\subset"'] \end{tikzcd}$$

Deduction of Proposition 1.7 from Theorem 1.6. Let

- $W^{i,j}_s(a^{i,j}_q)$ be the composition $S^{2l-1} \overset{W_2(a^{i,j}_q)}{\to} S^l_i \vee S^l_j \overset{\subset}{\to} V^l_s$;
- $W_s(a_q) : S^{2l-1} \to V^l_s$ be any PL map representing the sum of the maps $W^{i,j}_s(a_q)$;
- $W_s(a) : V^{2l-1}_m \to V^l_s$ be the map whose restriction to the q-th sphere is $W_s(a_q)$;
- $W(b) : V^{2l-1}_m \to S^l$ be the map whose restriction to the q-th sphere is $W(b_q)$.

By Theorem 1.6 and using $(\alpha_1 + \alpha_2) \circ \gamma = \alpha_1 \circ \gamma + \alpha_2 \circ \gamma$, for $l > 1$ we have

(a2s) $\widehat{x} \circ W_s(a) \simeq W(Q_x(a))$, where $Q_x(a) := \sum\limits_{1 \le i < j \le s} a^{i,j} x_i x_j$.

(b1s) $W_s(a) \simeq W_s(a')$ only when $a = a'$;

(b2s) for l odd $(\lambda \circ \widehat{x} + \mu \circ \widehat{y}) \circ W_s(2a) \simeq W_2(2R_{x,y}(a))$, where $R_{x,y}(a) := \sum\limits_{1 \le i < j \le s} a^{i,j}(x_i y_j - x_j y_i)$.

Proof that (SYM) $\Rightarrow$ (LD) for l even. Take an integer solution $x = (x_1, \dots, x_s)$. Let $\varkappa := \widehat{x}$. Then by (a2s) $\varkappa \circ W_s(a^q) \simeq W(Q_x(a^q)) = W(b_q)$ for each q. Thus $\varkappa \circ W_s(a) \simeq W(b)$.

Proof that (LD) $\Rightarrow$ (SYM) for l even. Take the PL map $\varkappa : V^l_s \to S^l$. Let $x_j := \deg(\varkappa|_{S^l_j})$. Then by Theorem 1.5.b $\varkappa \simeq \widehat{x}$. Take any q. Then by (a2s)

$$W(Q_x(a^q)) \simeq \widehat{x} \circ W_s(a^q) \simeq \varkappa \circ W_s(a^q) \simeq W(b_q).$$

Hence by (a1) of Theorem 1.6 $Q_x(a^q) = b_q$.

Proof that (SKEW) $\Rightarrow$ (LD) for $l > 1$ odd and all $a^q_{i,j}$ even. Take an integer solution $(x, y) = (x_1, \dots, x_s, y_1, \dots, y_s)$. Let $\varkappa := \lambda \circ \widehat{x} + \mu \circ \widehat{y}$. Then by (b2s) $\varkappa \circ W_s(a^q) \simeq W(R_{x,y}(a^q)) = W_2(b_q)$ for each q. Thus $\varkappa \circ W_s(a) \simeq W_2(b)$.

Proof that (LD) $\Rightarrow$ (SKEW) for $l >$ odd and all $a^q_{i,j}$ even. Let $x_j := \deg(u \circ \varkappa|_{S^l_j})$ and $y_j := \deg(v \circ \varkappa|_{S^l_j})$. Then by Theorem 1.5.c $\varkappa \simeq \lambda \circ \widehat{x} + \mu \circ \widehat{y}$. Take any q. Then by (b2s)

$$W_2(R_{x,y}(a^q)) \simeq (\lambda \circ \widehat{x} + \mu \circ \widehat{y}) \circ W_s(a^q) \simeq \varkappa \circ W_s(a^q) \simeq W_2(b_q).$$

Hence by (b1s) $R_{x,y}(a^q) = b_q$. $\square$

Proposition 1.8 (proved below in §1)**.** *For a simplicial map $g : P \to Q$ between complexes there is an effectively constructible triple* $(\mathrm{Cyl}\, g; P, Q)$ *of a complex* $\mathrm{Cyl}\, g$ *(called the* mapping cylinder *of g) and its subcomplexes isomorphic to P, Q such that*

(a) for any complex Y a simplicial map $\beta : P \to Y$ extends to $\mathrm{Cyl}\, g$ *if and only if there is a PL map $\varkappa : Q \to Y$ such that $\varkappa \circ g \simeq \beta$.*

(b) g extends to a complex $X \supset P$ if and only if the identity map of Q extends to $X \cup_P \mathrm{Cyl}\, g$;

Proof of the 'extendability is undecidable' Theorem 1.2. Take a simplicial subdivision of $\beta_l(b)$ as a given map, and $X = \mathrm{Cyl}\, W_s(a)$. Apply Propositions 1.7 and 1.8.a (in the latter take $P = V_m^{2l-1}$, $Q = V_s^l$, $Y = Y_l$, $g = W_s(a)$, and $\beta = \beta_l(b)$). We obtain that

• extendability of $\beta_l(b)$ to X is equivalent to (SYM) for l even;

• when all $a_q^{i,j}$ are even, extendability of $\beta_l(b)$ to X is equivalent to (SKEW) for l odd.

The latter is undecidable by Lemma 1.4.a,b'. □

Sketch of a construction of $\mathrm{Cyl}\, g$ *in Proposition 1.8.* For a map $f : P \to Q$ between subsets $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ define the *mapping cylinder* $\mathrm{Cyl}\, f$ to be the union of $0 \times Q \times 1 \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} = \mathbb{R}^{p+q+1}$ and segments joining points $(u, 0, 0) \in \mathbb{R}^{p+q+1}$ to $(0, f(u), 1) \in \mathbb{R}^{p+q+1}$, for all $u \in P$. See [CKM+, Figure in p. 14]. We identify P with $P \times 0 \times 0$ and Q with $0 \times Q \times 1$.

Define the map $\mathrm{ret}\, g : \mathrm{Cyl}\, g \to Q$ by mapping to $g(u)$ the segment containing $(u, 0, 0)$.

For a simplicial map $g : P \to Q$ between complexes denote by $|g| : |P| \to |Q|$ the corresponding PL map between their bodies. Then $\mathrm{Cyl}\, |g|$ is the body of certain complex

• whose vertices are the vertices of P and the vertices of Q;

• whose simplices are the simplices of P, the simplices of Q and another simplices that are not hard to define.

Example 1.9. *(a) For the 2-winding* $\widehat{2} : S^1 \to S^1$ *(i.e., for the quotient map* $S^1 \to \mathbb{R}P^1$*)* $\mathrm{Cyl}\, \widehat{2}$ *is the Möbius band (i.e. the complement to a 2-disk in* $\mathbb{R}P^2$*).*

(a') For the Hopf map $\eta : S^3 \to S^2$ *(i.e., for the quotient map* $S^3 \to \mathbb{C}P^1$*)* $\mathrm{Cyl}\, \eta$ *is the complement to a 4-ball in* $\mathbb{C}P^2$ *(i.e. the 'complexified' Möbius band).*

(b) For the commutator map $f : S^1 \to S^1 \vee S^1$ *(i.e.,* $f = aba^{-1}b^{-1}$*)* $\mathrm{Cyl}\, f$ *is the complement to a 2-disk in* $S^1 \times S^1$*.*

(b') The cylinder of the map $W(1) : S^{2l-1} \to S^l \vee S^l$ *is the complement to a* $2l$*-ball in* $S^l \times S^l$ *(this follows by footnote 5).*

Proof of Proposition 1.8. (a), 'only if'. Let $\varkappa$ be the restriction to $Q \subset \mathrm{Cyl}\, g$ of given extension.

(a), 'if'. Let the required extension be $\varkappa \circ \mathrm{ret}\, g$.

(b), 'only if'. Let the required extension be $\mathrm{ret}\, g$ on $\mathrm{Cyl}\, g$ and the given extension on X.

(b), 'if'. Let $r : X \cup_P \mathrm{Cyl}\, g \to Q$ be given extension. The composition $P \times [0,1] \to \mathrm{Cyl}\, g \to Q$ of the quotient map and r is a homotopy between $r|_P$ and g. Since $r|_P$ extends to X, by the Borsuk Homotopy Extension Theorem[3] it follows that g extends to X. □

Remark 1.10. Proposition 5.2 of [CKM+] asserts the equivalence of (SKEW) and extendability of $W_2(b)$ to $\mathrm{Cyl}\, W_s(a)$ (which follows by Propositions 1.7 and 1.8.a). The proof of Proposition 5.2 was not formally presented in [CKM+], it is written that the proposition follows from the text before. The phrase 'For this system, the above equation is exactly the one from (Q-SKEW)' before [CKM+, Proposition 5.2] is incorrect. Indeed, 'the above equation' is an equation in $\pi_{2k-1}(S^k \vee S^k)$ not in $\mathbb{Z}$ (and not in the direct summand $\pi_{2k-1}(S^{2k-1}) \cong \mathbb{Z}$ of $\pi_{2k-1}(S^k \vee S^k)$), so 'the above equation' is not 'exactly the one from (Q-SKEW)'. So the phrase 'We get the following:' before [CKM+, Proposition 5.2] is not justified. For its justification one needs to prove that multiplication by 2 of 'the system of s equations in $\pi_{2k-1}(S^k \vee S^k)$' produces an equivalent system. This is not so because the group $\pi_{2k-1}(S^k \vee S^k)$ (at one place denoted by $\pi_{2k-1}(S^d \vee S^d)$) can have elements of order 2 for some k. Thus the 'if' part of Proposition 5.2 is not proved in [CKM+]. This gap is easy to recover; e.g. it is recovered here.[4]

[3]This theorem states that if (K, L) is a polyhedral pair, $Q \subset \mathbb{R}^d$, $F : L \times I \to Q$ is a homotopy and $g : K \to Q$ is a map such that $g|_L = F|_{L \times 0}$, then F extends to a homotopy $G : K \times I \to Q$ such that $g = G|_{K \times 0}$.

[4]I am grateful to M. Čadek for confirming that [CKM+, Proposition 5.2] is incorrect but is easily correctible. If the (minor) gap would be recovered in the arXiv update of [CKM+], I would be glad to remove this footnote. I am also grateful to L. Vokřínek for the following explanation why Remark 1.10 is not proper: *There is a misprint in the statement of* [CKM+, Proposition 5.2]*; namely, the assumption that all coefficients* $a_{ij}^{(q)}$ *in* [CKM+, (Q-SKEW)]*=(SKEW) should be even is missing. With this assumption in place, I am not aware of*

2. Known proof of Theorem 1.6

Construction of $W_2(a)$. Decompose

$$S^{2l-1} = \partial(D^l \times D^l) = S^{l-1} \times D^l \cup_{S^{l-1}\times S^{l-1}} D^l \times S^{l-1}.$$

Define the *Whitehead map* $w : S^{2l-1} \to S^l \vee S^l$ as the 'union' of the compositions

$$S^{l-1} \times D^l \overset{\mathrm{pr}_2}{\to} D^l \overset{c}{\to} S^l \overset{\lambda}{\to} S^l \vee S^l \quad \text{and} \quad D^l \times S^{l-1} \overset{\mathrm{pr}_1}{\to} D^l \overset{c}{\to} S^l \overset{\mu}{\to} S^l \vee S^l.$$

Here pr_j is the projection onto the j-the factor, and c is contraction of the boundary to a point.[5] It is easy to modify this 'topological' definition to obtain an effectively constructible PL map w. Define $W_2(a)$ in the same way as w except that λ is replaced with $\lambda \circ \widehat{a}$.

Denote by lk the *linking coefficient* of two collections of oriented closed polygonal lines in $\mathbb{R}^3$, or, more generally, of two integer l-cycles in $\mathbb{R}^{2l-1}$. See definition e.g. in [ST80], [Sk, §4].

Sketch of a proof of (a1). For a PL map $\psi : S^{2l-1} \to S^l$ define $H(\psi) := \mathrm{lk}(\psi^{-1}y_1, \psi^{-1}y_2)$, where $y_1, y_2 \in S^l$ are distinct 'random' (or *regular*) values of ψ, and ψ^{-1} is 'oriented' preimage. More precisely, take subdivisions of S^{2l-1} and of S^l for which ψ is simplicial. Then take y_1, y_2 outside the image of any $(l-1)$-simplex of the subdivision of S^{2l-1}. This is a well-defined homotopy invariant of ψ (*Hopf invariant*).

For l even we have $HW(b) = \pm 2b$.[6] Hence $W(b) \simeq W(b')$ only when $b = b'$. □

Sketch of a proof of (b1). For a PL map $\psi : S^{2l-1} \to S^l \vee S^l$ define $H_\vee(\psi) := \mathrm{lk}(\psi^{-1}y_1, \psi^{-1}y_2)$, where $y_1 \in S^l \vee *$, $y_2 \in * \vee S^l$ are 'random' (or *regular*) values of ψ, and ψ^{-1} is 'oriented' preimage. More precisely, take subdivisions of S^{2l-1} and of $S^l \vee S^l$ for which ψ is simplicial. Then take y_1, y_2 outside the image of any $(l-1)$-simplex of the subdivision of S^{2l-1}. This is a well-defined homotopy invariant of ψ (*Whitehead invariant*).

Clearly, $H_\vee W_2(a) = \pm a$. Hence $W_2(a) \simeq W_2(a')$ only when $a = a'$. □

Sketch of a proof of (a2) and (b2). [7] For a complex X and maps $f, g : S^l \to X$ define a map $[f, g] : S^{2l-1} \to X$ in the same way as w except that $S^2 \vee S^2, \lambda, \mu$ are replaced by X, f, g. Then $w = [\lambda, \mu]$, $W_2(a) = [\lambda \circ \widehat{a}, \mu]$, and $W(b) = [\widehat{b}, \widehat{1}]$. This construction defines a map $[\cdot,\cdot] : \pi_l(X) \times \pi_l(X) \to \pi_{2l-1}(X)$ (called *Whitehead product*). For $l > 1$ we have

$$[\alpha_1 + \alpha_2, \gamma] = [\alpha_1, \gamma] + [\alpha_2, \gamma] \quad \text{and} \quad [\alpha, \gamma] = (-1)^l [\gamma, \alpha].$$

Then

$$(\widehat{x}_1 \vee \widehat{x}_2) \circ W_2(a) \simeq [\widehat{x}_1 \circ \widehat{a}, \widehat{x}_2] \simeq W(ax_1x_2).$$

For l odd denoting the homotopy class of a map by the same letter as the map we have

$$(\lambda \circ \widehat{x} + \mu \circ \widehat{y}) \circ W_2(a) \overset{(1)}{=} a[x_1 \lambda + y_1 \mu, x_2 \lambda + y_2 \mu] \overset{(2)}{=} a[x_1 \lambda, y_2 \mu] + a[y_1 \mu, x_2 \lambda] \overset{(3)}{=} W_2(a(x_1y_2 - x_2y_1)).$$

Here equalities (1), (2), (3) hold because $(\lambda \circ \widehat{x} + \mu \circ \widehat{y})|_{S^l_j} = x_j \lambda + y_j \mu$, because $[\lambda, \lambda] = -[\lambda, \lambda]$, $[\mu, \mu] = -[\mu, \mu]$, and a is even,[8] and because $[\mu, \lambda] = -[\lambda, \mu]$, respectively. □

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any gap. Remark 1.10 is incorrect in assuming that [CKM+] *uses multiplication by 2 in the homotopy group* $\pi_{2k-1}(S^k \vee S^k)$*; instead, the system of equations* [CKM+, (Q-SKEW)]*=(SKEW) with values in* $\mathbb{Z}$ *gets multiplied.* This does not show that Remark 1.10 is not proper, because Remark 1.10 concerns only the text [CKM+], not any other non-existent text, cf. [Sk21d, Remark 2.3.d]. The correction suggested by L. Vokřínek is proper; a list of this and all the induced corrections to [CKM+] would be helpful (or current lack of such a list is helpful) to see how proper is to call this a misprint. Cf. [Sk21d, Remark 2.3.abc].

[5] Observe that $S^l \times S^l \cong D^{2l}/\sim$, where $x \sim y \Leftrightarrow \big(x, y \in S^{2l-1}$ and $w(x) = w(y)\big)$.

[6] For l odd we have $H(\psi) = 0$ for any ψ.

[7] For $l = 2$ the equality (a2) alternatively follows because using the definition of the degree and simple properties of linking coefficients, we see that $H((\widehat{x}_1 \vee \widehat{x}_2) \circ W_2(a)) = 2ax_1x_2$, and because the Freudenthal-Pontryagin Theorem states that *if $H(\varphi) = H(\psi)$ for maps $\varphi, \psi : S^3 \to S^2$, then $\varphi \simeq \psi$.*

[8] For $l \in \{3, 7\}$ the equality (b2) holds for $2a$ replaced by a, because $W(1) = [\widehat{1}, \widehat{1}]$ is null-homotopic.

3. Appendix: is embeddability of complexes undecidable in codimension > 1?

Realizability of hypergraphs or complexes in the d-dimensional Euclidean space $\mathbb{R}^d$ is defined similarly to the realizability of graphs in the plane. E.g. for 2-complex one 'draws' a triangle for every three-element subset. There are different formalizations of the idea of realizability.

A complex (V, F) is **simplicially** (or linearly) **embeddable** in $\mathbb{R}^d$ if there is a set V' of distinct points in $\mathbb{R}^d$ corresponding to V such that for any subsets $\sigma, \tau \subset V'$ corresponding to elements of F the convex hull $\langle\sigma\rangle$ is a simplex of dimension $|\sigma| - 1$ and $\langle\sigma\rangle \cap \langle\tau\rangle = \langle\sigma \cap \tau\rangle$.

A complex is **PL** (piecewise linearly) **embeddable** in $\mathbb{R}^d$ if some its subdivision is simplicially embeddable in $\mathbb{R}^d$.[9]

For classical and modern results on embeddability and their discussion see e.g. surveys [Sk06], [Sk18, §3], [Sk, §5].

Theorem 3.1 (embeddability is undecidable in codimension 1)**.** *For every fixed d, k such that $5 \le d \in \{k, k+1\}$ there is no algorithm recognizing PL embeddability of k-complexes in $\mathbb{R}^d$.*

This is deduced in [MTW, Theorem 1.1] from the Novikov theorem on unrecognizability of the d-sphere. Cf. [NW97, Remark 3].

Conjecture 3.2 (embeddability is undecidable in codimension > 1)**.** *For every fixed d, k such that $8 \le d \le \frac{3k+1}{2}$ there is no algorithm recognizing PL embeddability of k-complexes in $\mathbb{R}^d$.*

Conjecture 3.2 easily follows from its 'extreme' case $2d = 3k+1 = 6l+4$ [FWZ, Corollaries 4 and 6]. The extreme case is implied by the equivalence $(SKEW) \Leftrightarrow (Em)$ of Conjecture 3.14 below.[10]

Conjecture 3.2 is stated as a theorem in [FWZ]. The proof in [FWZ] contains a gap described below. Their idea is to elaborate the following remark to produce the reduction (described below) to the 'retractability is undecidable' Theorem 1.1.

Remark 3.3. Homotopy classifications of maps $S^{2l-1} \to S^l$ and $S^{2l-1} \to S^l \vee S^l$ are related to isotopy classification of links of $S^{2l-1} \sqcup S^{2l-1}$ and of $S^{2l-1} \sqcup S^{2l-1} \sqcup S^{2l-1}$ in $\mathbb{R}^{3l}$ [Ha62l] (including higher-dimensional Whitehead link and Borromean rings [Sk06, §3]). E.g. the *generalized linking coefficients* of the Whitehead link and of the Borromean rings are (the homotopy classes) of the Whitehead maps $W(1) : S^{2l-1} \to S^l$ and $W_2(1) : S^{2l-1} \to S^l \vee S^l$ from Theorem 1.6. Analogous results for $l = 1$ do illustrate some ideas, see a description accessible to non-specialists in [Sk20, §3.2].

We use the notation of §1. In this section $a = ((a^{i,j})_1, \dots, (a^{i,j})_m)$, $1 \le i < j \le s$, and $b = (b_1, \dots, b_m)$ are arrays of integers. Define the *double mapping cylinder* $X(a, b)$ to be the union of $\mathrm{Cyl}\, W_s(a)$ and $\mathrm{Cyl}\, W_2(b) \supset Y$, in which $V_m^{2l-1} \subset \mathrm{Cyl}\, W_s(a)$ is identified with $V_m^{2l-1} \subset \mathrm{Cyl}\, W_2(b)$.

Assume that $S^{2l+1} \vee S^{2l+1}$ is standardly embedded into S^{3l+2}. Take a small oriented $(l+1)$-disks $D_+, D_- \subset S^{3l+2}$

- intersecting at a point in $\partial D_+ \cup \partial D_-$;
- whose intersections with $S^{2l+1} \vee S^{2l+1}$ are transversal and consist of exactly one point $D_+ \cap (S^{2l+1} \vee S^{2l+1}) \in S^{2l+1} \vee *$ and $D_- \cap (S^{2l+1} \vee S^{2l+1}) \in * \vee S^{2l+1}$.

Define *the meridian $\Sigma^l \vee \Sigma^l$ of $S^{2l+1} \vee S^{2l+1}$ in S^{3l+2}* to be $\partial D_+ \cup \partial D_-$.

Conjecture 3.4. *For any odd integer l and all $a_q^{i,j}$ even there is a $(2l+1)$-complex $G \supset S^l \vee S^l$ such that any of the following properties is equivalent to (SKEW):*

[9] The related different notion of being topologically embeddable is not required in this text.

[10] The extreme case is also implied by the equivalence between $(SKEW1)$ of Conjecture 3.11.a and the analogue of $(Em2)$ from Conjecture 3.14 for 'almost embedding' replaced by 'embedding'. The extreme case for l even is also implied by the equivalence between $(SYM1)$ of Conjecture 3.11.b and the analogue of $(Em1)$ from Conjecture 3.13 for 'almost embedding' replaced by 'embedding'.

(Ex) a PL homeomorphism of $S^l \vee S^l \to \Sigma^l \vee \Sigma^l$ of $S^{2l+1} \vee S^{2l+1}$ in S^{3l+2} extends to a PL map $X(a,b) \to S^{3l+2} - (S^{2l+1} \vee S^{2l+1})$.

(Ex') a PL homeomorphism of $S^l \vee S^l \to \Sigma^l \vee \Sigma^l$ extends to a PL embedding $X(a,b) \to S^{3l+2} - (S^{2l+1} \vee S^{2l+1})$.

(Em) $X(a,b) \cup_{S^l \vee S^l} G$ embeds into S^{3l+2}.

All the implications except $(Em) \Rightarrow (Ex')$ are correct results of [FWZ].

The implication $(Ex') \Rightarrow (Ex)$ is clear.

The equivalence of (Ex) and (SKEW) follows by Propositions 1.7 and 1.8.ab because there is a *strong deformation retraction* $S^{3l+2} - (S^{2l+1} \vee S^{2l+1}) \to \Sigma^l \vee \Sigma^l$.

The implication $(Ex) \Rightarrow (Ex')$ is implied by the following version of *the Zeeman-Irwin Theorem* [Sk06, Theorem 2.9].

Lemma 3.5. *For any PL map $f : X(a,b) \to S^{3l+2} - (S^{2l+1} \vee S^{2l+1})$ there is a PL embedding $f' : X(a,b) \to S^{3l+2} - (S^{2l+1} \vee S^{2l+1})$ such that the restrictions of f and f' to $S^l \vee S^l \subset X(a,b)$ are homotopic.*

The idea of [FWZ] to prove the implication $(Em) \Rightarrow (Ex')$ is to construct the complex G, and use a modification of the following Lemma 3.6.

Lemma 3.6 ([SS92, Lemma 1.4]). *For any integers $0 \le l < k$ there is a k-complex F_- containing subcomplexes $\Sigma^k \cong S^k$ and $\Sigma^l \cong S^l$, PL embeddable into $\mathbb{R}^{k+l+1}$ and such that for any PL embedding $f : F_- \to \mathbb{R}^{k+l+1}$ the images $f\Sigma^k$ and $f\Sigma^l$ are linked modulo 2.*

Lemma 30 of [FWZ] is a modification of Lemma 3.6 with 'linked modulo 2' replaced by 'linked with linking coefficient ± 1'. The proof of [FWZ, p. 778, end of proof of Lemma 30] used the following incorrect statement: *If $f : D^p \to \mathbb{R}^{p+q}$ and $g : S^q \to \mathbb{R}^{p+q}$ are PL embeddings such that $|f(D^p) \cap g(D^q)| = 1$, then the linking coefficient of $f|_{S^{p-1}}$ and g is ± 1.*

Example 3.7. *For any integers $p, q \ge 2$ and c there are PL embeddings $f : D^p \to \mathbb{R}^{p+q}$ and $g : S^q \to \mathbb{R}^{p+q}$ such that $|f(D^p) \cap g(S^q)| = 1$ and the linking coefficient of $f|_{S^{p-1}}$ and g is c.*

Proof. Take PL embeddings $f_0 : S^{p-1} \to \mathbb{R}^{p+q-1}$ and $g_0 : S^{q-1} \to \mathbb{R}^{p+q-1}$ whose linking coefficient is c. Take points $A, B \in \mathbb{R}^{p+q} - \mathbb{R}^{p+q-1}$ on both sides of $\mathbb{R}^{p+q-1}$. Then $f = f_0 * A$ and $g = g_0 * \{A, B\}$ are the required embeddings. $\square$

The modification [FWZ, Lemma 30] of Lemma 3.6 is presumably incorrect.

Theorem 3.8 ([KS20, Theorem 1.6]). *For any integers $1 < l < k$ and z there is a PL almost embedding $f : F_- \to \mathbb{R}^{k+l+1}$ such that the linking coefficients of oriented $f\Sigma^k$ and $f\Sigma^l$ is $2z+1$.*

Remark 3.9. (a) Lemma 3.5 is essentially a restatement of [FWZ, Theorem 10] accessible to non-specialists. Analogous lemma for $X(a,b)$ replaced by $2l$-dimensional $(l-2)$-connected manifold is (a particular case of) the Zeeman-Irwin Theorem. The required modification of the Zeeman-Irwin proof is not hard. It is based on a version of engulfing similar to [Sk98, §2.3] (such a version was possibly suggested by C. Zeeman to C. Weber [We67, §2, the paragraph before remark 1]).

(b) Proposition 34 of [FWZ] is a detailed general position argument for the following statement: *If Z is a subcomplex of a complex X and $2\dim Z < d$, then any PL map of X to a PL d-manifold is homotopic to a PL map the closure of whose self-intersection set misses Z.* (This should be known, at least in folklore, but I do not immediately see a reference.)

(c) Lemma 41 of [FWZ] is a version of the following theorem: *Any PL map of $S^n \times I$ to an $(2n+3-m)$-connected m-manifold Q is homotopic to a PL embedding* (this is a particular case of [Hu69, Theorem 8.3]). The novelty of [FWZ, Lemma 41] is the property $S(g_1) \subset S(g)$. This property is not checked in [FWZ, proof Lemma 41] but does follow from $C \cap g(\mathrm{Cl}(A \times [0,1] - \sigma)) = g(\widetilde{I})$; the latter holds because of the 'metastable dimension restriction' $2(3l+2) \ge 3(2l+1)$.

(d) In the proof of [FWZ, Lemma 42] the property $S(g_1) \subset S(g)$ is not checked. This property ensures that we can make new improvements without destroying the older ones. Cf. [Sk98, line 5 after the display formula in p. 2468]. This property presumably holds because of the 'metastable dimension restriction' $2(3l+2) \geq 3(2l+1)$.

A PL map $g : K \to \mathbb{R}^d$ of a complex K is called an **almost embedding** if $g\alpha \cap g\beta = \emptyset$ for any two disjoint simplices $\alpha, \beta \subset K$.

Conjecture 3.10 (almost embeddability is undecidable)**.** *For every fixed d, k such that*
(a) $5 \leq d \in \{k, k+1\}$;   (b) $8 \leq d \leq \frac{3k+1}{2}$
there is no algorithm recognizing almost embeddability of k-complexes in $\mathbb{R}^d$.

Conjecture 3.10 easily follows from its 'extreme' case $2d = 3k+1 = 6l+4$ analogously to [FWZ, Corollaries 4 and 6]. The extreme case for l even is implied by the equivalence $(SYM1) \Leftrightarrow (Em1)$ of the following Conjectures 3.11.b and Proposition 3.13. The extreme case for any l is implied by the equivalence $(SKEW1) \Leftrightarrow (Em2)$ of the following Conjectures 3.11.a and 3.14.

Conjecture 3.11. *(a) For some fixed integers m, s there is no algorithm which for given arrays $a = (a_q^{i,j})$, $1 \leq i < j \leq s$, $1 \leq q \leq m$ and $b = (b_1, \dots, b_m)$ of integers decides whether*
(SKEW1) there are integers $x_1, \dots, x_s, y_1, \dots, y_s, z$ such that

$$\sum_{1 \leq i < j \leq s} a_q^{i,j}(x_i y_j - x_j y_i) = (2z+1)b_q, \quad 1 \leq q \leq m.$$

(b) For some fixed integers m, s there is no algorithm which for given arrays $a = (a_q^{i,j})$, $1 \leq i < j \leq s$, $1 \leq q \leq m$, and $b = (b_1, \dots, b_m)$ of integers decides whether
(SYM1) there are integers $x_1, \dots, x_s, z$ such that

$$\sum_{1 \leq i < j \leq s} a_q^{i,j} x_i x_j = (2z+1)b_q, \quad 1 \leq q \leq m.$$

Remark 3.12. B. Moroz conjectured and E. Kogan sketched a proof that Conjecture 3.11.a is equivalent to:

(*) for some fixed positive integers m, s there is no algorithm which for a given system of m Diophantine equations in s variables decides whether the system has a solution in rational numbers with odd denominators.

Since m equations are equivalent to 1 equation (sum of squares) and since work of J. Robinson characterizes the rational numbers with odd denominators among all rational numbers in a Diophantine way, (*) is in turn is equivalent to:

(**) for some fixed positive integer s there is no algorithm which for a given polynomial equation with integer coefficients in s variables decides whether the system has a solution in rational numbers.

The statement (**) is an open problem.

An **odd (almost) embedding** is a PL (almost) embedding $f : S^l \to S^{3l+2} - S^{2l+1}$ such that $f(S^l)$ is linked modulo 2 with S^{2l+1}.

Proposition 3.13. *For any even l there is a $(2l+1)$-complex $G_1 \supset S^l$ such that any of the following properties is equivalent to (SYM1):*
(Ex1) some odd almost embedding extends to a PL map of $X(a, b)$.
(Ex'1) some odd almost embedding extends to a PL embedding of $X(a, b)$.
(Em1) $X(a, b) \cup_{S^l} G_1$ embeds into S^{3l+2}.

All the implications except $(Em1) \Rightarrow (Ex'1)$ (and their analogues for 'almost embedding' replaced by 'embedding') are proved analogously to the corresponding correct implications of

Conjecture 3.4. The implication $(Em1) \Rightarrow (Ex'1)$ (and its analogue) follows by Theorem 3.8 (by the conjecture in [KS20, Remark 1.7.b]) analogously to [FWZ].

An **odd (almost) embedding** is a PL (almost) embedding $f : S^l_1 \vee S^l_2 \to S^{3l+2} - S^{2l+1}_1 \vee S^{2l+1}_2$ such that the mod 2 linking coefficient of $f(S^l_i)$ and S^{2l+1}_j equals to the Kronecker delta $\delta_{i,j}$.

Conjecture 3.14. *For any odd $l > 1$ and all $a^{i,j}_q$ even there is a $(2l+1)$-complex $G_2 \supset S^l \vee S^l$ such that any of the following properties is equivalent to (SKEW1):*

(Ex2) some odd almost embedding extends to a PL map of $X(a,b)$.

(Ex'2) some odd almost embedding extends to a PL embedding of $X(a,b)$.

(Em2) $X(a,b) \cup_{S^l \vee S^l} G_2$ embeds into S^{3l+2}.

All the implications except $(Em2) \Rightarrow (Ex'2)$ (and their analogues for 'almost embedding' replaced by 'embedding') are proved analogously to the corresponding correct implications of Conjecture 3.4. The implication $(Em2) \Rightarrow (Ex'2)$ (and its analogue) would follow by a 'wedge' analogue of Theorem 3.8 (and of the conjecture in [KS20, Remark 1.7.b]) analogously to [FWZ].

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