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Extended Affine Root Systems II (Flat Invariants)

By

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Contents

Notations. Let F be an A-module over an algebra A. Let I be a nondegenerate A-symmetric bilinear form on F. Then we shall use the notations:

F^* := the dual A-module of F,
$\langle\ ,\ \rangle$:= the pairing map between F and F^*,
I := the A-linear map $F \to F^*$ defined by the relation:

$$I(x, y) = \langle I(x), y\rangle \qquad \text{for } {}^\forall x, y \in F,$$

I^* := the symmetric A-bilinear form on F^* defined by:

$$I^*(I(x), I(y)) = I(x, y) \qquad \text{for } {}^\forall x, y \in F.$$

By this convention, $I^*: F^* \to \mathrm{F}$ is defined as the inverse map of I. Similar notations will be used for the case when I is degenerate.

§ 1. Introduction

(1.1) The purpose of the present paper is to introduce and to describe *the flat structure on the ring of invariant θ-functions for an extended affine root system.* (The extended

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affine root system is introduced in [25], and the results of the present paper were anounced in [26]. But we do not assume the knowledge of the results.)

A brief description of the result

i) An extended affine root system (or EARS for short) R is a root system associated to a positive semi-definite Killing form with radical of rank 2. The extended Weyl group $\tilde{W}_R$ for R is an extention of a finite Weyl group W_f by a Heisenberg group $\tilde{H}_R$. A Coxeter element $\tilde{c}$ is defined in the group, whose power $\tilde{c}^{(l_{\max}+1)}$ generates the center of the Heisenberg group. A survey on the root system and the Coxeter element will be given in § 2.

ii) The $\tilde{W}_R$-invariant ring S^W is a ring of W_f-invariant θ-functions on a line bundle L over a family X of Abelian variety, which is associated to the Heisenberg group $\tilde{H}_R$ and on which W_f acts. The Gauss-Manin connection ∇ is introduced on the module of derivations Der_{S^W} of S^W as the Levi-Civita connection with respect to the Killing form. Then Der_{S^W} admits a good filtration (a Hodge filtration). (See §'s 3–6.)

iii) The flat structure on S^W is roughly a certain particular system of homogeneous generators of the algebra S^W, whose linear spann is uniquely characterized by admitting a $\mathbb{C}$-inner product, denoted by J. The goal of the present paper is the construction of the flat structure, achieved in § 11, for the root system with codimension 1 (cf (2.4.2)). In the construction, the fact that the fixed point set of a Coxeter element is regular w.r.t. the Weyl group action, studied in §'s 7–9, plays an essential role.

As a consequence, $\mathrm{Spec}(S^W)$ becomes a graded affine linear space with some additional structures J and N. For more details on the structures, see *Note* 1 below and § 11 (11.5) Theorem.

Note 1. Precisely, the flat structure is formulated in terms of a triplet $\{J, N, \nabla^{\#}\}$ on a module $\mathscr{G}$, where $\mathscr{G}$ (called the small tangent module) is the leading term of the tangent module Der_{S^W} w.r.t. the Hodge filtration (cf (9.7.1)), J is a nondegenerate metric on $\mathscr{G}$, N is a semisimple endomorphism of $\mathscr{G}$ (defining the grading on $\mathscr{G}$), and $\nabla^{\#}$ is an integrable torsion free connection on $\mathscr{G}$ such that $\nabla^{\#}J = 0$ and $\nabla^{\#}N = 0$. Actually $\{J, N, \nabla^{\#}\}$ is defined as the "leading term" of the triplet $\{\tilde{I}_W, \theta_l, \nabla\} := \{$Killing form, multiplication of the coordinate θ_l of the highest level, the Gauss-Manin connection$\}$ on Der_{S^W}.

Note 2. The line bundle L over X (for fixed $\tau \in \mathbb{H}$) coincides with the one considered by Looijenga [10] by using Appell-Humbert theorem, and the ring S^W is studied by authers, Looijenga [10], Kac & Peterson [7, 8], Bernstein & Schwarzman [3, 4], for which a Chevalley's theorem was shown. We recall the result in § 4.

Note 3. Such flat structure was firstly constructed for the invariant ring of a finite reflection group [23]. So the present paper may be regarded as its generalized version to a parabolic geometry. A hyperbolic version is yet to be studied. Another generalization of the flat structure to certain unitary reflection groups was given by Orlik-Solomon [18].

(1.2) Let us describe briefly a background of the present paper from the theory of period mappings. This is logically unnecessary to read but explains a motivation and a role of the present paper.

There is a theory of period mappings for primitive forms (cf. *Remark* 1, [15, 24]). As a consequence of the theory, the space of deformation of a hypersurface isolated singularity, on which the period map is defined, carries a flat structure, i.e. the triplet $(J, N, \nabla^{\#})$ (cf. (1.1) *Note* 1.).

As a version to the classical *Jacobi's inversion problem*, we ask an intrinsic description for the domain of periods, action of the monodromy group on it, and the ring of monodromy invariant functions on the domain. Since the space of deformation is described as the quotient of the period domain by the monodromy action, we ask further for the description of the flat structure in terms of the monodromy invariant ring.

As the classical case, the period domain and the monodromy group for simple singularity are described in terms of classical root systems (Brieskorn [6], Slodowy [28, 29], cf. Arnold [2]). Then the flat structure on the Weyl group invariant polynomial ring is constructed in [23] only in terms of the finite reflection group and a Coxeter transformation, independent of the period mapping. (Cf [1, 7, 27, 33]).

This fact causes naturally a general problem:

Develop a theory of suitably generalized root system with a good Coxeter transformation and a theory of flat invariants for the root system in a self-contained way, to answer the Jacobi's inversion problem for further cases of period mappings.* (* This means to use only the data of a root system as for the building block but not of a period map).

The present paper answers the problem for the case of *extended affine root systems*. An extended affine root system is a root system in a generalized sense, which belongs to a positive semi-definite quadratic form with rank 2 radical, whose Coxeter transformation are carefully studied in the previous paper [25].

Starting with an extended affine root system R with a datum of a marking G on R (cf (2.1), *Remark* 1), we construct in this paper the invariant theory for (R, G), which finally leads to the flat structure on the invariant ring S^W. Since the invariant space $\mathrm{Spec}(S^W)$ with the flat structure is identified with the space of deformation for a simply elliptic singularity (cf. **Appendix**, [21]), this is an answer to the Jacobi's inversion problem for that case.

Remark 1. A period map for a universal unfolding $\varphi: \mathscr{X} \to S$ is determined by a primitive form $\zeta^{(0)}$, which is an element of the middle relative de-Rahm cohomology group $\mathbf{R}^d\varphi_*(\Omega^{\cdot}_{\mathscr{X}/S})$ satisfying certain infinite system of higher residue bilinear equations. The primitive form induces an isomorphism of Hodge filter $F^0(\mathbf{R}^d\varphi_*(\Omega^{\cdot}_{\mathscr{X}/S}))$ with the tangent bundle Der_S of the space of the deformation S and an integrable connection on Der_S, whose integration is the period map. So our aim is to reconstruct the data on $\mathbf{R}^d\varphi_*(\Omega^{\cdot}_{\mathscr{X}/S})$ from a root system.

In the case of simple elliptic singularity, the primitive form is easily written down explicitly as

$$\zeta^{(0)} = \omega \Big/ \int_a \mathrm{Res}_{E_\infty}[\omega]$$

where $\omega := \mathrm{Res}_{\mathscr{X}}\left[\frac{dx\,dy\,dz}{F}\right] \in \mathbb{R}^d\varphi_*(\Omega^{\cdot}_{\mathscr{X}/S}(*))$ for $F = F(x, y, z, t)$: the defining polynomial of the unfolding $\mathscr{X}$, E_∞ is the elliptic curve at infinity ([21]) and a is a homology class $\in H_1(E_\infty)$, so that the denominator of $\zeta^{(0)}$ is the elliptic integral of the first kind.

The ambiguity of $\zeta^{(0)}$ is the choice of $a \in H_1(E_\infty)$. Since the $H_1(E_\infty)$ is identified with the radical of the root system, after all, this ambiguity in the period map is reflected in the present paper as a choice of a marking in (2.1), which is an element $a \in \mathrm{rad}\, I \simeq H_1(E_\infty)$ of the radical of the root system.

Remark 2. The period mappings for simply elliptic singularities are treated also by some works of E. Looijenga [9, 10, 11]. (Cf. H. Pinkham [20], P. Slodowy [30].) So the present paper may be considered as its completion by introducing the flat structure, for which purpose the introduction of the Coxeter transformation theory for extended affine root system was essential.

(1.3) Let us give a brief view of the contents of this paper.

The notion of an extended affine root system and the results on Coxeter elements for the root system are recalled in §2. From the data of a root system, a family L of line bundles over Abelian variety over the complex upper half plane $\mathbb{H}$ is constructed in §3 and a Chevalley theorem on the invariant ring S^W is recalled in §4.

The Killing form $\tilde{I}$ for the root system induces metric $\tilde{I}_W$ on the tangent bundle Der_{S^W} of the quotient space $L/W_f = \mathrm{Spec}\,(S^W)$. $\tilde{I}_W$ degenerates along the discriminant loci $\Theta_A^2 = 0$. A precise study of the degeneration of $\tilde{I}_W$ leads to logarithmic forms in §5 and the logarithmic (Gauss-Manin) connection ∇ in §6.

The results on Coxeter element in §2 is applied to show the non-vanishing of leading coefficients for discriminant Θ_A^2 in §7. Also the Killing form $\tilde{I}_W$ is used to normalize a unit factor for a generator θ_l of S^W in §8. Both facts lead to a key step: the construction of the non-degenerate metric J on the "small tangent bundle $\mathscr{G}$" in §9 (under the (9.1)).

This metric J may be considered as the leading term of the Killing form $\tilde{I}_W$ on Der_{S^W} according to a graduation $\mathrm{Der}_{S^W} = \bigoplus_{i \geq 0} \mathscr{G}_i$ studied in §§9, 10. Then also taking the leading term of the connection ∇ by this graduation, one obtains a non-singular connection $\nabla^{\#}$ on the "small tangent bundle $\mathscr{G}$" such that $\nabla^{\#} J = 0$ in §11. This determines the flat structure on S^W.

(1.4) The first draft of the paper was written with [25] in the Winter semester 1983–84, when the author was a visitor at the Max Planck Institute für Mathematik at Bonn. He also expresses his gratitude to Prof. T. Springer for constant encouragement.

§2. Extended Affine Root Systems

A summary on Coxeter transformation theory for extended affine root systems is given from [25]. Main results are summarized in **Lemma**'s $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. For details and proofs, one is refered to the original paper.

(2.1) Let us start with a generalized concept of a root system, which we call also a root system for simplicity.

Let F be a real vector space of finite rank with a symmetric bilinear form I: $F \times F \to \mathbb{R}$. For an non-isotropic element $\alpha \in F$ (i.e. $I(\alpha, \alpha) \neq 0$), put $\alpha^\vee := 2\alpha/I(\alpha, \alpha) \in F$. The reflection w_α w.r.t. α is an element of $O(I) := \{g \in GL(F): I(x, y) = I(g(x), g(y))\}$ given by,

$$w_\alpha(u) := u - I(u, \alpha^\vee)\alpha \qquad (^\forall u \in F).$$

Then $\alpha^{\vee\vee} = \alpha$ and $w_\alpha^2 = \mathrm{id}$.

Definition. 1. *A set R of non-isotropic elements of F is a root system belonging to (F, I), if it satisfies the axioms* i)–iv):

i) *The additive group generated by R in F, denoted by $Q(R)$, is a full sub-lattice of F. I.e, the embedding $Q(R) \subset F$ induces the isomorphism: $Q(R) \otimes_{\mathbf{Z}} \mathbf{R} \simeq F$.*

ii) $I(\alpha, \beta^\vee) \in \mathbb{Z}$ *for* $^\forall\alpha, \beta \in R$.

iii) $w_\alpha(R) = R$ *for* $^\forall\alpha \in R$.

iv) *If $R = R_1 \cup R_2$ with $R_1 \perp R_2$, then either R_1 or R_2 is void.*

2. *A root system R belonging to (F, I) is a k-extended affine root system (k-EARS for short), if I is positive semi-definite and the radical*: $\mathrm{rad}(I) := \{x \in F: I(x, y) = 0$ *for* $^\forall y \in F\} = F^\perp$, *is of rank k over $\mathbf{R}$.*

3. *A marking G for a k-EARS is a decreasing sequence*:

$$G_0 = \mathrm{rad}(I) \supset G_1 \supset \cdots \supset G_k = \{0\}$$

of subspaces of $\mathrm{rad}(I)$ *such that* $G_i \cap Q(R) \simeq \mathbf{Z}^{k-i}$.

The pair (R, G) will be called a marked extended affine root system (or a marked EARS). Two marked EARS's are isomorphic, if there exists a linear isomorphism of the ambient vector spaces, inducing the bijection of the sets of roots and the markings.

Note 1. i) If R is a root system belonging to (F, I), then $R^\vee := \{\alpha^\vee : \alpha \in R\}$ is also a root system belonging to (F, I).

ii) For a root system belonging to (F, I), there exists a real number $c > 0$ such that the bilinear form cI defines an even lattice structure on $Q(R)$ (i.e. $cI(x, x) \in 2\mathbf{Z}$ for $x \in Q(R)$). The smallest such c is denoted by $I_R : I$ and the bilinear form $(I_R : I)I$ is denoted by I_R.

iii) For a root system R belonging to (F, I), there exists a positive integer $t(R)$ such that

$$I_R \otimes I_{R^\vee} = t(R) I \otimes I.$$

$t(R)$ is called the tier number of R ($R^\vee$) ([25, (1.10 ~ 12)]).

Note 2. i) A 0-EARS is shown to be finite and hence a root system in the classical sense [5]. We shall call it a finite root system.

ii) A 1-EARS is shown to satisfy the axioms for an affine root system in the sense of Macdonald [12]. We shall call it an affine root system.

Notation. Since we are mainly concerned with 2-EARS's in this paper, we simply say EARS instead of 2-EARS. In the case, a datum of a marking G is a rank 1 subspace G_1 of $\mathrm{rad}(I)$ defined over $\mathbf{Q}$. We shall denote the space G_1 by the same notation G. A generator of $G \cap Q(R) \simeq \mathbf{Z}$, which is unique up to a sign, is denoted by a.

$$G \cap Q(R) = \mathbb{Z}a \qquad \text{and} \qquad G = \mathbb{R}a\,.$$

(2.2) **The basis $\alpha_0, \ldots, \alpha_l$ for (R, G).**

Let (R, G) be a marked EARS. The image of R by the projection $F \to F/\mathrm{rad}\,(I)$ (resp. $F \to F/G$) is a finite (resp. affine) root system, which we shall denote by R_f (resp. R_a). In the present paper, we assume that the affine root system R_a is reduced. (I.e. $\alpha = c\beta$ for $\alpha, \beta \in R_a$ and $c \in \mathbb{R}$ implies $c \in \{\pm 1\}$.)

Put,

$$l := \mathrm{rank}_{\mathbb{R}}\,(F/\mathrm{rad}\,(I))\,, \qquad (\text{i.e. } \mathrm{rank}_{\mathbb{R}}(F) = l + 2)\,. \tag{2.2.1}$$

Once and for all in this paper, we fix $l + 1$ elements,

$$\alpha_0, \ldots, \alpha_l \in R$$

such that their images in R_a form a basis for R_a [12]. (I.e. the images are normal vectors of walls of an affine Weyl chamber of R_a directing inside of the chamber.) We shall call them a basis for (R, G). Such basis is unique up to isomorphisms of (R, G). There exists positive integers $n_0, \ldots, n_l$ such that the sum:

$$b := \sum_{i=0}^{l} n_i \alpha_i \tag{2.2.2}$$

belongs to $\mathrm{rad}\,(I)$. By a permutation of basis, we may assume [12],

$$n_0 = 1\,. \tag{2.2.3}$$

Then the images of $\alpha_1, \ldots, \alpha_l$ in R_f form a positive basis for R_f and the image of $-\alpha_0$ in R_f is the highest root w.r.t. the basis.

Put,

$$L := \bigoplus_{i=1}^{l} \mathbb{R}\alpha_i \tag{2.2.4}$$

on which I is positive definite and $R \cap L$ is a finite root system with the positive basis $\alpha_1, \ldots, \alpha_l$ (cf. *Note* 2).

We have a direct sum decomposition of the vector space:

$$F = L \oplus \mathrm{rad}\,(I)\,, \tag{2.2.5}$$

and the lattice:

$$Q(R) = \bigoplus_{i=0}^{l} \mathbb{Z}\alpha_i \oplus \mathbb{Z}a = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i \oplus \mathbb{Z}a \oplus \mathbb{Z}b\,, \tag{2.2.6}$$

$$Q(R) \cap \mathrm{rad}\,(I) = \mathbb{Z}a \oplus \mathbb{Z}b\,, \tag{2.2.7}$$

$$Q(R) \cap L = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i\,. \tag{2.2.8}$$

Note 1. The choice of the basis $\alpha_0, \ldots, \alpha_l$ is done for the sake of explicite calculations, but it does not affect the result of the present paper. A change of the basis $\alpha_0, \dots, \alpha_l$ induces a change (a, b) to $(a, b + ma)$ for some $m \in \mathbf{Z}$.

Note 2. Under the assumption that R_f is reduced, the set $R \cap L$ is bijective to R_f (i.e. $(R \cap L, L) \simeq (R_f, F/\mathrm{rad}(I))$.) Hence we shall sometimes confuse $(R \cap L, L)$ with $(R_f, F/\mathrm{rad}(I))$ [25].

(2.3) **The Weyl group W_R.**

The Weyl group W_R for R is defined as the group generated by the reflexions w_α for $^\forall\alpha \in R$. The projection $p: F \to F/\mathrm{rad}(I)$ induces a homomorphism $p_*: W_R \to W_{R_f}$. One gets a short exact sequence:

(2.3.1) $$0 \longrightarrow H_R \xrightarrow{E} W_R \xrightarrow{p_*} W_{R_f} \longrightarrow 1.$$

Here

(2.3.2) $$H_R := (\mathrm{rad}(I) \otimes_{\mathbf{R}} F/\mathrm{rad}(I)) \cap E^{-1}(W_R)$$

is a finite index subgroup in the lattice $(\mathbf{Z}a \oplus \mathbf{Z}b) \otimes_{\mathbf{Z}} \left(\bigoplus_{i=1}^{l} \mathbf{Z}\alpha_i^\vee \right)$.

A notational remark. The group H_R (2.3.2) was denoted by T in the [25] to indicate that it is a translation group. Since we shall use the notation "T" as for a subalgebra (8.1.1) of S^W, the notation for the group is changed as (2.3.2) to indicate its relation with the Heisenberg group $\tilde{H}_R$ (2.7.1). (Cf. (3.4.4)).

The map E, called the Eichler-Siegel transformation, is a semi-group homomorphism defined as follows ([25, (1.14) $\sim$ (1.15)]).

(2.3.3) $$E: F \otimes_{\mathbf{R}} F/\mathrm{rad}(I) \longrightarrow \mathrm{End}(F)$$

(2.3.4) $$E\left(\sum_i \xi_i \otimes \eta_i\right)(u) := u - \sum_i \xi_i I(\eta_i, u) \qquad \text{for} \quad u \in F.$$

Here a semi-group structure $\circ$ on $F \otimes_{\mathbf{R}} F/\mathrm{rad}(I)$ is defined by,

$$\left(\sum_i u_i \otimes v_i\right) \circ \left(\sum_j w_j \otimes x_j\right) := \sum_i u_i \otimes v_i + \sum_j w_j \otimes x_j - \sum_{i,j} I(v_i, w_j) u_i \otimes x_j.$$

The semi-group structure $\circ$ coincides with the natural addition of vectors on the subspace $\mathrm{rad}(I) \otimes (F/\mathrm{rad}(I))$ and hence on H_R. The inverse map $E^{-1}|W_R$ is an injective homomorphism,

(2.3.5) $$E^{-1}: W_R \longrightarrow F \otimes (F/\mathrm{rad}(I)) \simeq L \otimes L \oplus \mathbf{R}a \otimes L \oplus \mathbf{R}b \otimes L,$$

by which the reflection w_α goes to $\alpha \otimes \alpha^\vee$. Hence the image $E^{-1}(W_R)$, generated by $\alpha \otimes \alpha^\vee$ $(\alpha \in R)$, is included in $Q(R) \otimes (Q(R)^\vee)/\mathrm{rad}(I))$.

Let us write the map E^{-1} componentwisely.

(2.3.6) $$E^{-1}(g) = \xi(g) + a \otimes p(g) + b \otimes q(g).$$

Here ξ, p and q are maps

(2.3.7) $$\xi\colon W_R \longrightarrow \left(\bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i\right) \otimes_{\mathbb{Z}} \left(\bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i^\vee\right) \subset L \otimes L\,,$$

(2.3.8) $$p\colon W_R \longrightarrow \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i^\vee \subset L\,,$$

(2.3.9) $$q\colon W_R \longrightarrow \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i^\vee \subset L\,,$$

The action of $g = E(\xi + a \otimes p + b \otimes q)$ on $x \in F$ is given by

(2.3.10) $$g(x) = E_0(\xi)(x_L) - aI(p, x) - bI(q, x)\,,$$

where $E_0\colon L \otimes L \to \mathrm{End}\,(L)$ is the Eichler-Siegel transformation defined similarly as (2.3.4) and x_L is the L part of x in (2.2.5). Under these notations, we have:

i) *The map* $E_0(\xi)\colon W_R \to \mathrm{End}\,(L)$ *is factorized by* $W_R \to W_{R_f}$ *so that the subgroup* H_R *is characterize by* $\{g \in W_R\colon \xi(g) = 0\}$.

ii) *The subgroup of* W_R *generated by* $w_{\alpha_1}, \ldots, w_{\alpha_l}$ *is isomorphic to* W_{R_f} *by the homomorphism* p_* *so that the exact sequence* (2.3.1) *splits into a semi-direct product:* $W_R = W_{R_f} \ltimes H_R$.

(2.4) **The Dynkin graph.**

For a marked EARS (R, G), we associate a diagram $\Gamma_{R,G}$, called the Dynkin graph for (R, G), in which all data on (R, G) are coded. The graph is constructed in the following steps i)–iv).

i) *Let* Γ *be the graph for the affine root system* R_a.

I.e. a) *The set of the vertices* $|\Gamma|$ *is* $\{\alpha_0, \ldots, \alpha_l\}$.

b) *Edges of* Γ *is given according to a convention in* iv) b).

ii) *The exponent for each vertex* $\alpha_i \in |\Gamma|$ *is defined by*

(2.4.1) $$m_i := \frac{I_R(\alpha_i, \alpha_i)}{2k(\alpha_i)}\, n_i\,,$$

where $k(\alpha) := \inf\,\{n \in \mathbb{N}\colon \alpha + na \in R\}$.

iii) *Put*

$$m_{\max} := \max\,\{m_0, \ldots, m_l\}\,,$$

$$|\Gamma_m| := \{\alpha_i \in |\Gamma|\colon m_i = m_{\max}\}\,,$$

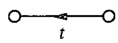
$$|\Gamma_m^*| := \{\alpha_i + k(\alpha_i)a\colon \alpha_i \in |\Gamma_m|\}\,.$$

iv) *The graph* $\Gamma_{R,G}$ *is defined as the graph for* $|\Gamma| \cup |\Gamma_m^*|$.

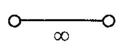
I.e. a) *The set of the vertices* $|\Gamma_{R,G}| := |\Gamma| \cup |\Gamma_m^*|$.

b) *Two vertices* $\alpha, \beta \in |\Gamma_{R,G}|$ *are connected by the convention:*

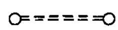
α β	
○ ○	*if* $I(\alpha, \beta^\vee) = 0$ ($\Leftrightarrow I(\beta, \alpha^\vee) = 0$),
○——○	*if* $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -1$,



if $I(\alpha, \beta^\vee) = -1, I(\beta, \alpha^\vee) = -t$,



if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -2$,


if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = 2$.

v) *A complete list of Dynkin Diagrams and exponents for extended affine root systems is given in the Table* 1.

Definition. *For a marked EARS* (R, G), *the codimension, denoted as* cod (R, G), *is defined as follows.*

(2.4.2) $$\mathrm{cod}\,(R, G) := \#\{0 \le i \le l : m_i = m_{\max}\} = \#|\Gamma_m|.$$

Note 1. Converse to the above construction of the graph $\Gamma_{R,G}$ from a (R, G), the isomorphism class of a marked EARS (R, G) is constructed from the data of the diagram $\Gamma_{R,G}$ by a help of Coxeter transformation theory (cf [25, §9]).

Note 2. The exponents m_i's introduced in ii) are half integers, which might have a common factor. We have:

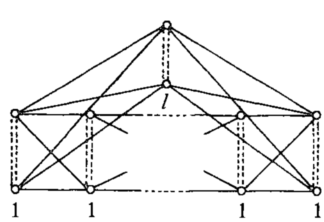
The smallest common denominator for the rational numbers $m_i/m_{\max}$ $(i = 0, \dots, l)$ *is equal to* $l_{\max} + 1$ ([25, (8.4)]), *where* $l_{\max} :=$ max $\{$*# of vertices in a connected component of* $\Gamma \backslash \Gamma_m\}$. So we sometimes normalize the exponents as follows.

(2.4.3) $$\tilde{m}_i := m_i \frac{l_{\max} + 1}{m_{\max}} \qquad i = 0, \dots, l.$$

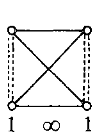
Note 3. The name codimension for cod (R, G) has an origin in the period mapping theory. Namely it expresses the codimension (= # of independent defining equations) of the singularity for (R, G) in an ambient space (Cf. **Appendix**)

Table 1. Dynkin Diagrams for Extended Affine Root Systems and Their Exponents [25].

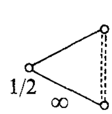
$A_l^{(1,1)}$



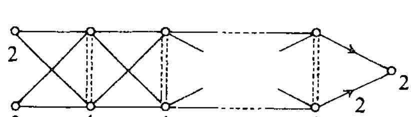
$(l \ge 2)$



$A_l^{(1,1)*}$

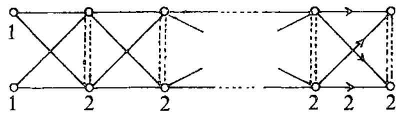


$B_l^{(1,1)}$



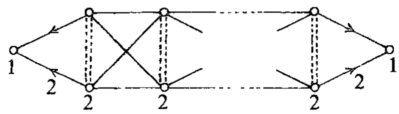
$(l \ge 3)$

$B_l^{(1,2)}$



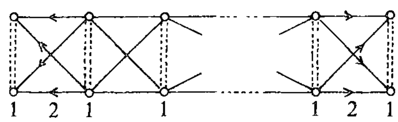
$(l \geq 3)$

$B_l^{(2,1)}$



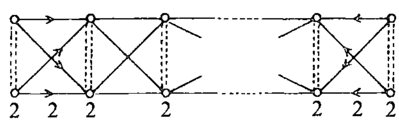
$(l \geq 2)$

$B_l^{(2,2)}$



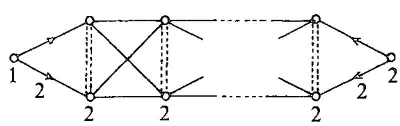
$(l \geq 2)$

$C_l^{(1,1)}$



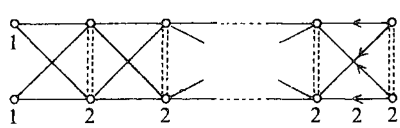
$(l \geq 2)$

$C_l^{(1,2)}$



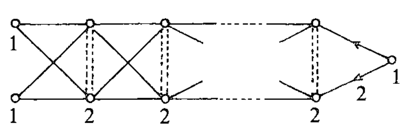
$(l \geq 2)$

$C_l^{(2,1)}$



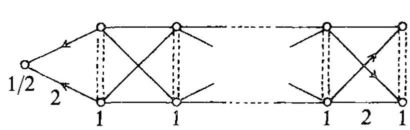
$(l \geq 3)$

$C_l^{(2,2)}$



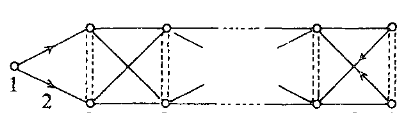
$(l \geq 3)$

$B_l^{(2,2)}$



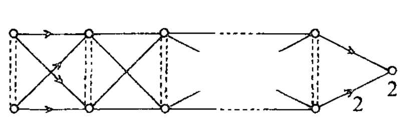
$(l \geq 2)$

$C_l^{(1,1)}$

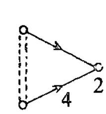


$(l \geq 2)$

$BC_l^{(2,1)}$

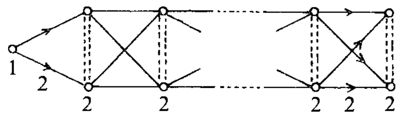


$(l \geq 2)$,

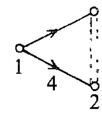


$(l = 1)$

$BC_l^{(2,1)}$

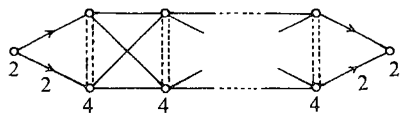


$(l \geq 2)$,



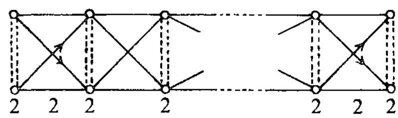
$(l = 1)$

$BC_l^{(2,2)}(1)$

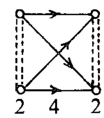


$(l \geq 2)$

$BC_l^{(2,2)}(2)$

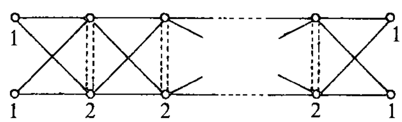


$(l \geq 2)$,



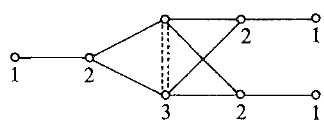
$(l = 1)$

$D_l^{(1,1)}$

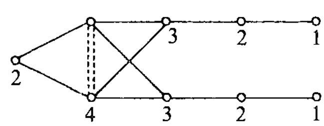


$(l \geq 4)$

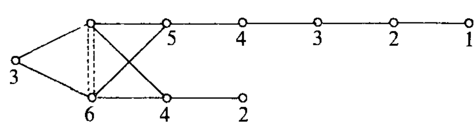
$E_6^{(1,1)}$



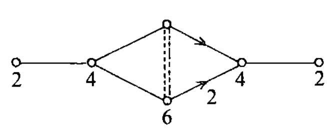
$E_7^{(1,1)}$



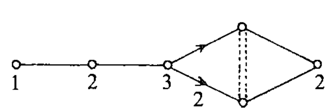
$E_8^{(1,1)}$



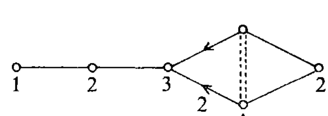
$F_4^{(1,1)}$



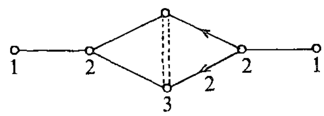
$F_4^{(1,2)}$



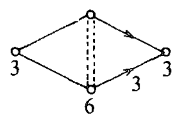
$F_4^{(2,1)}$



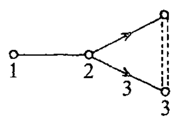
$F_4^{(2,2)}$



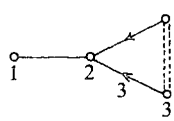
$G_2^{(1,1)}$



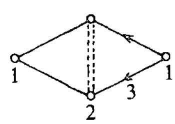
$G_2^{(1,3)}$



$G_2^{(1,1)}$



$G_2^{(2,3)}$



(2.5) **The Coxeter transformation for** (R, G).

A Coxeter transformation $c \in W_R$ is by definition [25, (9.7)] a product of reflexions w_α for $\alpha \in |\Gamma_{R,G}|$ with a restriction on the order of the product that $w_{\alpha*}$ comes next to w_α for $\alpha \in |\Gamma_m|$. The following **Lemma**'s $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are basic results for the Coxeter transformation, which will be used essentially in this paper.

Lemma $\mathbb{A}$ ([25, (9.7)]). *A Coxeter transformation c is semi-simple of finite order* $= l_{\max} + 1$. *The set of eigenvalues of c is given by*:

$$1 = \exp(0) \qquad \text{and} \qquad \exp(2\pi\sqrt{-1}m_i/m_{\max}) \qquad (i = 0, \ldots, l).$$

Particularly, $\{$*the multiplicity of eigenvalue* $1\} = 1 + \operatorname{cod}(R, G)$.

Lemma $\mathbb{B}$ ([25, (10.1)]). *Let c be a Coxeter transformation for* (R, G). *Then*

$$R \cap \operatorname{Image}(c - \mathrm{id}_F) = \phi .$$

To state the **Lemma** $\mathbb{C}$ in (2.8), we recall some more concepts.

(2.6) **The hyperbolic extention** $(\tilde{F}, \tilde{I})$. There exists uniquely (up to a linear isomorphism) a real vector space $\tilde{F}$ of rank $l + 3$ with

i) an inclusion map $F \subset \tilde{F}$ as a real vector space,

ii) a symmetric form $\tilde{I}: \tilde{F} \times \tilde{F} \to \mathbb{R}$ such that $\tilde{I}|_F = I$ and $\operatorname{rad}(\tilde{I}) = \mathbb{R}a$.

The pair $(\tilde{F}, \tilde{I})$ will be called a hyperbolic extension for (F, I).

Denote by $\tilde{w}_\alpha$ the reflextion for $\alpha \in R$ as an element of $\tilde{F}$ and by $\tilde{W}_R$ the subgroup of the isometries of $(\tilde{F}, \tilde{I})$ generated by them. The restriction $\tilde{w}_\alpha|_F$ is w_α. So we have a surjection $\tilde{W}_R \twoheadrightarrow W_R$ and then a short exact sequence:

$$0 \longrightarrow \tilde{K}_R \xrightarrow{\tilde{E}} \tilde{W}_R \longrightarrow W_R \longrightarrow 1 \tag{2.6.1}$$

where $\tilde{K}_R$ is an infinite cyclic group generated by

$$k := (I_R : I)\frac{l_{\max} + 1}{m_{\max}} a \otimes b\,, \tag{2.6.2}$$

and $\tilde{E}$: $F \otimes (F/G) \to \mathrm{End}\,(\tilde{F})$ is the Eichler-Siegel transformation,

$$\tilde{E}\left(\sum_i \xi_i \otimes \eta_i\right)(u) := u - \sum_i \xi_i \tilde{I}(\eta_i, u) \qquad \text{for} \quad u \in \tilde{F}\,. \tag{2.6.3}$$

Notational remark. The notations $\tilde{W}_R$, $\tilde{K}_R$ and $\tilde{H}_R$ (2.7.1) are inexact in the sense that they depends not only on the root system R but also on the marking G. The confusion can be avoidable by the context.

A precise description of the group $\tilde{W}_R$ is as follows ([25, (1.18) ~ (1.20)]). The inverse map

$$\tilde{E}^{-1}: \tilde{W}_R \longrightarrow F \otimes F/G \simeq \begin{matrix} (F \otimes L) \\ \oplus(F \otimes \mathbb{R}b) \end{matrix} = \begin{matrix} L \otimes L \oplus \mathbb{R}a \otimes L \oplus \mathbb{R}b \otimes L \\ \oplus L \otimes b \oplus \mathbb{R}b \otimes b \oplus \mathbb{R}a \otimes b \end{matrix} \tag{2.6.4}$$

is an injective semigroup homomorphism. Due to the fact that $\tilde{W}_R$ preserves the metric $\tilde{I}$, we obtain the following decomposition:

$$\begin{aligned} \tilde{E}^{-1}(g) = \xi(g) + a \otimes p(g) + b \otimes q(g) - E_0(\xi(g))q(g) \otimes b \\ + \frac{1}{2} I(q(g), q(g)) b \otimes b + r(g)(I_R : I)\frac{l_{\max} + 1}{m_{\max}} a \otimes b\,. \end{aligned} \tag{2.6.5}$$

Here ξ, p, q are defined in (2.3) and r is a map,

$$r: \tilde{W}_R \longrightarrow \mathbb{Z}\,. \tag{2.6.6}$$

That $\tilde{E}$ is a semi-group homomorphism implies the relation ([25, (1.20), (1.19)]): For $^\forall g_1, g_2 \in \tilde{W}_R$,

$$r(g_1 g_2) = r(g_1) + r(g_2) - \frac{m_{\max}}{(I_R : I)(l_{\max} + 1)} I(p(g_1), E_0(\xi(g_2)q(g_2)))\,. \tag{2.6.7}$$

Under these notations, we have:

i) *The map* $E(\xi + a \otimes p + b \otimes q)$: $\tilde{W}_R \to \mathrm{End}\,(F)$ *is factorized by* $\tilde{W}_R \to W_R$.

ii) *An element* $g \in \tilde{W}_R$ *belongs to the cyclic group* $\tilde{K}_R$, *if and only if* $\xi(g) = p(g) = q(g) = 0$.

Remark. The following i), ii) and iii) show that a $\tilde{W}_R$-invariant symmetric bilinear form on $\tilde{F}$, whose restriction on F is not zero, is a constant multiple of $\tilde{I}$ up to an automorphism of $\tilde{F}$ of the form $\tilde{E}(z)$ for a $z \in$ Center of $F \otimes F/\mathrm{rad}\,(\tilde{I})$.

i) *$\{\tilde{W}_R$-invariant symmetric bilinear forms on $\tilde{F}\} = \mathbb{R}\tilde{I} + \mathbb{R}\tilde{I}_\infty$, where $\tilde{I}_\infty$ is a form on $\tilde{F}$ characterized by $\tilde{I}_\infty(\tilde{\lambda}, \tilde{\lambda}) = 1$ and $\tilde{I}_\infty | F \times \tilde{F} = 0$.*

ii) *$Z(GL(\tilde{F}) \cap \tilde{E}(F \otimes F/G))$ (:= the center of $GL(\tilde{F}) \cap \tilde{E}(F \otimes F/G)) = \tilde{E}(\mathrm{rad}\,(I) \otimes \mathrm{rad}\,(I)/G) = \tilde{E}(\mathbb{R}b \otimes b + \mathbb{R}a \otimes b)$.*

iii) *Let I_1 be a $\tilde{W}_R$-invariant symmetric form on $\tilde{F}$ such that $I_1 | F \times F \neq 0$. Then there exists an element $g \in Z(GL(\tilde{F}) \cap \tilde{E}(F \otimes F/G))$ such that $I_1(x, y) = c\tilde{I}(g(x), g(y))$ for a $c \in \mathbb{R}$.*

Proof. i) Use the expression (2.6.5).

ii) An element $z \in F \otimes F$ belongs to its center $\Leftrightarrow z \in F^\perp \otimes F^\perp$.

iii) Due to i), $I_1 = c\tilde{I} - d\tilde{I}_\infty$ for $c, d \in \mathbb{R}$ with $c \neq 0$. Then we may choose $g = \tilde{E}\left(\frac{d}{2c} b \otimes b\right)$ as follows. Due to ii), g belongs to $Z(GL(\tilde{F}) \cap \tilde{E}(F \otimes F/G))$. Since $g(x) := x - \frac{d}{2c}\tilde{I}(x, b)b$ etc., we have,

$$\begin{aligned}\tilde{I}(g(x), g(y)) &= \tilde{I}\left(x - \frac{d}{2c}\tilde{I}(x, b)b,\ y - \frac{d}{2c}\tilde{I}(y, b)b\right)\\ &= \tilde{I}(x, y) - 2\frac{d}{2c}\tilde{I}(x, b)\tilde{I}(y, b)\\ &= \frac{1}{c}(c\tilde{I} - d\tilde{I}(b) \otimes \tilde{I}(b))(x, y)\,.\end{aligned}$$

Here $\tilde{I}(b) \otimes \tilde{I}(b)$ satisfies the characterization for $\tilde{I}_\infty$, we have shown the statement. □

For $\tilde{E}(\mathbb{R}a \otimes b) \subset O(\tilde{F}, \tilde{I})$, the group $Z(GL(\tilde{F}) \cap \tilde{E}(F \otimes F/G))/\tilde{E}(\mathbb{R}a \otimes b)$ ($\simeq \mathbb{R}b \otimes b$) acts on the space $\tilde{I} + \mathbb{R}\tilde{I}_\infty$ as the affine transformation group.

(2.7) **Heisenberg group $\tilde{H}_R$.**

Put

(2.7.1) $$\tilde{H}_R := \{\tilde{E}^{-1}(g) \colon g \in \tilde{W}_R \text{ such that } \xi(g) = 0\}\,.$$

Then, 1) $\tilde{H}_R$ is a Heisenberg group with the short exact sequence,

(2.7.2) $$0 \longrightarrow \tilde{K}_R \longrightarrow \tilde{H}_R \longrightarrow H_R \longrightarrow 1\,,$$

where $\tilde{H}_R \to H_R$ is induced from the projection $F \otimes F/G \to F \otimes F/\mathrm{rad}\,(I)$. (Cf. (2.3.2) and its following *Notational Remark*).

2) The extension class of the sequence is calculated from (2.6.7). ([25, (1.11.3)])

(2.7.3) $$\frac{m_{\max}}{t(R)(l_{\max} + 1)\,\mathrm{Im}\,(\tau)} J \otimes I_{R^\vee} \in \mathrm{Ext}^2\,(H_R, \tilde{K}_R) \simeq \bigwedge^2 \mathrm{Hom}_{\mathbb{Z}}\,(H_R, \mathbb{Z})\,,$$

where

i) $\tilde{K}_R$ is identified with $\mathbb{Z}$ through (2.6.6),

ii) J is the skew symmetric form on $\mathbb{Z}a + \mathbb{Z}b$ by $J(a, b) = -J(b, a) = -1$.

iii) H_R is embedded in $(\mathbb{Z}a + \mathbb{Z}b) \otimes \left(\bigotimes_{i=1}^{l} \mathbb{Z}\alpha_i^\vee\right)$ (cf (2.4)) so that $J \otimes I_{R^\vee}$ induces a skew symmetric bilinear form on H_R.

3) We have the following commutative diagram (cf (2.3.1)):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}_R & \longrightarrow & \tilde{H}_R & \underset{\tilde{E}}{\longrightarrow} & H_R & \underset{E}{\longrightarrow} & 1 \\
 & & \| & & \cap & & \cap & & \\
0 & \longrightarrow & \tilde{K}_R & \underset{\tilde{E}}{\longrightarrow} & \tilde{W}_R & \longrightarrow & W_R & \underset{p_*}{\longrightarrow} & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W_{R_f} & = & W_{R_f} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

(2.8) **The hyperbolic Coxeter transformation.**

A hyperbolic Coxeter transformation $\tilde{c} \in \tilde{W}_R$ is a product of reflexions $\tilde{w}_\alpha$ for $\alpha \in |\Gamma_{R,G}|$ in the same ordering as for the Coxeter transformation c ([25, (11.2)]).

Lemma C ([25, (11.3)]). i) *The power* $\tilde{c}^{(l_{\max}+1)}$ *of the hyperbolic Coxeter transformation* $\tilde{c}$ *is a generator of* $\tilde{K}$.

ii) $\tilde{K}$ *is generated by* $(I_R : I)\dfrac{l_{\max}+1}{m_{\max}}a \otimes b$.

This is equivalent to:

Lemma C′ ([25, (11.4.1)]). *There exists a projection map* $p: \tilde{F} \to F$ *such that for* $^\forall\tilde{\lambda} \in \tilde{F}$,

(2.8.1) $$\tilde{c}(\tilde{\lambda}) = \tilde{\lambda} + (c - id_F)p(\tilde{\lambda}) + \tilde{I}(b, \tilde{\lambda})\frac{(I_R : I)}{m_{\max}}a\,.$$

§3. A Family of Polarized Abelian Variety over H

(3.1) Let (R, G) be a marked EARS and let $(\tilde{F}, \tilde{I})$ be its hyperbolic extention. We define complex affine half spaces as follows.

(3.1.1) $$\tilde{\mathbf{E}} := \{x \in \mathrm{Hom}_{\mathbf{R}}(\tilde{F}, \mathbf{C}) \colon a(x) = 1 \quad \text{and} \quad \mathrm{Im}(b(x)) > 0\},$$

(3.1.2) $$\mathbf{E} := \{x \in \mathrm{Hom}_{\mathbf{R}}(F, \mathbf{C}) \colon a(x) = 1 \quad \text{and} \quad \mathrm{Im}(b(x)) > 0\},$$

(3.1.3) $$\mathbf{H} := \{x \in \mathrm{Hom}_{\mathbf{R}}(\mathrm{rad}(I), \mathbf{C}) \colon a(x) = 1 \quad \text{and} \quad \mathrm{Im}(b(x)) > 0\},$$

where $\dim_{\mathbf{C}} \tilde{\mathbf{E}} = l + 2$, $\dim_{\mathbf{C}} \mathbf{E} = l + 1$ and $\dim_{\mathbf{C}} \mathbf{H} = 1$.

Note. A change of the basis $\alpha_0, \ldots, \alpha_l$ does not affect the definition of the spaces $\tilde{\mathbf{E}}$, $\mathbf{E}$ and $\mathbf{H}$ due to (2.2) *Note* 1.

(3.2) Elements of $\tilde{F}$ (resp. F and $\mathrm{rad}(I)$) are $\mathbb{C}$-valued affine linear functionals on the space $\tilde{\mathbb{E}}$ (resp. $\mathbb{E}$ and $\mathbb{H}$). Put,

$$\tau := b/a \tag{3.2.1}$$

as a complex valued function on $\tilde{\mathbb{E}}$, $\mathbb{E}$ and $\mathbb{H}$. Then $\mathbb{H}$ is identified with the complex upper half plane by τ.

A notational remark. For a convenience, the same notation τ is used for the following different meanings:

i) A coordinate function of the complex upper half plane $\mathbb{H}$ as defined in (3.2.1).

ii) A point of $\mathbb{H}$.

iii) The value in $\mathbb{C}$ of the function (3.2.1) at a point of $\mathbb{H}$.

The inclusion maps: $\tilde{F} \supset F \supset \mathrm{rad}(I)$ induces the projections:

$$\tilde{\mathbb{E}} \xrightarrow{\tilde{\pi}} \mathbb{E} \xrightarrow{\pi} \mathbb{H}. \tag{3.2.2}$$

By the projections, $\tilde{\mathbb{E}}$ and $\mathbb{E}$ are regarded as a total space of a family of complex affine spaces $\tilde{\mathbb{E}}_\tau := (\pi \circ \tilde{\pi})^{-1}(\tau)$ and $\mathbb{E}_\tau := \pi^{-1}(\tau)$ of dimension $l+1$ and l parametrized by $\tau \in \mathbb{H}$.

(3.3) Obviously by definition, the tangent and co-tangent spaces of $\tilde{\mathbb{E}}$, $\mathbb{E}$ and $\mathbb{H}$ at a point x of them are naturally given by:

$$T_x(\tilde{\mathbb{E}}) \simeq \mathbb{C} \otimes (\tilde{F}/G)^*,\ T_x(\mathbb{E}) \simeq \mathbb{C} \otimes (F/G)^*,\ T_x(\mathbb{H}) \simeq \mathbb{C} \otimes (\mathrm{rad}(I)/G)^*, \tag{3.3.1}$$

$$T_x^*(\tilde{\mathbb{E}}) \simeq \mathbb{C} \otimes (\tilde{F}/G),\ T_x^*(\mathbb{E}) \simeq \mathbb{C} \otimes (F/G),\ T_x^*(\mathbb{H}) \simeq \mathbb{C} \otimes (\mathrm{rad}(I)/G)\,. \tag{3.3.1*}$$

In particular, the relative tangent spaces of $\mathbb{E}$ and $\tilde{\mathbb{E}}$ relative to the projections π and $\tilde{\pi} \circ \pi$ of (3.2.2) are given by

$$V_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} V \qquad \text{for} \quad V := (F/\mathrm{rad}(I))^*\,, \tag{3.3.2}$$

$$\tilde{V}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \tilde{V} \qquad \text{for} \quad \tilde{V} := (\tilde{F}/\mathrm{rad}(I))^*\,. \tag{3.3.2$^\sim$}$$

Actually, the natural inclusions $V \subset F^* \subset \mathrm{Hom}_{\mathbb{R}}(F, \mathbb{C})$ or $\tilde{V} \subset \tilde{F}^* \subset \mathrm{Hom}_{\mathbb{R}}(\tilde{F}, \mathbb{C})$ induce the action of $V_{\mathbb{C}}$ or $\tilde{V}_{\mathbb{C}}$ on $\mathbb{E}$ or $\tilde{\mathbb{E}}$ as the translation groups of the affine fibers $\mathbb{E}_\tau$ or $\tilde{\mathbb{E}}_\tau$ over $\tau \in \mathbb{H}$ respectively.

The bilinear forms I (2.1) and $\tilde{I}$ (2.6) induces perfect pairings

$$I\colon F/\mathrm{rad}(I) \times F/\mathrm{rad}(I) \longrightarrow \mathbb{R}\,, \tag{3.3.3}$$

$$\tilde{I}\colon F/G \times \tilde{F}/\mathrm{rad}(I) \longrightarrow \mathbb{R}\,, \tag{3.3.3$^\sim$}$$

which induce cannonical isomorphisms,

$$I^*\colon V := (F/\mathrm{rad}(I))^* \xrightarrow{\sim} F/\mathrm{rad}(I) \tag{3.3.4}$$

$$\tilde{I}^*\colon \tilde{V} := (\tilde{F}/\mathrm{rad}(I))^* \xrightarrow{\sim} F/G \tag{3.3.4$^\sim$}$$

of real vector spaces. Therefore we shall regard F/G and $F/\mathrm{rad}(I)$ as the real part of the translation groups of $\tilde{\mathbb{E}}_\tau$ and $\mathbb{E}_\tau$ for $\tau \in \mathbb{H}$ respectively.

The bilinear forms on V and $\tilde{V}$ induced from I and $\tilde{I}$ through the isomorphisms (3.3.4) will be denoted by I_V^* and $I_{\tilde{V}}^*$ respectively.

(3.4) The pairing $\mathrm{Hom}_{\mathbf{R}}(\mathrm{rad}(I), \mathbf{C}) \times \mathrm{rad}(I) \to \mathbf{C}$ induces the real map:

$$\begin{aligned} \varphi\colon \mathbf{H} \times \mathrm{rad}(I) &\longrightarrow \mathbf{C} \\ \tau \times ua + vb &\longmapsto u + v\tau \,. \end{aligned} \tag{3.4.1}$$

The restriction $\varphi_\tau := \varphi|_{\tau \times \mathrm{rad}(I)}$ (for $\tau \in \mathbf{H}$) induces real isomorphism

$$\varphi_\tau\colon \mathrm{rad}(I) \simeq \mathbf{C}\,. \tag{3.4.2}$$

Particularly the lattice $Q(R) \cap \mathrm{rad}(I) = \mathbf{Z}a + \mathbf{Z}b$ is embedded in $\mathbf{C}$:

$$\varphi_\tau\colon \mathbf{Z}a + \mathbf{Z}b \simeq \mathbf{Z} + \mathbf{Z}\tau\,. \tag{3.4.3}$$

Taking on account of (3.3.2), (3.3.4) and (3.4.2), we obtain a family of natural isomorphisms:

$$\varphi_\tau \otimes I\colon \mathrm{rad}(I) \otimes_{\mathbf{R}} (F/\mathrm{rad}(I)) \simeq V_{\mathbf{C}} \tag{3.4.4}$$

$$\varphi_\tau \otimes \tilde{I}\colon \mathrm{rad}(I) \otimes_{\mathbf{R}} (F/G) \simeq \tilde{V}_{\mathbf{C}} \tag{3.4.4$\tilde{\ }$}$$

depending on the parameter $\tau \in \mathbf{H}$. Hence the H_R (2.3.2) is regarded as the lattice of the complex vector space $V_{\mathbf{C}}$ through $\varphi_\tau \otimes I$.

(3.5) The actions of the groups W_R and $\tilde{W}_R$ on F and $\tilde{F}$ fixes the rad (I) pointwisely. Hence the dual actions of W_R and $\tilde{W}_R$ induce actions on $\mathbf{E}$ and $\tilde{\mathbf{E}}$ respectively. They are equivariant with the projections $\tilde{\pi}$ and π (3.2.2). To avoid confusions, the dual action of $g \in W_R$ (resp. $\tilde{W}_R$) on $\mathbf{E}$ (resp. $\tilde{\mathbf{E}}$) will be denoted by g^*.

Lemma.

1. *The actions of W_R (resp. $\tilde{W}_R$) on $\mathbf{E}$ (resp. $\tilde{\mathbf{E}}$) are properly discontinuous.*

2. *The action of H_R on $\mathbf{E}$ coincides with the translation by the natural embedding $H_R \subset V_{\mathbf{C}}$ (cf. (2.3.2) and (3.4.4)).*

Put $X := \mathbf{E}/H_R$ and denote by π/H_R the map induced from π:

$$\pi/H_R\colon X \longrightarrow \mathbf{H}\,. \tag{3.5.1}$$

The fiber $X_\tau := (\pi/H_R)^{-1}(\tau)$ over $\tau \in \mathbf{H}$ is isogeneous to an l-times product of elliptic curves of the same modulous τ.

3. *The action of $\tilde{H}_R$ on $\tilde{\mathbf{E}}$ is fixed point free. Put $L^* := \tilde{\mathbf{E}}/\tilde{H}_R$. The map $\tilde{\pi}/\tilde{H}_R$ induced from $\tilde{\pi}$:*

$$\tilde{\pi}/\tilde{H}_R\colon L^* \longrightarrow X\,, \tag{3.5.2}$$

defines a principal $\mathbf{C}^*$-bundle over X. Let L be the associated complex line bundle over X, which is, as a set, a union

$$L = L^* \cup X\,. \tag{3.5.3}$$

The finite Weyl group W_{R_f} is acting on L and X equivariantly.

4. *The Chern class $c(L|_{X_\tau})$ of the line bundle over $X_\tau := (\pi/H_R)^{-1}(\tau)$ for $\tau \in \mathbf{H}$ is given by,*

$$c(L|_{X_\tau}) = \mathrm{Im}(H) \in \bigwedge^2 \mathrm{Hom}_{\mathbf{Z}}(H_R, \mathbf{C}) \simeq H^2(X_\tau, \mathbf{C})\,, \tag{3.5.4}$$

where H is an Hermitian form on $V_{\mathbf{C}}$ given by

$$H(z, w) := -\frac{m_{\max}}{t(R)(l_{\max}+1)\,\mathrm{Im}(\tau)} I_{R^\vee}(z, \bar{w})\,. \tag{3.5.5}$$

Proof. 1. Since the subgroups H_R and $\tilde{H}_R$ are of finite index in W_R and $\tilde{W}_R$, it is enough to show the properness of the actions of H_R and $\tilde{H}_R$. This will be shown explicitly in the next 2 and 3.

2. Recalling (2.3.10), the dual action g^* of $g = E(\xi + a \otimes p + b \otimes q) \in W_R$ on $x \in \mathbb{E}$ is calculated as:

$$\begin{aligned}\langle g^*(x), y\rangle &:= \langle x, g(y)\rangle = \langle x, E(\xi + a\otimes p + b\otimes q)(y)\rangle \\ &= \langle x, E_0(\xi)(y) - aI(p, y) - bI(q, y)\rangle \\ &= \langle E_0(\xi)^*(x), y\rangle - I(p, y) - \tau(x)I(q, y) \\ &= \langle E_0(\xi)^*(x) - I(p) - \tau(x)I(q), y\rangle\end{aligned}$$

and hence

$$g^*(x) = E_0(\xi)^*x - I(p) - \tau(x)I(q)\,. \tag{3.5.6}$$

Particularly for $g \in H_R$, $\xi(g) = 0$ and $E_0(0) = \mathrm{id}_L$ so that the action of H_R on $\mathbb{E}$ coincides with that through the embedding $H_R \subset V_{\mathbf{C}}$ (3.4.4).

3. Let us fix a base $\tilde{\lambda} \in \tilde{F}\backslash F$ normalized as

$$\tilde{I}(\tilde{\lambda}, b) = 1\,, \tag{3.5.7}$$

$$\tilde{I}(\tilde{\lambda}, \alpha_i) = 0 \qquad \text{for} \quad i = 1, \ldots, l\,. \tag{3.5.8}$$

Considering this $\tilde{\lambda}$ as a complex coordinate for $\tilde{\mathbb{E}}$, we obtain,

$$(\tilde{\lambda}, \tilde{\pi})\colon \tilde{\mathbb{E}} \xrightarrow{\sim} \mathbf{C} \times \mathbb{E}\,. \tag{3.5.9}$$

Using the formula (2.6.5), the action of $g \in \tilde{W}_R$ on $\tilde{\lambda}$ is given as:

$$g(\tilde{\lambda}) = \tilde{\lambda} + E_0(\xi)q(g) - \frac{1}{2}I(q(g), q(g))b - r(g)(I_R : I)\frac{l_{\max}+1}{m_{\max}}a\,. \tag{3.5.10}$$

Particularly the generator k of $\tilde{K}_R$ acts on $\tilde{\lambda}$ by

$$\tilde{E}(k)(\tilde{\lambda}) = \tilde{\lambda} - (I_R : I)\frac{l_{\max}+1}{m_{\max}}a\,.$$

Hence the complex function λ on $\tilde{\mathbb{E}}$ defined by

$$\lambda := \exp\left(2\pi\sqrt{-1}\,\frac{m_{\max}}{(I_R : I)(l_{\max}+1)}\tilde{\lambda}\right), \tag{3.5.11}$$

is $\tilde{K}$ invariant, giving a fiber coordinate for the $\mathbf{C}^*$ bundle:

$$(\lambda, \tilde{\pi})\colon \tilde{\mathbb{E}}/\tilde{K}_R \xrightarrow{\sim} \mathbf{C}^* \times \mathbb{E}\,. \tag{3.5.12}$$

Of course this direct decomposition of $\tilde{\mathbb{E}}/\tilde{K}_R$ is not canonical but it depends on the choice of $\tilde{\lambda}$.

The action of $g = a \otimes p(g) + b \otimes q(g) \in \tilde{H}_R/\tilde{K}_R \simeq H_R$ on λ is calculated by (3.5.10) as:

$$\tilde{E}(g)(\lambda) = \lambda e_g \tag{3.5.13}$$

where

$$e_g := \exp\left(2\pi\sqrt{-1}\,\frac{m_{\max}}{(I_R : I)(l_{\max}+1)}\left(q(g) - \frac{1}{2}I(q(g), q(g))\tau\right)\right). \tag{3.5.14}$$

This implies that $\tilde{\mathbb{E}}/\tilde{H}_R \simeq (\mathbb{C}^* \times \mathbb{E})/H_R$ is a principal $\mathbb{C}^*$-bundle over $\mathbb{E}/H_R$ associated to the Hermitian form (3.5.5) (cf. [14, Chap. I.2]).

(3.6) *Note.* Since the line bundle L^{-1} is ample relative to $\mathbb{H}$, one may blow down the zero section $X \subset L$ of L to $\mathbb{H}$ (i.e. X_τ is blow down to a point for $\tau \in \mathbb{H}$). The blow down space, denoted as

$$\mathbb{L}\ (\simeq L^* \cup \mathbb{H})\,, \tag{3.6.1}$$

is a family of affine algebraic variety of dimension $l+1$ with an isolated singularity parameterized by the space $\mathbb{H}$.

§4. The Ring S^W of Invariants

In this section, we recall and reformulate a Chevalley type theorem (4.5), studied by Looijenga [10], Schwarzman & Bernstein [3, 4], Kac & Peterson [7, 8] and others.

(4.1) For a non-negative integer k, let,

$$S_k := \Gamma(X, \mathcal{O}(L^{-\otimes k})) \tag{4.1.1}$$

be the module of holomorphic sections of the $-k$-th power of the line bundle L over X defined in (3.5) **Lemma 3**. An element $\theta \in S_k$ is realized as a holomorphic function on $\mathbb{E}$ satisfying the relation:

$$g^*(\theta) = (e_g)^{-k}\theta \qquad \text{for} \quad {}^\forall g \in H_R\,, \tag{4.1.2}$$

where e_g is as defined in (3.5.14). Due to [14, Chap. I], it is easily seen that S_k is a $\Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$-free module of finite rank, where $\Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ is the algebra of holomorphic functions on $\mathbb{H}$.

For an element $\theta \in S_k$, put,

$$\tilde{\theta} := (\lambda)^k\theta\,, \tag{4.1.3}$$

where λ is the function (3.5.11) on $\tilde{\mathbb{E}}$. Then (3.5.13) and (4.1.2) implies that $\tilde{\theta}$ is a $\tilde{H}_R$-invariant holomorphic function on $\tilde{\mathbb{E}}$ with a degree condition:

$$E(\tilde{\theta}) = k\tilde{\theta}\,, \tag{4.1.4}$$

for the Euler operator:

(4.1.5) $$E := \lambda \frac{d}{d\lambda}.$$

(4.2) *Note.* Define a graded algebra over $\Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ by

(4.2.1) $$S := \bigoplus_{k=0}^{\infty} S_k .$$

This algebra is the ring of polynomial functions of the affine variety $\mathbb{L}$ over $\mathbb{H}$ (3.6.1). In an algebraic geometric expression,

$$\operatorname{Spec}(S) \simeq \mathbb{L} .$$

(4.3) The group $W_{R_f} \simeq \tilde{W}_R/\tilde{H}_R$ acts on L and X equivariantly. So it acts also on the space of sections $S_k := \Gamma(X, \mathcal{O}(L^{-\otimes k}))$, $k = 0, 1, \ldots$. Put,

(4.3.1) $S_k^W :=$ the set of W_{R_f} invariant elements of S_k,

(4.3.2) $S_k^{-W} :=$ the set of W_{R_f} anti-invariant elements of S_k.

By the correspondence $\theta \leftrightarrow \tilde{\theta}$ (4.1.3), the set S_k^W (resp. S_k^{-W}) is regarded as the set of all $\tilde{W}_R$-invariant (resp. anti-invariant) holomorphic functions on $\tilde{\mathbb{E}}$, satisfying the degree condition (4.1.1), since the action of $\tilde{W}_R/\tilde{H}_R$ on S_k is identified with the pull-back action on the functions on $\tilde{\mathbb{E}}$.

Put,

(4.3.3) $$S^W := \bigoplus_{k=0}^{\infty} S_k^W$$

(4.3.4) $$S^{-W} := \bigoplus_{k=0}^{\infty} S_k^{-W} .$$

Naturally S^W is a $\Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$-algebra and S^{-W} is an S^W-module.

Remark. A basis of the module S_k^W and their product are described in terms of representation theory [8].

Let $\mathfrak{g}(A)$ be an affine Lie algebra. For an dominant integral weight $\Lambda \in P_+$ of level k, the character $Ch_{L(\Lambda)}$ of its highest weight representation is rewritten by a θ-function which gives an element of S_k^W. Irreducible decomposition of tensor representation gives the product rule.

(4.4) We prepare one more concept: the Jacobian $\tilde{J}(\theta_1, \ldots, \theta_{l+2})$ for a system of sections $\theta_i \in S_{k_i}$ $(i = 1, \ldots, l+2)$ as an element of $S_k \left(k = \sum_{i=1}^{l+2} k_i\right)$ given by the following relation.

$$d\tilde{\theta}_1 \wedge \cdots \wedge d\tilde{\theta}_{l+2} = \tilde{J}(\theta_1, \ldots, \theta_{l+2})\,(d\tau \wedge d\alpha_1 \wedge \cdots \wedge d\alpha_l \wedge d\tilde{\lambda}) .$$

The Jacobian is well defined, since $\omega := d\tau \wedge d\alpha_1 \wedge \cdots \wedge d\alpha_l \wedge d\tilde{\lambda}$ is $\tilde{H}_R$-invariant. Moreover, since the form ω is $\tilde{W}_R$ anti-invariant, if $\theta_i \in S_{k_i}^W$ $(i = 1, \ldots, l+2)$ then $J(\theta_1, \ldots, \theta_{l+2}) \in S_k^{-W}\left(k = \sum_{i=1}^{l+2} k_i\right)$.

(4.5) We recall a Chevalley type theorem for an EARS.

Theorem [3, 4, 8, 10].

1. *S^W is a polynomial algebra over $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$, freely generated by $l+1$ homogeneous elements $\theta_0, \ldots, \theta_l$ of degree $\tilde{m}_i := m_i \frac{(l_{\max}+1)}{m_{\max}}$ $(i=0,\ldots,l)$, where m_i $(i=0,\ldots,l)$ is the set of exponents for the root system (R, G) ((2.4.3)).*

2. *S^{-W} is a free S^W-module of rank 1, generated by an element $\Theta_A := \tilde{J}(\tau, \theta_0, \ldots, \theta_l)$ homogeneous of degree $\frac{(l+1+\operatorname{cod}(R,G))(l_{\max}+1)}{2}$.*

3. *The zero-loci of Θ_A on $\tilde{\mathbf{E}}$ is equal to the union $\bigcup_{\alpha \in R} H_\alpha$ of the complex hyperplanes H_α defined by the equation $\alpha = 0$ for $\alpha \in R$.*

Definition. *The invariant function,*

$$\Theta_A^2 \in S_k^W \qquad (k := (l+1+\operatorname{cod}(R,G))(l_{\max}+1))$$

will be called the discriminant for (R, G).

(4.6) The generator system $\theta_0, \ldots, \theta_l$ of the algebra S^W in the **Theorem** has an ambiguity of a change, $\theta_i' = P_i(\theta_0, \ldots, \theta_l)$ $(i = 0, \ldots, l)$ by weighted homogeneous polynomials with coefficients in $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$.

Notation. We fix a homogeneous generator system $\theta_0, \ldots, \theta_l$ of the algebra $S^W = \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})[\theta_0, \ldots, \theta_l]$ with an ordering

$$\deg(\theta_0) = 1 \leq \deg(\theta_1) \leq \cdots \leq \deg(\theta_l) = l_{\max} + 1 .$$

As a convention, we put

$$\theta_{-1} := \tau \qquad \text{and} \qquad \deg(\theta_{-1}) = 0 .$$

A goal of the paper is to find an intrinsic system of generators of the algebra S^W for the case $\operatorname{cod}(R, G) = 1$ which we shall call the flat generator system.

(4.7) *Note.* Since W_{R_f} is a finite group, the quotient space $\mathbf{L}/W_{R_f}$ is a family of algebraic variety over $\mathbf{H}$, whose polynomial function ring is the invariant ring S^W (cf. (3.6) and (4.2)).

$$\operatorname{Spec}(S^W) = \mathbf{L}/W_{R_f} \; (= (\tilde{\mathbf{E}}/\tilde{W}_R) \cup \mathbf{H}) .$$

The action of W_{R_f} on $\mathbf{L}$ is as a reflexion group, where the reflexion hyperplane of w_α for $\alpha \in R(\operatorname{mod} H_R)$ is given by a "hyperplane" $\mathbf{H}_\alpha$ in $\mathbf{L}$ as the image of the hyperplane $H_{\tilde{\alpha}} := \{x \in \tilde{\mathbf{E}}: \tilde{\alpha}(x) = 0\}$ for an inverse image $\tilde{\alpha} \in R$ of α. The branching loci of the map $\mathbf{L} \to \mathbf{L}/W_{R_f}$ is a hypersurface defined by $\Theta_A^2 = 0$, which will be called the discriminant loci.

§5. The Metrics $\tilde{I}_W$, $\tilde{I}_W^*$ and Logarithmic Forms on S^W

The form $\tilde{I}$ induces metric $\tilde{I}_W$ on the co-tangent space of $\operatorname{Spec}(S^W)$.

(5.1) Let us denote by $\mathcal{O}_{\mathbb{E}}$, $\Omega^1_{\mathbb{E}}$ and $\mathrm{Der}_{\mathbb{E}}$ the sheaf of germs of holomorphic functions, 1-forms and vector fields on $\mathbb{E}$ respectively.

Recalling an expression of tangent and cotangent spaces of $\tilde{\mathbb{E}}$ in (3.3.1) and (3.3.2), we have the cannonical isomorphisms:

$$\Omega^1_{\mathbb{E}} \simeq \mathcal{O}_{\mathbb{E}} \otimes_{\mathbb{R}} (\tilde{F}/G) \quad \text{and} \quad \mathrm{Der}_{\mathbb{E}} \simeq \mathcal{O}_{\mathbb{E}} \otimes_{\mathbb{R}} (\tilde{F}/G)^* . \tag{5.1.1}$$

The vector space $\tilde{F}/G$ carries a nondegenerate symmetric bilinear form induced from $\tilde{I}$. By extending $\tilde{I}$ to $\Omega^1_{\mathbb{E}}$ by $\mathcal{O}_{\mathbb{E}}$-bilinearly, we obtain a form:

$$\tilde{I}_{\mathbb{E}}: \Omega^1_{\mathbb{E}} \times \Omega^1_{\mathbb{E}} \to \mathcal{O}_{\mathbb{E}}, \qquad \omega_1 \times \omega_2 \mapsto \sum_{i,j=1}^{l+2} \frac{\omega_1}{\partial X_i} \frac{\omega_2}{\partial X_j} \tilde{I}(X_i, X_j) , \tag{5.1.2}$$

where X_i $(i = 1, \dots, l+2)$ are basis of $\tilde{F}/G$ and $\omega = \sum_i \frac{\omega}{\partial X_i} dX_i$.

(5.2) Put,

$$\mathrm{Der}_{S^W} := \text{the module of } \mathbb{C}\text{-derivations of the algebra } S^W , \tag{5.2.1}$$

$$\Omega^1_{S^W} := \text{the module of 1-forms for the algebra } S^W . \tag{5.2.2}$$

They are dual S^W-modules by the natural pairing: $\langle\ ,\ \rangle$ with the dual basis:

$$\mathrm{Der}_{S^W} = \bigoplus_{i=-1}^{l} S^W \frac{\partial}{\partial \theta_i} \qquad \text{and} \qquad \Omega^1_{S^W} = \bigoplus_{i=-1}^{l} S^W \, d\theta_i \text{ by}$$

using a generator system θ_i's of (4.6) *Notation*. There is a natural lifting map: $\Omega^1_{S^W} \to \Omega^1_{\mathbb{E}}$, $d\tilde{\theta} \mapsto \sum_i \frac{\partial \tilde{\theta}}{\partial X_i} dX_i$, so that the form $\tilde{I}_{\mathbb{E}}$ (5.1.2) induces an S^W-bilinear form:

$$\tilde{I}_W: \Omega^1_{S^W} \times \Omega^1_{S^W} \longrightarrow S^W . \tag{5.2.3}$$

(The values of $\tilde{I}_W$ lie in S^W, since the form $\tilde{I}_{\mathbb{E}}$ is $\tilde{W}_R$-invariant.)

Let us show a simple but crucial formulae for all what follows.

Formula.

$$\tilde{I}_W(d\tau, d\tilde{\theta}) = 2\pi\sqrt{-1}\frac{m_{\max}}{(I_R : I)(l_{\max}+1)} E(\tilde{\theta}) \qquad \textit{for} \quad \tilde{\theta} \in S^W , \tag{5.2.4}$$

$$\det(\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j)_{i,j=-1,\dots,l}) = \Theta_A^2 , \tag{5.2.5}$$

where E is the Euler operator (4.1.5).

Proof. Recall that $\tau = b/a$ and $a = 1$. So $d\tau = db/a = db$ and hence

$$\begin{aligned} \tilde{I}_W(d\tau, d\tilde{\theta}) &= \tilde{I}_W\left(db/a, \sum_{i=-1}^{l} dX_i \frac{\partial \tilde{\theta}}{\partial X_i}\right) \\ &= \tilde{I}_W(db, db)\frac{\partial \tilde{\theta}}{\partial b} + \sum_{i=1}^{l} \tilde{I}_W(db, d\alpha_i)\frac{\partial \tilde{\theta}}{\partial \alpha_i} + \tilde{I}_W(db, d\tilde{\lambda})\frac{\partial \tilde{\theta}}{\partial \tilde{\lambda}} \end{aligned}$$

$$= \tilde{I}(b, \tilde{\lambda})\frac{\partial \lambda}{\partial \tilde{\lambda}}\frac{\partial \tilde{\theta}}{\partial \lambda}$$

$$= 2\pi\sqrt{-1}\frac{m_{\max}}{(I_R : I)(l_{\max}+1)}\lambda\frac{\partial \tilde{\theta}}{\partial \lambda}$$

$$= 2\pi\sqrt{-1}\frac{m_{\max}}{(I_R : I)(l_{\max}+1)}E(\tilde{\theta}).$$

(cf (3.5.7), (3.5.8), (3.5.11) and (4.1.5).)

(5.2.5) is simply a reformulation of (4.5) Theorem 2, since by Definition (4.5.1) of the Jacobian J, we have,

$$\det (\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j)_{i,j=-1,\ldots,l}) = J(\theta_{-1}, \ldots, \theta_l)^2 \det ((I(\alpha_i, \alpha_j))_{ij}). \qquad \square$$

The formula (5.2.5) implies that the form $\tilde{I}_W$ is degenerate along the discriminant. Let us make this statement more precise by introducing logarithmic forms as follows.

(5.3) Consider the S^W-linear map induced from $\tilde{I}_W$ of (5.2.3).

$$\tilde{I}_W\colon \Omega^1_{S^W} \longrightarrow \mathrm{Der}_{S^W}, \qquad \omega \mapsto \tilde{I}_W(\omega, \cdot) := \sum_{i=0}^{l} \tilde{I}_W(\omega, d\theta_i)\, d\theta_i .$$

Assertion. *The above map $\tilde{I}_W$ induces an S^W-isomorphism:*

(5.3.1) $$\tilde{I}_W\colon \Omega^1_{S^W} \xrightarrow{\sim} \mathrm{Der}_{S^W}(-\log(\Theta_A^2)),$$

where

(5.3.2) $$\mathrm{Der}_{S^W}(-\log(\Theta_A^2)) := \{\delta \in \mathrm{Der}_{S^W};\ \delta\Theta_A^2 \in \Theta_A^2 S^W\}.$$

Proof. For an $\omega \in \Omega^1_{S^W}$, the element $\tilde{I}_W(\omega, d\Theta_A)$ is anti-invariant. Hence it is divisible by Θ_A ((4.4) Theorem 2.). Then $\tilde{I}_W(\omega, d\Theta_A^2) = 2\Theta_A \tilde{I}_W(\omega, d\Theta_A)$ is divisible by Θ_A^2. This implies $\tilde{I}_W(\omega) \in \mathrm{Der}_{S^W}(-\log(\Theta_A^2))$ by definition.

To show that the (5.3.1) is an S^W-isomorphism, we recall the following simple criterium for free basis of logarithmic modules.

Criterium ([22]). *Let elements* $\delta_i = \sum_{j=-1}^{l} g_{ij}\frac{\partial}{\partial \theta_j} \in \mathrm{Der}_{S^W}(-\log(\Theta_A^2))$ $(i = -1, \ldots, l)$ *be given. Then* $\mathrm{Der}_{S^W}(-\log(\Theta_A^2))$ *is an S^W-free module with the free basis* δ_i $(i = -1, \ldots, l)$ *if and only if* $\det((g_{ij})_{ij})$ *is a unit multiple of* Θ_A^2.

We apply the criterium for $\delta_i := \tilde{I}_W(d\theta_i) = \sum_{j=-1}^{l} \tilde{I}_W(d\theta_i, d\theta_j)\frac{\partial}{\partial \theta_j}$ $(i = -1, \ldots, l)$. Since $\det(\tilde{I}_W(d\theta_i, d\theta_j)) = \Theta_A^2$ (5.2.5), the condition in the criterium is satisfied. Hence $\tilde{I}_W(d\theta_i)$ $(i = -1, \ldots, l)$ form an S_W-free basis of $\mathrm{Der}_{S^W}(-\log(\Theta_A^2))$. $\square$

Corollary. Let us denote by $\tilde{I}_W^*$ the S^W-bilinear form on the module Der_{S^W} *dual to the form $\tilde{I}_W$* (5.2.3). *Then the pairing,*

(5.3.3) $$\tilde{I}_W^*\colon \mathrm{Der}_{S^W} \times \mathrm{Der}_{S^W}(-\log(\Theta_A^2)) \longrightarrow S^W$$

is S^W-perfect.

Proof. By definition, $\tilde{I}_W^*(x, y) := \langle x, \tilde{I}_W^{-1}(y)\rangle$. The isomorphy (5.3.1) implies the perfectness (i.e. $\det(\tilde{I}_W^*(x_i, y_j)_{ij}) =$ unit in S^W for a basis x_i and y_j of Der_{S^W} and $\mathrm{Der}_{S^W}(-\log(\Theta_A^2))$.). □

Remark. There is a natural lifting $\mathrm{Der}_{S^W}(-\log \Theta_A^2) \to \mathrm{Der}_{\mathbb{E}}$ such that $\tilde{I}_W^*$ (5.3.3) is a pull-back of $\tilde{I}_{\mathbb{E}}^*$ on $\mathrm{Der}_{\mathbb{E}}$ (cf. (6.3) Assertion).

(5.4) We have shown the isomorphy: $\tilde{I}_W: \Omega_{S^W}^1 \simeq \mathrm{Der}_{S^W}(-\log(\Theta_A^2))$. Particularly important is the correspondence between $d\tau$ and the Euler operator E, which was calculated in (5.2.4).

$$\tilde{I}_W(d\tau) = 2\pi\sqrt{-1}\frac{m_{\max}}{(I_R : I)(l_{\max}+1)}E. \tag{5.4.1}$$

$$\tilde{I}_W^*(E) = \frac{(I_R : I)(l_{\max}+1)}{2\pi\sqrt{-1}m_{\max}}d\tau. \tag{5.4.2}$$

The Euler operator E (4.1.5) is expressed as an element of Der_{S^W} by

$$E = \sum_{i=0}^{l} \deg(\theta_i)\theta_i\frac{\partial}{\partial\theta_i}. \tag{5.4.3}$$

(5.5) Put

$$\Omega_{S^W}^1(\log(\Theta_A^2)) := \{\omega \in \Omega_{S^W}^1(*\Theta_A^2) \colon \Theta_A^2\omega \in \Omega_{S^W}^1 \text{ and } \Theta_A^2\, d\omega \in \Omega_{S^W}^2\}.$$

The natural pairing between forms and vector fields induces a perfect pairing: $\mathrm{Der}_{S^W}(-\log(\Theta_A^2)) \times \Omega_{S^W}^1(\log(\Theta_A^2)) \to S^W$ (cf. [22]).

The Assertion (5.3) implies the following commutative diagram.

$$\begin{array}{ccc} \Omega_{S^W}^1 & \subset & \Omega_{S^W}^1(\log(\Theta_A^2)) \\ \wr\downarrow \tilde{I}_W & & \wr\downarrow \tilde{I}_W \\ \mathrm{Der}_{S^W}(-\log(\Theta_A^2)) & \subset & \mathrm{Der}_{S^W}. \end{array} \tag{5.5.1}$$

Remark. As a (2, 0)-type tensor field on $\mathrm{Spec}(S^W)$, $\tilde{I}_W^*$ (5.3.3) is expressed as:

$$\tilde{I}_W^* = \sum_{i=-1}^{l} d\theta_i \otimes \tilde{I}_W^{-1}\left(\frac{\partial}{\partial\theta_i}\right), \tag{5.5.2}$$

implying that $\tilde{I}_W^*$ has logarithmic poles along the discriminant.

(5.6) *Note.* The above logarithmic modules are that for the discriminant: $\Theta_A^2 = 0$ in the space $\mathbb{L}/W_{R_f} := \mathrm{Spec}(S^W)$. Instead of this, one may study the logarithmic modules for a system of hyperplanes:

$$\bigcup_{\alpha\in R_f} \mathbb{H}_\alpha = \{\Theta_A = 0\}$$

in $\mathbb{L} := \mathrm{Spec}(S)$ as follows. (cf. (3.6), (4.2), (4.7)). Put

$$\mathrm{Der}_S(-\log\Theta_A) := \{\delta \in \mathrm{Der}_S \colon \delta\Theta_A \in \Theta_A S\},$$

$$\Omega_S^1(\log\Theta_A) := \{\omega \in \Omega_S^1(*\Theta_A) \colon \Theta_A\omega \in \Omega_S^1 \text{ and } \Theta_A\, d\omega \in \Omega_S^2\}.$$

which are dual S-modules by the natural pairing. Then a slightly modified argument in (5.3) shows the followings.

Assertion. *The bilinear form $\tilde{I}_{\mathbf{E}}$ of* (5.1.2) *induces a non-degenerate S-bilinear form*:

$$\tilde{I}_{\mathbf{L}}: \Omega_S^1 \times \Omega_S^1 \to S\,.$$

Then this $\tilde{I}_{\mathbf{L}}$ (: $\Omega_S^1 \to \mathrm{Der}_S$) induces the following S-isomorphisms:

$$(5.6.1)\qquad \begin{array}{ccccc} \Omega^1_{S^W}\otimes S & \subset & \Omega^1_S & \subset & \Omega^1_S(\log \Theta_A) \\ \downarrow \wr\ \tilde{I}_{\mathbf{L}} & & \downarrow \wr\ \tilde{I}_{\mathbf{L}} & & \downarrow \wr\ \tilde{I}_{\mathbf{L}} \\ \mathrm{Der}_S(-\log \Theta_A) & \subset & \mathrm{Der}_S & \subset & \mathrm{Der}_{S^W}\otimes S\,. \end{array}$$

Particularly this implies that $\mathrm{Der}_S(-\log \Theta_A)$ *and* $\Omega_S^1(\log \Theta_A)$ *are S-free modules of rank* $l+2$, *with the free basis* $\tilde{I}_{\mathbf{L}}(d\theta_i)$ *and* $\tilde{I}_{\mathbf{L}}^{-1}\left(\frac{\partial}{\partial\theta_i}\right)$ *for* $i=-1,\ldots,l$ *respectively.*

(5.7) *Remark.* Similar diagrams to (5.5.1) and (5.6.1) were shown for a finite reflexion group W in [23]. The diagrams relate the exponents m_i ($i=1,\ldots,l$) of the group W with the degrees m_i-1 ($i=1,\ldots,l$) of free basis of $\mathrm{Der}_S(-\log \Delta)$. (Cf the generalized Schephard-Todd-Brieskorn formula. Terao [32], Orlik-Solomon [16, 17]).

As an analogue to this fact in case of extended affine root systems, we ask to clarify the relationship among the following three polynomials:

i) The Möbius function for the lattice defined by the system of hyperplanes $\mathbf{H}_\alpha(\alpha\in R)$ in $\mathbf{L}$.

ii) The Poincare polynomial for the topological space $\mathbf{L}\setminus\bigcup_{\alpha\in R}\mathbf{H}_\alpha$.

iii) $P(T):=\prod_{i=-1}^{l}(1+\tilde{m}_iT)$.

(5.8) *Remark.* The tangent spaces of $\mathrm{Spec}(S^W)$, $\mathbf{L}\setminus\mathbf{H}$ and $\tilde{\mathbf{E}}$ are the complexification (3.3.2) of a real vector space $(\tilde{F}/G)^*$, which carries a nondegenerate symmetric bilinear form $\tilde{I}^*$. Its complex bilinear extension gives $\tilde{I}_W^*$ (5.3.3), $\tilde{I}_{\mathbf{L}}^*$ (5.6) and $\tilde{I}_{\mathbf{E}}^*$ (6.3.2).

Let $\tilde{H}_W^*$, $\tilde{H}_{\mathbf{L}}^*$ and $\tilde{H}_{\mathbf{E}}^*$ be the sesquilinear extension of $\tilde{I}^*$ on the tangent spaces of $\mathrm{Spec}(S^W)$, $\mathbf{L}\setminus\mathbf{H}$ and $\tilde{\mathbf{E}}$ respectively. They are Hermitian forms on the tangent bundles, whose sign is $(l+1,1)$. Since the group action of $\tilde{H}_R$ on $\tilde{\mathbf{E}}$ has no fixed point, $\tilde{H}_{\mathbf{E}}^*$ and $\tilde{H}_{\mathbf{L}}^*$ are regular everywhere on $\tilde{\mathbf{E}}$ and $\mathbf{L}\setminus\mathbf{H}$ respectively. But $\tilde{H}_W^*$ is singular along the discriminant loci, having logarithmic poles.

The geometric significance of the Hermitian forms $\tilde{H}_W^*$, $\tilde{H}_{\mathbf{L}}^*$ and $\tilde{H}_{\mathbf{E}}^*$ is not clear and is yet to be studied.

§6. The Logarithmic Connection ∇ on Der_{S^W}

A metric connection ∇ on the tangent bundle of $\mathrm{Spec}(S^W)$ is introduced, which has logarithmic poles along the discriminant. ∇ may be called the Gauss-Manin connection for the root system, since it is to be identified with the Gauss-Manin connection for an unfolding of a simple elliptic singularity after all.

(6.1) **Lemma.** *There exists uniquely a connection ∇ on the S^W module* Der_{S^W}, *which is integrable, torsion free and metric w.r.t.* $\tilde{I}_W^*$ (5.3.3), *having the logarithmic poles along the discriminant in the following sense.*

There exists uniquely a $\mathbb{C}$*-bilinear map*

$$\nabla\colon \mathrm{Der}_{S^W}(-\log \Theta_A^2) \times \mathrm{Der}_{S^W}(-\log \Theta_A^2) \longmapsto \mathrm{Der}_{S^W}(-\log \Theta_A^2)$$

$$\delta \quad \times \quad \xi \quad \longmapsto \quad \nabla_\delta \xi$$

such that

i) ∇ *is a connection.* (*I.e.* $\nabla_\delta(\xi)$ *is* S^W*-linear for the variable* δ *and satisfy the Leibnitz rule for the variable* ξ.)

ii) ∇ *is integrable.* $[\nabla_{\delta_1}, \nabla_{\delta_2}] = \nabla_{[\delta_1,\delta_2]}$.

iii) ∇ *is torsion free.* $\nabla_\delta \xi - \nabla_\xi \delta = [\delta, \xi]$.

iv) ∇ *is metric w.r.t.* $\tilde{I}_W^*$. $\delta_1 \tilde{I}_W^*(\delta_2, \delta_3) = \tilde{I}_W^*(\nabla_{\delta_1}\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, \nabla_{\delta_1}\delta_3)$.

Proof of (6.1) *Lemma.* The Proof is divided in the following (6.2) ~ (6.4). The existence and the uniqueness of a connection ∇ satisfying i), iii) and iv) defined on the complement of the discriminant loci in $\mathrm{Spec}(S^W)$ is well known as the Levi-Civita connection for the metric $\tilde{I}_W^*$. (Recall that $\tilde{I}_W^*$ is non-degenerate outside of the discriminant.) Since we need to show the integrability ii) of ∇ and to give a description of the singularities of ∇ along the discriminant, we proceed to an explicit description of ∇.

(6.2) First we proceed to an explicit construction of ∇.

1. *The uniqueness of* ∇. Assuming an existence of ∇ satisfying i), iii) and iv), let us show its uniqueness.

For $\delta_1, \delta_2, \delta_3 \in \mathrm{Der}_{S^W}(\log \Theta_A^2)$,

$$\begin{aligned}
\tilde{I}_W^*(\nabla_{\delta_1}\delta_2, \delta_3) &= \delta_1 \tilde{I}_W^*(\delta_2, \delta_3) - \tilde{I}_W^*(\delta_2, \nabla_{\delta_1}\delta_3)\\
&= \delta_1 \tilde{I}_W^*(\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, [\delta_3, \delta_1]) - \tilde{I}_W^*(\delta_2, \nabla_{\delta_3}\delta_1)\\
&= \delta_1 \tilde{I}_W^*(\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_W^*(\delta_2, \delta_1) + \tilde{I}_W^*(\nabla_{\delta_3}\delta_2, \delta_1)\\
&= \delta_1 \tilde{I}_W^*(\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_W^*(\delta_2, \delta_1) - \tilde{I}_W^*([\delta_2, \delta_3], \delta_1)\\
&\quad + \tilde{I}_W^*(\nabla_{\delta_2}\delta_3, \delta_1)\\
&= \delta_1 \tilde{I}_W^*(\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_W^*(\delta_2, \delta_1) - \tilde{I}_W^*([\delta_2, \delta_3], \delta_1)\\
&\quad + \delta_2 \tilde{I}_W^*(\delta_3, \delta_1) - \tilde{I}_W^*(\delta_3, \nabla_{\delta_2}\delta_1)\\
&= \delta_1 \tilde{I}_W^*(\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_W^*(\delta_2, \delta_1) - \tilde{I}_W^*([\delta_2, \delta_3], \delta_1)\\
&\quad + \delta_2 \tilde{I}_W^*(\delta_3, \delta_1) + \tilde{I}_W^*(\delta_3, [\delta_1, \delta_2]) - \tilde{I}_W^*(\delta_3, \nabla_{\delta_1}\delta_2)\,.
\end{aligned}$$

Hence altogether, one obtains the formula:

$$\begin{aligned}
(6.2.1)\quad \tilde{I}_W^*(\nabla_{\delta_1}\delta_2, \delta_3) = \{&\delta_1 \tilde{I}_W^*(\delta_2, \delta_3) + \tilde{I}_W^*(\delta_2, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_W^*(\delta_2, \delta_1)\\
&- \tilde{I}_W^*([\delta_2, \delta_3], \delta_1) + \delta_2 \tilde{I}_W^*(\delta_3, \delta_1) + \tilde{I}_W^*(\delta_3, [\delta_1, \delta_2])\}/2\,.
\end{aligned}$$

The right hand of the formula (6.2.1) does not contain ∇. Hence the left hand is independent of a choice of ∇. Since the discriminant of $\tilde{I}_W^*$ is Θ_A^{-2} and Θ_A^2 is a non zero divisor in S^W, $\nabla_{\delta_1}\delta_2$ is uniquely determined from (6.2.1) (cf. (5.3) Cor).

2. *The existence of* ∇. Put $F(\delta_1, \delta_2, \delta_3) :=$ the right hand of (6.2.1). Let the variables δ_1 and δ_2 run over $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$ and δ_3 over Der_{S^W}. Since $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$ is closed under the bracket product, the value of F defines an element of S^W (cf. (5.3.3)). Noting that F is S^W-linear in the variable δ_3 and that $\tilde{I}_W^*$ is a perfect (5.3.3), one may write $F(\delta_1, \delta_2, \delta_3) = \tilde{I}_W^*(G(\delta_1, \delta_2), \delta_3)$ for some $G(\delta_1, \delta_2) \in \mathrm{Der}_{S^W}(-\log \Theta_A^2)$. It is easy to check that G is S^W-linear in δ_1 and that $G(\delta_1, \theta\delta_2) = \theta G(\delta_1, \delta_2) + (\delta_1\theta)\delta_2$ for $\theta \in S^W$. This implies the existence of a connection ∇ such that $G(\delta_1, \delta_2) = \nabla_{\delta_1}\delta_2$. It is also a routine work to check that ∇ is torsion free and metric w.r.t. $\tilde{I}_W^*$.

(6.3) To show the integrability of ∇, we extend the domain of ∇.

The set $\Gamma(\tilde{\mathbb{E}}, \mathrm{Der}_{\tilde{\mathbb{E}}})$ of all holomorphic vector fields on $\tilde{\mathbb{E}}$ will be denoted by $\Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}})$ for short. The set of all holomorphic functions on $\tilde{\mathbb{E}}$ will be denoted by $\Gamma(\mathcal{O}_{\tilde{\mathbb{E}}})$.

Recalling (3.3.1), one has a natural isomorphism:

(6.3.1) $$\Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}}) \simeq \Gamma(\mathcal{O}_{\tilde{\mathbb{E}}}) \otimes_{\mathbb{R}} (\tilde{F}/G)^*,$$

where $(\tilde{F}/G)^* = \mathbb{R}\dfrac{\partial}{\partial X_1} \oplus \cdots \oplus \mathbb{R}\dfrac{\partial}{\partial X_{l+2}}$, if $\tilde{F}/G = \mathbb{R}X_1 \oplus \cdots \oplus \mathbb{R}X_{l+2}$.

Since $\tilde{I}$ induces a non-degenerate form on $\tilde{F}/G$ (2.6) ii), let us denote by $\tilde{I}^*$ the dual form on $(\tilde{F}/G)^*$. Applying (6.3.1), one gets a $\Gamma(\mathcal{O}_{\mathbb{E}})$-bilinear map $\tilde{I}_{\tilde{\mathbb{E}}}^*$ on $\Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}})$, which is
dual to $\tilde{I}_{\tilde{\mathbb{E}}}$ (5.1.2).

(6.3.2) $$\tilde{I}_{\tilde{\mathbb{E}}}^*: \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}}) \times \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}}) \to \Gamma(\mathcal{O}_{\tilde{\mathbb{E}}}).$$

Recall that the invariant ring S^W is naturally embedded in $\Gamma(\mathcal{O}_{\tilde{\mathbb{E}}})$

(4.3). Put

$$\varphi: S^W \subset \Gamma(\mathcal{O}_{\tilde{\mathbb{E}}}).$$

Assertion. *There exists a natural injective* $\mathbb{C}$*-linear map*

(6.3.3) $$\iota: \mathrm{Der}_{S^W}(-\log \Theta_A^2) \to \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}})$$

such that

i) ι *is* φ*-linear*: $\iota(\theta\delta) = \varphi(\theta)\iota(\delta)$ *for* $\theta \in S^W$ *and* $\delta \in \mathrm{Der}_{S^W}(\log \Theta_A^2)$.

ii) ι *is equivariant with the derivation action*:

$$\varphi(\delta\theta) = \iota(\delta)\varphi(\theta) \qquad \textit{for} \quad \theta \in S^W \textit{ and } \delta \in \mathrm{Der}_{S^W}(\log \Theta_A^2).$$

iii) ι *is a Lie algebra homomorphism.*

iv) *The pull back of the form* $\tilde{I}_{\tilde{\mathbb{E}}}^*$ *through the map* ι *is* $\tilde{I}_W^*$.

Proof. By definition, S^W is a subalgebra of $\Gamma(\mathcal{O}_{\tilde{\mathbb{E}}})$ consisting of $\tilde{W}_R$-invariant functions, which are polynomials in the variable λ. (cf. (4.3)). The map ι is defined as the composition of the following maps:

(6.3.*) $$\iota\colon \mathrm{Der}_{S^W}(-\log \Theta_A^2) \xrightarrow[\tilde{I}_W^*]{} \Omega^1_{S^W} \longrightarrow \Gamma(\Omega^1_{\tilde{\mathbb{E}}}) \xrightarrow[\tilde{I}_{\mathbb{E}}]{} \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}})$$

i) ι is S^W-linear and injective, since each step in (6.3.*) is so.

ii) That ι is equivariant with derivation, means the formula:

(6.3.4) $$\delta(\theta) = \iota(\delta)(\theta) \quad \text{for} \quad \delta \in \mathrm{Der}_{S^W}(-\log \Theta_A^2) \quad \text{and} \quad \theta \in S^W .$$

$(\iota(\delta)(\theta) = \tilde{I}_{\mathbb{E}}(\tilde{I}_W^*(\delta))(\theta) = \tilde{I}_{\mathbb{E}}(\tilde{I}_W^*(\delta), \mathrm{d}\theta) = \tilde{I}_W(\tilde{I}_W^*(\delta), \mathrm{d}\theta) = \langle \delta, \mathrm{d}\theta \rangle = \delta(\theta).)$

iii) Applying (6.3.4) repeatedly, one obtains a relation,

(6.3.5) $$\iota[\delta, \xi](\theta) = [\iota(\delta), \iota(\xi)](\theta)$$

for $\delta, \xi \in \mathrm{Der}_{S^W}(-\log \Theta_A^2)$ and $\theta \in S^W$. Since the Jacobian $J(\theta_{-1}, \dots, \theta_l)$ is a nonzero divisor in $\Gamma(\mathcal{O}_{\tilde{\mathbb{E}}})$, the (6.3.5) is enough to show that ι is a Lie algebra homomorphism.

iv) This is checked as follows.

$$\begin{aligned}\tilde{I}_{\mathbb{E}}^*(\iota(\delta), \iota(\xi)) &= \tilde{I}_{\mathbb{E}}^*(\tilde{I}_{\mathbb{E}}(\tilde{I}_W^*(\delta)), \tilde{I}_{\mathbb{E}}(\tilde{I}_W^*(\xi))) = \tilde{I}_{\mathbb{E}}(\tilde{I}_W^*(\delta), \tilde{I}_W^*(\xi)) \\ &= \tilde{I}_W(\tilde{I}_W^*(\delta), \tilde{I}_W^*(\xi)) = \tilde{I}_W^*(\delta, \xi) .\end{aligned}$$ □

(6.4) The same argument as in (6.2) shows the existence and the uniqueness of connection,

(6.4.1) $$\tilde{\nabla}\colon \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}}) \times \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}}) \longrightarrow \Gamma(\mathrm{Der}_{\tilde{\mathbb{E}}})$$

such that i) $\tilde{\nabla}$ is torsion free, and ii) $\tilde{\nabla}$ is metric w.r.t. $\tilde{I}_{\mathbb{E}}^*$. Such $\tilde{\nabla}$ is explicitly given by a similar formula as (6.2.1).

(6.4.2) $$\begin{aligned}\tilde{I}_{\mathbb{E}}^*(\tilde{\nabla}_{\delta_1}\delta_2, \delta_3) = \{&\delta_1 \tilde{I}_{\mathbb{E}}^*(\delta_2, \delta_3) + \tilde{I}_{\mathbb{E}}^*(\delta_2, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_{\mathbb{E}}^*(\delta_2, \delta_1) \\ &- \tilde{I}_{\mathbb{E}}^*([\delta_2, \delta_3], \delta_1) + \delta_2 \tilde{I}_{\mathbb{E}}^*(\delta_3, \delta_1) + \tilde{I}_{\mathbb{E}}^*(\delta_3, [\delta_1, \delta_2])\}/2 .\end{aligned}$$

Let X_i $(i = 1, \dots, l+2)$ be linear coordinates of $\tilde{\mathbb{E}}$ so that $\frac{\partial}{\partial X_i}$ generates $\mathrm{Der}_{\tilde{\mathbb{E}}}$ as $\Gamma(\mathcal{O}_{\tilde{\mathbb{E}}})$ module. Noting that $\tilde{I}_{\mathbb{E}}^*\left(\frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j}\right)$'s are constants, and substituting δ_1, δ_2 and δ_3 of (6.4.2) by $\frac{\partial}{\partial X_i}$ $(i = 1, \dots, l+2)$, we obtain $\tilde{I}_{\mathbb{E}}\left(\tilde{\nabla}_{\frac{\partial}{\partial X_i}} \frac{\partial}{\partial X_j}\right) = 0$ for $i, j = 1, \dots, l+2$. Recalling the fact that $\tilde{I}_{\mathbb{E}}^*$ is non-degenerate, we conclude that $\tilde{\nabla}\left(\frac{\partial}{\partial X_i}\right) = 0$. This in particular implies that $\tilde{\nabla}$ is integrable.

By comparing the formulae (6.2.1) and (6.4.2), we see that the restriction $\tilde{\nabla} | \mathrm{Der}_{S^W}(-\log \Theta_A^2) \times \mathrm{Der}_{S^W}(-\log \Theta_A^2)$ coincides with ∇. Then the integrability of $\tilde{\nabla}$ implies that of ∇.

This completes a *Proof* of the (6.1) **Lemma.** □

(6.5) **Lemma bis.** *The connection* ∇ *in* (6.1) **Lemma**, *naturally extends to*:

$$\nabla\colon \mathrm{Der}_{S^W}(-\log \Theta_A^2) \times \mathrm{Der}_{S^W} \to \mathrm{Der}_{S^W} .$$

$$\nabla\colon \mathrm{Der}_{S^W} \times \mathrm{Der}_{S^W}(-\log \Theta_A^2) \to \mathrm{Der}_{S^W} .$$

Proof. In (6.2.1), if one of the variables δ_1, δ_2 for ∇ and δ_3 belongs to $\mathrm{Der}_{S^W}(\log \Theta_A^2)$ and the remaining variable in Der_{S^W}, then the value of the right hand of (6.2.1) belongs to S^W for the same reasons as in the proof of the lemma. Then the perfectness of $\tilde{I}_W^*$ (5.3.3) implies that the range of ∇ is Der_{S^W}. □

(6.6) *The Euler operator E (5.4.3) is horizontal. I.e.*

(6.6.1)
$$\nabla_\delta E = 0 \qquad \textit{for} \quad {}^\forall\delta \in \mathrm{Der}_{S^W}.$$

Proof. Substitute δ_2 by E in the formula (6.2.1). Applying (5.2.4) and the facts in (5.4), one has

$$\tilde{I}_W^*(\tilde{\nabla}_{\delta_1} E, \delta_3) = \delta_1 \tilde{I}_W^*(E, \delta_3) + \tilde{I}_W^*(E, [\delta_3, \delta_1]) - \delta_3 \tilde{I}_W^*(E, \delta_1) = 0.$$ □

Remark. The fact (6.6.1) (coming from the fact that the intersection form I is degenerate) makes the flat structure for EARS's in the present paper much more distinct and harder, compared with that for finite root systems [23] and for indefinite exceptional root systems. Namely, as an element of $\mathrm{Hom}_{S^W}(\mathrm{Der}_{S^W}, \mathrm{Der}_{S^W})$,

$$\nabla E = \mathrm{id}_{\mathrm{Der}_{S^W}}, \qquad \text{or} \qquad -\mathrm{id}_{\mathrm{Der}_{S^W}},$$

according as the finite root system case or the exceptional root system case.

(6.7) The following relation:

$$\delta\langle \xi, \omega\rangle = \langle \nabla_\delta \xi, \omega\rangle + \langle \xi, \nabla_\delta^* \omega\rangle$$

for $\delta, \xi \in \mathrm{Der}_{S^W}$ and $\omega \in \Omega_{S^W}^1$ defines a dual connection,

$$\nabla^*\colon \mathrm{Der}_{S^W} \times \Omega_{S^W}^1 \to \Omega_{S^W}^1(\log \Theta_A^2),$$

which is integrable, metric, torsion free, having a logarithmic singularities along the discriminant. Here "torsion free" means the commutativity of the following diagram:

$$\begin{array}{ccc} \Omega_{S^W}^1 & \xrightarrow{\ \nabla^*\ } & \Omega_{S^W}^1(\log \Theta_A^2) \otimes \Omega_{S^W}^1 \\ & {\scriptstyle d}\searrow \quad \swarrow{\scriptstyle \wedge} & \\ & \Omega_{S^W}^2(\log \Theta_A^2) & \end{array}$$

The explicit formula (6.2.1) is rewritten as:

(6.7.1)
$$\begin{aligned} \tilde{I}_W(\nabla_\delta^* \omega_2, \omega_3) = \{&\delta \tilde{I}_W(\omega_2, \omega_3) + \langle \omega_2, [\tilde{I}_W \omega_3, \delta]\rangle - \tilde{I}_W(\omega_3)(\delta\omega_2) \\ &- \tilde{I}_W^*([\tilde{I}_W \omega_2, \tilde{I}_W \omega_3], \delta) - \langle \omega_3, [\tilde{I}_W \omega_2, \delta]\rangle + \tilde{I}_W(\omega_2)(\delta\omega_3)\}/2. \end{aligned}$$

§7. The Fixed Points by the Coxeter Transformation

Leading coefficients (as functions in $\tau \in \mathbb{H}$) of the polynomial expression of Θ_A^2 is shown to have no common zeros. This is an important consequence of the **Lemma**'s $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$.

(7.1) Let $c \in W_R$ be the Coxeter transformation for (R, G) (2.5). Put, (7.1.1) $$\mathbb{E}^c := \{x \in \mathbb{E} : c^*(x) = x\}\,.$$

Since $c^*(x) = x \Leftrightarrow \langle f, c^*(x) - x\rangle = 0$ for ${}^\forall f \in F \Leftrightarrow \langle cf - f, x\rangle = 0$ for ${}^\forall f \in F$, the space $\mathbb{E}^c$ is the common zeros of the functionals $(c - \mathrm{id}_F)F$ and the equation $a(x) - 1 = 0$ with the inequality $\mathrm{Im}(b(x)) > 0$. Therefore

$$\dim_{\mathbb{C}} \mathbb{E}^c = \#\,\{\text{eigen value 1 of } c\} - 1 = \mathrm{cod}\,(R, G)\,.$$

(Cf. (2.5) Lemma A). Since $c(a) = a$ and $c(b) = b$, we have a surjective map,

(7.1.2) $$\mathbb{E}^c \to \mathbb{H}\,.$$

(7.2) **Lemma.** *For any $\tau \in \mathbb{H}$, we have,*

(7.2.1) $$\mathbb{E}^c \cap \mathbb{E}_\tau \not\subset \bigcup_{\alpha \in R} H_\alpha\,.$$

Proof. Since the right hand of (7.2.1) is a locally finite union of hyperplanes and the left hand is a linear subspace over $\mathbb{C}$, it is enough to show $\mathbb{E}^c \cap \mathbb{E}_\tau \not\subset H_\alpha$ for ${}^\forall \alpha \in R$. Suppose the contrary: $\mathbb{E}^c \cap \mathbb{E}_\tau \subset H_\alpha$ for some $\alpha \in R$. Then the Hilbert-Nullstellensatz for the polynomial ring $\mathbb{C}[F]$ of functions on $\mathbb{C} \otimes_{\mathbb{R}} F^*$ implies that α belongs to the ideal $\mathscr{I}(\mathbb{E}^c \cap \mathbb{E}_\tau)$ generated by $(c - \mathrm{id}_F)F$, $a - 1$ and $b - \tau$. Hence

(*) $$\alpha = \sum_{i=1}^{k} g_i(c - \mathrm{id}_F)f_i + g(a - 1) + g'(b - \tau)$$

for some $k \in \mathbb{N}$ and $f_i \in F$ and g_i, g and $g' \in \mathbb{C}[F]$ $(i = 1, \ldots, k)$. Since $(c - \mathrm{id}_F)F \cap \mathrm{rad}(\mathrm{I}) = \{0\}$, we may assume that $(c - \mathrm{id}_F)f_i$ $(i = 1, \ldots, k)$ and a and b are linearly independent. Let us show by a descending induction on $d := \max\{\deg(g_1), \ldots, \deg(g_k), \deg(g), \deg(g')\}$, which we can reduce, that g_i's, g and g' are constants.

Suppose $d > 0$. Let us denote by g_i^*, g^* and g'^* the degree $= \mathrm{d}$ part of the corresponding polynomials. Then obviously, we have:

(**) $$0 = \sum_{i=1}^{k} g_i^*(c - \mathrm{id}_F)f_i + g^*a + g'b^*\,.$$

The linear independence of $(\mathrm{c} - \mathrm{id}_F)f_i$'s, a and b implies then an existence of a skew symmetric matrix $\mathscr{H}$ of the size $k + 2$ whose entries are homogeneous polynomials in $\mathbb{C}[F]$ of degree $d - 1$ such that $(g_1{}^*, \ldots, g_k^*, g^*, g'^*) = ((c - \mathrm{id}_F)f_1, \ldots, (c - \mathrm{id}_F)f_k, a, b)\mathscr{H}$. By multiplying ${}^t((c - \mathrm{id}_F)f_1, \ldots, (c - \mathrm{id}_F)f_k, a - 1, b - \tau)$ from the right, we obtain

(***) $$\sum_{i=1}^{k} g_i^*(c - \mathrm{id}_F)f_i + g^*(a - 1) + g'^*(b - \tau)$$
$$= ((c - \mathrm{id}_F)f_1, \ldots, (c - \mathrm{id}_F)f_k, a, b)\mathscr{H}\ {}^t((c - \mathrm{id}_F)f_1, \ldots, (c - id_F)f_k, a - 1, b - \tau)$$
$$= (0, \ldots, 0, 1, \tau)\mathscr{H}\ {}^t((c - \mathrm{id}_F)f_1, \ldots, (c - \mathrm{id}_F)f_k, a - 1, b - \tau)$$
$$= \sum_{i=1}^{k} h_i^*(c - \mathrm{id}_F)f_i + h^*(a - 1) + h'^*(b - \tau)$$

where $(h_1^*, \ldots, h_k^*, h^*, h'^*)$ is a linear combination of the $k + 1$ and $k + 2$-th low vectors of $\mathscr{H}$. By this equality (***), one may replace the highest degree = d part of (*) by a degree d − 1 expression.

Let us assume that g_i's and g and g' are constant. Take the homogeneous of degree 1 part of (*) and take its imaginary part.

$$0 = \sum_{i=1}^{k} \mathrm{Im}(g_i)(c - \mathrm{id}_F)f_i + \mathrm{Im}(g)a + \mathrm{Im}(g')b .$$

The linear independence of $(c - \mathrm{id}_F)f_i$'s, a and b implies all the coefficients are zero and hence the constants g_i's, g and g' are real. The constant part of the (*) implies $g + g'\tau = 0$. This is possible only when $g = g' = 0$, since $\mathrm{Im}(\tau) > 0$. Thus the final expression of (*) implies $\alpha \in (c - \mathrm{id}_F)F$, which contradicts the **Lemma B.** These complete the proof of the **Lemma.** □

Remark. Recalling that $\tilde{c}(\lambda) - \lambda \equiv \dfrac{(I_R : I)}{m_{\max}} a \pmod{(c - \mathrm{id}_F)F}$ for the hyperbolic Coxeter transformation $\tilde{c}$ ((2.8) **Lemma C**), we see easily that $\tilde{c}$ does not have a fixed point on $\tilde{\mathbb{E}}$.

(7.3) Put

(7.3.1) $$\tilde{\mathbb{E}}^c := \tilde{\pi}^{-1}(\mathbb{E}^c)$$

(7.3.1) $$\tilde{\mathbb{E}}^c/\tilde{W}_R := \text{the image of } \tilde{\mathbb{E}}^c \text{ in } \tilde{\mathbb{E}}/\tilde{W}_R ,$$

which are irreducible varieties of $\dim_{\mathbb{C}} = \mathrm{cod}\,(R, G) + 1$.

Assertion. *Let* $\theta_0, \ldots, \theta_l$ *be a generator system of* S^W *as in* (4.6) *such that* $1 \leq \deg(\theta_i) < l_{\max} + 1$ *for* $1 \leq i \leq l - \mathrm{cod}\,(R, G)$. *Then*

(7.3.3) $$\tilde{\mathbb{E}}^c/\tilde{W}_R = \text{the } \textit{common zeros of } \tilde{\theta}_0, \ldots, \tilde{\theta}_{l-\mathrm{cod}(R,G)} \textit{ on } \tilde{\mathbb{E}}/\tilde{W}_R .$$

Proof. Put $\tilde{\theta}_i = \lambda^{\deg(\theta_i)}\theta_i$. That $\tilde{\theta}_i$ vanishes on $\tilde{\mathbb{E}}^c$ is equivalent that θ_i vanishes on $\mathbb{E}^c$. Since $\tilde{\theta}_i$ is an $\tilde{W}_R$-invariant function on $\tilde{\mathbb{E}}$, it is invariant under the action of a hyperbolic Coxeter transformation $\tilde{c}$. Using **Lemma C** (2.8.1) and (3.5.11), one obtains,

$$c^*(\theta_i(z, \tau)) = \exp\left(2\pi\sqrt{-1}\deg(\theta_i)\left((c - \mathrm{id}_F)f' + \frac{1}{l_{\max} + 1}\right)\right)\theta_i(z, \tau) .$$

for some $f' \in F$. Since $(c - \mathrm{id}_F)f'|_{\mathbb{E}^c} = 0$, we obtain,

$$\theta_i(z, \tau)|_{\mathbb{E}^c} = \exp\left(2\pi\sqrt{-1}\frac{\deg(\theta_i)}{l_{\max} + 1}\right)\theta_i(z, \tau)|_{\mathbb{E}^c} .$$

Since $0 < \dfrac{\deg(\theta_i)}{l_{\max} + 1} = \dfrac{m_i}{m_{\max}} < 1$ and hence $\exp\left(2\pi\sqrt{-1}\dfrac{\deg(\theta_i)}{l_{\max} + 1}\right) \neq 1$ for $0 \leq i \leq l - \mathrm{cod}\,(R, G)$, we have $\theta_i|_{\mathbb{E}^c} = 0$ and hence $\tilde{\theta}_i|_{\tilde{\mathbb{E}}^c/\tilde{W}_R} = 0$.

On the other hand, $\tilde{\theta}_i$ $(0 \leq i \leq l - \mathrm{cod}\,(R, G))$ form a part of generator system of the algebra S^W so that their common zeros is an affine irreducible variety of dimension = $l + 2 - (l - \mathrm{cod}\,(R, G) + 1) = \mathrm{cod}\,(R, G) + 1 = \dim_{\mathbb{C}}(\tilde{\mathbb{E}}^c) = \dim_{\mathbb{C}}(\tilde{\mathbb{E}}^c/\tilde{W}_R)$. This implies (7.3.3). □

(7.4) **Corollary.** *For any value* τ_0 *of* $\mathbb{H}$,

$$\Theta_A^2 \notin (\theta_0, \ldots, \theta_{l-\mathrm{cod}(R,G)})S^W + (\tau - \tau_0)S^W .$$

Proof. The zero loci of Θ_A^2 in $\tilde{\mathbb{E}}$ is the union of the hyperplanes $\bigcup H_\alpha$ ((4.5) **Theorem 3.**). Hence if the contrary to the corollary happens, by comparing the zero loci, one gets an inclusion $\bigcup_{\alpha \in R} H_\alpha \supset \mathbb{E}^c \cap \mathbb{E}_\tau$, which contradicts (7.2) **Assertion.** □

§ 8. Normalization of the Unit Factor for Θ_A

The unit factors of the invariants $\theta_{l-\mathrm{cod}(R,G)+1}, \ldots, \theta_l$ are partly normalized up to a constant factor in $\mathbb{C}$ with the aid of $\tilde{I}_W$.

(8.1) Let $\theta_0, \ldots, \theta_l$ be a generator system for S^W as before (4.6). Put,

(8.1.1) $$T := \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})[\theta_0, \ldots, \theta_{l-\mathrm{cod}(R,G)}] .$$

Due to the strict inequality $\deg(\theta_0), \ldots, \deg(\theta_{l-\mathrm{cod}(R,G)}) < \deg(\theta_{l-\mathrm{cod}(R,G)+1}) = \cdots = \deg(\theta_l) = l_{\max} + 1$, T is intrinsically defined as a subalgebra of S^W independent of the choice of the generator system of the algebra. Let T^+ be the ideal of T consisting of all positive degree elements, generated by $\theta_0, \ldots, \theta_{l-\mathrm{cod}(R,G)}$. So

$$T/T^+ \simeq \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}) .$$

(8.2) Put,

$$M := S^W_{l_{\max}+1}/(S^W_{l_{\max}+1} \cap T^+S^W) ,$$

$$S^2M := S^W_{2(l_{\max}+1)}/(S^W_{2(l_{\max}+1)} \cap T^+S^W) ,$$

where M is a $\Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$ free module of rank $\mathrm{cod}(R, G)$ generated by $\theta_{l-\mathrm{cod}(R,G)+1}, \ldots, \theta_l$ and S^2M is its symmetric tensor product.

(8.3) Define a covariant differential operator,

(8.3.1) $$\nabla = \nabla_{\partial/\partial\tau}\colon S^2M \to S^2M$$

by the correspondence

(8.3.2) $$\theta \otimes \theta' \bmod T^+S^W \longmapsto \frac{(I_R : I)}{2\pi\sqrt{-1}\, m_{\max}} \tilde{I}_W(d\theta, d\theta') \bmod T^+S^W .$$

The well definedness of ∇:
First note that $S^W_{2(l_{\max}+1)} \cap T^+S^W = S^W_{2(l_{\max}+1)} \cap (T^+)^2S^W$. This implies that the correspondence (8.3.2) sends an equivalent class to an equivalent class. Let us define a map $D\colon S^W_{l_{\max}+1} \times S^W_{l_{\max}+1} \to S^W_{2(l_{\max}+1)}$ in the same way as (8.3.2).

The following i), ii) *and* iii) *for D, imply that D induces* ∇.

i) $D(\theta, \theta') = D(\theta', \theta)$,

ii) $D(\varphi(\tau)\theta, \theta') = D(\theta, \varphi(\tau)\theta')$,

iii) $D(\varphi(\tau)\theta, \theta') = \dfrac{\partial\varphi}{\partial\tau}\theta\theta' + \varphi D(\theta, \theta')$.

Proof. i) is obvious by definition. ii) is a consequence of i) and iii). Let us show the iii).

$$\begin{aligned} D(\varphi(\tau)\theta, \theta') &= \frac{(I_R : I)}{2\pi\sqrt{-1}\, m_{\max}} \tilde{I}_W(d\varphi\theta + \varphi d\theta, d\theta') \\ &= \frac{\partial\varphi}{\partial\tau}\theta \frac{(I_R : I)}{2\pi\sqrt{-1}\, m_{\max}} \tilde{I}_W(d\tau, d\theta') + \varphi \frac{(I_R : I)}{2\pi\sqrt{-1}\, m_{\max}} \tilde{I}_W(d\theta, d\theta') \\ &= \frac{\partial\varphi}{\partial\tau}\theta\theta' + \varphi D(\theta, \theta') . \end{aligned}$$

(Cf. (5.2.4) and note the fact $E\theta' = (l_{\max} + 1)\theta'$.) □

(8.4) The connection ∇ is integrable, since **H** is 1-dimensional over **C**. Furthermore, since **H** is simply connected, the horizontal sections of ∇ is defined globally on **H**. Hence we have:

Assertion. *There exists uniquely a* **C**-*vector subspace* N *of* S^2M *such that*

i) $\sum_i \tilde{I}_W(d\theta_i, d\theta_i') \in T^+S^W$ *for* $^\forall$ $\sum_i \theta_i \otimes \theta_i' \in N$.

ii) $N \otimes_{\mathbf{C}} \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}) \simeq S^2M$.

(8.5) *Remark.* Except for the case cod $(R, G) = 1$, there does not exist a connection ∇' on the module M such that the induced connection on the module S^2M coincides with ∇.

Proof. Suppose that there exists such connection ∇'. Let ξ_i $(i = 1, \ldots, \operatorname{cod}(R, G))$ be **C**-basis of the horizontal sections for ∇'. Then $\xi_i \otimes \xi_j$ for $1 \le i \le j \le \operatorname{cod}(R, G)$ form **C**-basis for the horizontal sections for ∇. This means explicitly that $\tilde{I}_W^*(d\xi_i, d\xi_j) \in T^+S^W$. Take ξ_i's as the part of generators $\theta_{l-\operatorname{cod}(R,G)+1}, \ldots, \theta_l$ for S^W. Then we have $\tilde{I}_W(d\theta_i, d\theta_j) \in T^+S^W$ for $0 \le i \le l$ and $l - \operatorname{cod}(R, G) + 1 \le j \le l$. (For $0 \le i \le l - \operatorname{cod}(R, G)$, the degree is not divisible by $l_{\max} + 1$.) This implies that $\det(\tilde{I}_W(d\theta_i, d\theta_j)_{i,j=-1,\ldots,l}) \in T^+S^W$ except for $\operatorname{cod}(R, G) = 1$. This is a contradiction to (5.2.5) and (7.4). □

§9. The Metric J on $\mathscr{G}$

From this section, we assume that cod $(R, G) = 1$, and the last coordinate θ_l is distinguished as $S^W = T[\theta_l]$. Then several structures on S^W-modules are reduced to corresponding structures on T-modules. This comparison leads to the studies in Sections 10, 11.

(9.1) From this section, we restrict our study to the case

(9.1.1) $$\operatorname{cod}(R, G) = 1 .$$

This includes the marked EARS's of types: $A_1^{(1,1)*}$, $B_2^{(2,1)}$, $B_3^{(1,1)}$, $C_2^{(1,2)}$, $C_3^{(2,2)}$, $BC_1^{(2,1)}$, $BC_1^{(2,4)}$, $BC_2^{(2,2)}(1)$, $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$, $E_8^{(1,1)}$, $F_4^{(1,1)}$, $F_4^{(1,2)}$, $F_4^{(2,1)}$, $F_4^{(2,2)}$, $G_2^{(1,1)}$, $G_2^{(1,3)}$, $G_2^{(3,1)}$, $G_2^{(3,3)}$.

(9.2) The assumption (9.1.1) implies that

$$\deg \theta_i < \deg \theta_l = l_{\max} + 1 \qquad (i = 0, 1, \dots, l-1).$$

for a generator system $\theta_0, \dots, \theta_l$ of $S^W = T[\theta_l]$ (4.6). Let us express the discriminant $\Theta_A^2 \in S^W$ as a polynomial in $T[\theta_l]$.

$$\Theta_A^2 = A_0\theta_l^{l+2} + A_1\theta_l^{l+1} + \cdots + A_{l+2}. \tag{9.2.1}$$

Here A_i is an element of $T = \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})[\theta_0, \dots, \theta_{l-1}]$ homogeneous of degree $i(l_{\max}+1)$ for $i = 0, \dots, l+2$. Note that $A_0 \in \Gamma(\mathbb{H}, \mathcal{O})$ and $A_i \in T^+$ for $i > 0$. Due to (7.4) *Corollary*, we have:

(9.2.2) $\quad A_0(\tau)$ *is not zero for* $\forall \tau \in \mathbb{H}$.

(9.3) The (9.1.1) implies $\operatorname{rank}_{\mathbb{C}} N = \operatorname{cod}(R, G)(\operatorname{cod}(R, G) + 1)/2 = 1$. Then the (8.4) **Assertion** is reformulated as follows.

Assertion. *There exists an element* $\theta_l \in S^W_{l_{\max}+1}$ *such that*
i) $\theta_l \neq 0 \bmod T^+S^W$
ii) $\tilde{I}_W(d\theta_l, d\theta_l) \equiv 0 \bmod T^+S^W$.
Such θ_l mod T^+S^W *is unique up to a constant factor in* $\mathbb{C}$.

We fix such θ_l as a generator of the algebra S^W in (4.6).

(9.4) As a graded module, the lowest degree elements of Der_{S^W} is generated by $\dfrac{\partial}{\partial\theta_l}$ over $\Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})$, where $\dfrac{\partial}{\partial\theta_l}$ is unique up to a constant factor after the normalization in (9.3), independent of a choice of the generators $\theta_0, \dots, \theta_l$ of S^W. We fix one choice and denote it:

$$\partial_l := \frac{\partial}{\partial\theta_l}. \tag{9.4.1}$$

The element $\theta_l \in S^W$ of (9.3) is characterized as

$$\partial_l\theta_l = \text{constant} \neq 0. \tag{9.4.2}$$

The subring T (8.1) of S^W is characterized as

$$T = \{\theta \in S^W : \partial_l\theta = 0\}. \tag{9.4.3}$$

(9.5) We define,

$$\mathscr{F} := \{\omega \in \Omega^1_{S^W} : L_{\partial_l}\omega = 0\}, \tag{9.5.1}$$

$$\mathscr{G} := \{\xi \in \mathrm{Der}_{S^W} : [\partial_l, \xi] = 0\}, \tag{9.5.2}$$

where L_δ means the Lie derivative w.r.t. a vector field δ. As immediate consequences of the definition, we have:

i) *$\mathscr{F}$ and $\mathscr{G}$ are T free modules of rank $l+2$, generated by $d\theta_{-1}, d\theta_0, \ldots, d\theta_l$ and $\frac{\partial}{\partial\theta_{-1}}, \frac{\partial}{\partial\theta_0}, \ldots, \frac{\partial}{\partial\theta_l}$ respectively, dual to each other by the pairing $\langle,\rangle$. One has natural isomorphisms:*

$$(9.5.3) \qquad \mathscr{G} \otimes_T S^W \simeq \mathrm{Der}_{S^W}, \qquad \mathscr{F} \otimes_T S^W \simeq \Omega^1_{S^W}.$$

ii) *$\mathscr{G}$ is closed under the bracket product [,].*

iii) *There is a short exact sequence as T-Lie-algebra:*

$$(9.5.4) \qquad 0 \longrightarrow T\partial_l \longrightarrow \mathscr{G} \longrightarrow \mathrm{Der}_T \longrightarrow 0,$$

where $\mathrm{Der}_T :=$ derivations of the $\mathbf{C}$-algebra T.

The image of an element $u \in \mathscr{G}$ in Der_T is denoted by $\ddot{u}$. This correspondence is equivariant with their derivation action on T (i.e. $uf = \ddot{u}f$ for $f \in T$). We shall confuse u and $\ddot{u}$ so far they act on T.

A splitting factor $\mathscr{G}' \subset \mathscr{G}$ for (9.5.4) and a function $\theta_l \in S^W$ with $\partial_l\theta_l = 1$ correspond one to one by $\mathscr{G}' = \{\delta \in \mathscr{G} : \delta\theta_l = 0\}$.

iv) *$\mathscr{F}$ is involutive (i.e. $d\mathscr{F} \subset \mathscr{F} \wedge \mathscr{F}$).*

v) *Statements for $\mathscr{F}$ dual to that of* iii) *for $\mathscr{G}$ holds.*

vi) *For the reason of* (9.5.3) *and* (9.5.4), *we have called the module $\mathscr{G}$ as the "small tangent module" in the introduction.*

(9.6) Recall the form $\tilde{I}_W$ (5.2.3) and define a T bilinear form,

$$(9.6.1) \qquad J^*: \mathscr{F} \times \mathscr{F} \to T, \qquad \omega \times \omega' \mapsto \partial_l \tilde{I}_W(\omega, \omega').$$

Well definedness of J^*: Due to (9.4.3), the map $\partial_l\tilde{I}_W(\omega, \omega')$ is T-billinear in ω and ω'. So, we have only to show that its values lie in T. Due to (9.4.3), this is equivalent that for any $\omega, \omega' \in \mathscr{F}$, $(\partial_l)^2\tilde{I}_W(\omega, \omega') = 0$. Let us show this for the basis $d\theta_i$'s.

For $-1 \le i, j \le l$, if $(i, j) \ne (l, l)$, then we have

$$\deg((\partial_l)^2\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j)) = \deg(\theta_i) + \deg(\theta_j) - 2(l_{\max} + 1) < 0.$$

This implies $(\partial_l)^2\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j) = 0$. On the other hand, (9.3) **Assertion** ii) implies that $\tilde{I}_W(d\tilde{\theta}_l, d\tilde{\theta}_l)$ cannot contain a term θ_l^2. This implies $(\partial_l)^2\tilde{I}_W(d\tilde{\theta}_l, d\tilde{\theta}_l) = 0$. □

Assertion. *The T-bilinear form J^* is nondegenerate.*

Proof. The above argument for well definedness of J^* shows that all entries of the $(l+2) \times (l+2)$ matrix $(\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j))_{ij}$ are at most degree $= 1$ in θ_l. Recalling the fact $\det((\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j))_{ij}) = \Theta_A^2 = A_0\theta_l^{l+2} + A_1\theta_l^{l+1} + \cdots + A_{l+2}$ ((5.2.5), (9.2.1)), we obtain $\det((\partial_l\tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j))_{i,j=-1,\ldots,l}) = A_0(\tau)$. This does not vanish anywhere on $\mathbf{H}$ due to (9.2.2). □

Remark. Since $\tilde{I}_W$ is homogeneous of degree 0 and ∂_l is homogeneous of degree $-(l_{\max}+1)$, J^* is homogeneous of degree $-(l_{\max}+1)$. Therefore the intersection matrix $(J^*(d\theta_i, d\theta_j))_{ij=-1}l$ has a form,

$$\begin{bmatrix} 0,\ldots,0,0, & & \dfrac{(I_R:I)}{2\pi\sqrt{-1}m_{\max}} \\ 0, & & * \quad * \\ 0, & & * \qquad * \\ & * & \\ \dfrac{(I_R:I)}{2\pi\sqrt{-1}m_{\max}}, & & * \quad * \quad * \end{bmatrix}$$

where $*$ are the values at the entry (i,j) for $\deg(\theta_i)+\deg(\theta_j)\geq l_{\max}+1$. As we shall see in §11, for a flat generator system $\tau, \theta_0, \ldots, \theta_1$ of S^W, this matrix becomes a constant matrix and hence the "off anti-diagonal part" of the matrix becomes 0.

(9.7) The following decomposition plays an important role in all what follows.

Assertion. *As T-module, one has a direct sum decomposition.*

(9.7.1) $$\mathrm{Der}_{S^W} = \mathscr{G} \oplus \mathrm{Der}_{S^W}(-\log \Theta_A^2)\,.$$

Proof. Recall that an S^W-free basis of $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$ is given by $\tilde{I}_W(d\theta_i) = \sum_{j=-1}^{l} \tilde{I}_W(d\theta_i, d\theta_j)\frac{\partial}{\partial\theta_j}$ $(i=-1,\ldots,l)$ (5.3.1), whose coefficient matrix $\tilde{I}_W(d\theta_i, d\theta_j)$ is linear in θ_l. The leading coefficients $\partial_l \tilde{I}_W(d\tilde{\theta}_i, d\tilde{\theta}_j)$ form an invertible matrix $J^*(d\theta_i, d\theta_j)$. Then by induction on the degree in θ_l of the coefficients, any element of Der_{S^W} is uniquely expressed as a sum of elements of $\mathscr{G}$ and $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$. □

Corollary. *The bracket product with ∂_l induces a T-isomorphism*:

(9.7.2) $$[\partial_l,]: \mathrm{Der}_{S^W}(-\log \Theta_A^2) \simeq \mathrm{Der}_{S^W}\,.$$

Proof. Let us extend the map $[\partial_l,]$ to be a map from Der_{S^W} to itself, which is obviously T-linear due to (9.4.3). If we show the exactness of the following sequence, the proof is finished.

(9.7.3) $$0 \longrightarrow \mathscr{G} \longrightarrow \mathrm{Der}_{S^W} \xrightarrow{[\partial_l,]} \mathrm{Der}_{S^W} \longrightarrow 0\,.$$

By definition of $\mathscr{G}$ (9.5.2), the sequence is exact except for the surjectivity of $[\partial_l,]$. For an element $\delta = \sum_{i=-1}^{l} f_i \frac{\partial}{\partial\theta_i}$, we have,

$$[\partial_l, \delta] = \sum_{i=-1}^{l} \partial_l f_i \frac{\partial}{\partial\theta_i}\,,$$

which implies the surjectivity of (9.7.3) immediately. □

(9.8) Let θ_l be as in (9.3).

For an element $\delta \in \mathscr{G}$, define $\theta_l * \delta \in \mathscr{G}$ and $w(\delta) \in \mathrm{Der}_{S^W}(-\log \Theta_A^2)$ by the relation:

(9.8.1) $$\theta_l \delta = \theta_l * \delta + w(\delta)$$

according to the decomposition (9.7.1). The first factor $\theta_l * \delta$ depends on the choice of θ_l, since θ_l is unique only modulo addition of $T_{l_{\max}+1}$ (cf. (9.3)). The second factor $w(\delta)$ defines a unique (up to a constant factor) T-linear map, (9.8.2) $$w: \mathcal{G} \longrightarrow \mathrm{Der}_{S^W}(-\log \Theta_A^2)$$

From the definition, we have immediately the following:

Assertion.

i) *w sends T free basis of $\mathcal{G}$ to S^W-free basis of* $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$.

ii) *For a $\delta \in \mathcal{G}$, $w(\delta)$ is characterized as the unique element w of* $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$ *such that $[\partial_l, w] = \delta$.*

iii) $$w(\partial_l) = \frac{1}{l_{\max} + 1} E,$$

where E is the Euler operator (5.4.3).

iv) *For $\delta, \xi \in \mathcal{G}$, one has a relation*:

$$[w(\delta), w(\xi)] = \theta_l w([\delta, \xi]) + w(\theta_l * [\delta, \xi] + (\delta\theta_l)\xi - (\xi\theta_l)\delta - [\delta, \theta_l * \xi] - [\theta_l * \delta, \xi]).$$

Proof. i) We apply the (5.3) Criterium for the elements $\delta_i := w\left(\frac{\partial}{\partial \theta_i}\right)$, $i = -1, 0, \ldots, l$. Obviously by definition of the map w, the determinant D of the coefficients matrix of δ_i's is a monic polynomial in θ_l of degree $l_{\max} + 2$. In general D is divisible by Θ_A^2 (cf [22]). So D is a non-zero constant multiple of Θ_A^2, which implies *i*).

ii) By applying $[\partial_l, \,]$ on (9.8.1), this is trivial.

iii) This is obvious from (5.4.3), since

$$\theta_l \partial_l = \frac{1}{l_{\max} + 1}\left(E - \sum_{i=0}^{l-1} \deg(\theta_i)\theta_i \frac{\partial}{\partial \theta_i}\right)$$

and the uniqueness of the decomposition (9.8.1).

iv) Let us express $w(\delta) = \theta_l \delta - \theta_l * \delta$ and $w(\xi) = \theta_l \xi - \theta_l * \xi$. Then,

$$\begin{aligned}
[w(\delta), w(\xi)] &= [\theta_l \delta - \theta_l * \delta, \theta_l \xi - \theta_l * \xi] \\
&= \theta_l^2 [\delta, \xi] + \theta_l((\delta\theta_l)\xi - (\xi\theta_l)\delta - [\delta, \theta_l * \xi] - [\theta_l * \delta, \xi]) \\
&\quad + ((\theta_l * \xi)\theta_l)\delta - ((\theta_l * \delta)\theta_l)\xi + [\theta_l * \delta, \theta_l * \xi] \\
&= \theta_l w([\delta, \xi]) + w(\theta_l * [\delta, \xi] + (\delta\theta_l)\xi - (\xi\theta_l)\delta - [\delta, \theta_l * \xi] - [\theta_l * \delta, \xi]) \\
&\quad + \theta_l * (\theta_l * [\delta, \xi] + (\delta\theta_l)\xi - (\xi\theta_l)\delta - [\delta, \theta_l * \xi] - [\theta_l * \delta, \xi]) \\
&\quad + ((\theta_l * \xi)\theta_l)\delta - ((\theta_l * \delta)\theta_l)\xi + [\theta_l * \delta, \theta_l * \xi].
\end{aligned}$$

Since $\mathrm{Der}_{S^W}(-\log \Theta_A^2)$ is closed under the bracket product, the last two lines in the above calculation should vanish due to the unicity of the decomposition (9.7.1). This proves the formula. □

Remark. The map $\theta_l *: \mathcal{G} \to \mathcal{G}$ defined in (9.8.1) is self adjoint w.r.t. J:

(9.8.3) $$J(\delta, \theta_l * \xi) = J(\theta_l * \delta, \xi).$$

Proof. For $\delta, \xi \in \mathcal{G}$, we calculate, $\tilde{I}_W^*(w(\delta), w(\xi)) = J(\delta, w(\xi)) = J(\delta, \theta_l \xi - \theta_l * \xi) = \theta_l J(\delta, \xi) - J(\delta, \theta_l * \xi)$. By replacing the role of δ and ξ, this is equal to $\theta_l J(\delta, \xi) - J(\theta_l * \delta, \xi)$. □

(9.9) We now show that the datum J^* on $\mathscr{F}$ is enough to reconstruct the metric $\tilde{I}_W$.

Definition. *Let J be the symmetric T-bilinear form*

$$J: \mathscr{G} \times \mathscr{G} \to T \tag{9.9.1}$$

which is the dual of J^* (9.6.1).

Assertion. *By the map w (9.8.2), J and $\tilde{I}_W$ (or $\tilde{I}^*_W$) are related by the formulae:*

$$J(\delta, \delta') = \tilde{I}^*_W(\delta, w(\delta')) \qquad \text{for} \quad \delta, \delta' \in \mathscr{G} . \tag{9.9.2}$$

$$\tilde{I}_W := \sum_{i=-1}^{l} w(\delta_i) \otimes \delta_i^* \tag{9.9.3}$$

where $\delta_{-1}, \dots, \delta_l$ is a T-free basis of $\mathscr{G}$ and $\delta^*_{-1}, \dots, \delta^*_l$ is its dual basis w.r.t. the metric J.

Proof. The formulae (9.9.2) and (9.9.3) are equivalent, since if δ_i's and δ_i^*'s are dual basis for J, then (9.9.2) implies that δ_i^*'s and $w(\delta_i)$'s are dual basis w.r.t. $\tilde{I}^*_W$ and this fact is equivalent to (9.9.3).

So we show (9.9.2), which is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{J} & \mathscr{F} \\ \Big\downarrow{\scriptstyle w} & & \cap \\ \mathrm{Der}_{S^W}(-\log \Theta_A^2) & \xrightarrow[\tilde{I}^*_W]{} & \Omega^1_{S^W} . \end{array}$$

Since $J = J^{*-1}$ and $\tilde{I}^*_W = \tilde{I}_W^{-1}$ are isomorphisms in the above diagram, one has to show that $\tilde{I}_W(\omega) = w(J^*(\omega))$ for $\omega \in \mathscr{F}$. Since $\tilde{I}_W(\omega)$ is logarithmic, one has only to show $\tilde{I}_W(\omega) \equiv \theta_l J^*(\omega) \bmod \mathscr{G}$ by definition of w (9.8.1). This fact is checked by a calculation:

$$\begin{aligned} \theta_l J^*(\omega) &= \theta_l \sum_{j=-1}^{l} J^*(\omega, d\theta_j) \frac{\partial}{\partial \theta_j} = \sum_{j=-1}^{l} \theta_l \partial_l \tilde{I}_W(\omega, d\theta_j) \frac{\partial}{\partial \theta_j} \\ &\equiv \sum_{j=-1}^{l} \tilde{I}_W(\omega, d\theta_j) \frac{\partial}{\partial \theta_j} \qquad (\mathrm{mod}\ \mathscr{G}) = \tilde{I}_W(\omega) . \end{aligned}$$ □

Corollary (9.9.4). $w = \tilde{I}_W \circ J$,

i.e.

$$w: \mathscr{G} \xrightarrow{J} \mathscr{F} \subset \Omega^1_{S^W} \xrightarrow{\tilde{I}_W} \mathrm{Der}_{S^W}(\log \Theta_A^2) .$$

Proof. We need only recall the identity: $\tilde{I}_W^{-1} = \tilde{I}^*_W$. □

Remark. The above formulae (9.9.2), (9.9.3) and (9.9.4) will be used essentially in §10. The formulae are also interesting in connection with the period map theory. (Cf. (1.2)).

Namely, an unfolding of a singularity defines a family $\mathscr{X} \to S$ of complex analytic varieties. Then J is a metric on the tangent bundle of S, calculated by the residue symbols with the support on the critical set of the family. On the other hand, the form

I is the intersection form for the middle homology group of the regular fiber of the family. So the above formulae relate the Grothendiéck duality J with the Poincaré duality I [15], [24].

(9.10) The following **Assertion** plays an important role.

Assertion. *For $\delta \in \mathscr{G}$, we have,*

$$J(\partial_l, \delta) = \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}} \delta\tau . \tag{9.10.1}$$

Equivalently, for dual basis $\delta_i \in \mathscr{G}$ and $\omega_i \in \mathscr{F}$ $(i = 1, \ldots, l+2)$, we have:

$$\frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}} d\tau = \sum_{i=1}^{l+2} J(\partial_l, \delta_i)\omega_i . \tag{9.10.2}$$

Proof. It is enough to show (9.10.1). Recalling formulae (9.9.2), (9.8) **Assertion** iii), and (5.4.2), we calculate,

$$\begin{aligned} J(\partial_l, \delta) &= \tilde{I}_W^*(w(\partial_l), \delta) = \frac{1}{l_{\max}+1} \tilde{I}_W^*(E, \delta) \\ &= \frac{1}{l_{\max}+1} \frac{(I_R : I)(l_{\max}+1)}{2\pi\sqrt{-1}m_{\max}} \langle d\tau, \delta\rangle = \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}} \delta\tau . \end{aligned}$$ □

Corollary. *For any $\delta, \xi \in \mathscr{G}$, we have,*

$$J([\delta, \xi], \partial_l) = \delta J(\xi, \partial_l) - \xi J(\delta, \partial_l) . \tag{9.10.3}$$

(9.11) The following **Assertion** is one of the key facts in the construction of a graduation of Der_{S^W} in §10, for which the normalized ∂_l is used essentially.

Assertion. *Let ∇^* be the logarithmic connection defined on $\Omega^1_{S^W}$, which is dual to ∇ defined on* Der_{S^W} *in §6. Then*

$$\tilde{I}_W(\nabla^*_{\partial_l}(\omega)) \in \mathscr{G} \tag{9.11.1}$$

for $^\forall\omega \in \mathscr{F}$.

Proof. The fact (9.11.1) is equivalent to that

$$\tilde{I}_W(\nabla^*_{\partial_l}(\omega_2), \omega_3) \in T \qquad \text{for } ^\forall\omega_2, \omega_3 \in \mathscr{F} .$$

Recall a formula (6.7.1) for ∇^* and (6.2.1). Substituting ∂_l for δ:

$$\begin{aligned} \tilde{I}_W(\nabla^*_{\partial_l}\omega_2, \omega_3) = \{&\partial_l\tilde{I}_W(\omega_2, \omega_3) + \langle\omega_2, [\tilde{I}_W\omega_3, \partial_l]\rangle - \tilde{I}_W(\omega_3)(\partial_l\omega_2) \\ &- \tilde{I}_W^*([\tilde{I}_W\omega_2, \tilde{I}_W\omega_3], \partial_l) - \langle\omega_3, [\tilde{I}_W\omega_2, \partial_l]\rangle + \tilde{I}_W(\omega_2)(\partial_l\omega_3)\}/2 . \end{aligned}$$

The first term $\partial_l\tilde{I}_W(\omega_2, \omega_3)$ belongs to T due to the normalization (9.6.1). Next, if we put $\omega_i = J(\delta_i)$ for some $\delta_i \in \mathscr{G}$ $(i = 2, 3)$, then $[\tilde{I}_W\omega_i, \partial_l] = [w(\delta_i), \partial_l] = \delta_i$ and therefore $\langle\omega_j, [\tilde{I}_W\omega_i, \partial_l]\rangle = \delta_i\omega_j \in T$. The remaining terms are

$$\begin{aligned}
&-\tilde I_W^*([\tilde I_W\omega_2, \tilde I_W\omega_3], \partial_l) + \tilde I_W(\omega_2)(\partial_l\omega_3) - \tilde I_W(\omega_3)(\partial_l\omega_2)\\
&\quad = -\tilde I_W^*([w(\delta_2), w(\delta_3)], \partial_l) + w(\delta_2)(\partial_l J(\delta_3)) - w(\delta_3)(\partial_l J(\delta_2))\\
&\quad = -\tilde I_W^*(\theta_l w([\delta_2, \delta_3])) + w(\xi), \partial_l) + w(\delta_2)J(\partial_l, \delta_3) - w(\delta_3)J(\partial_l, \delta_2)\\
&\quad \equiv \theta_l(-J([\delta_2, \delta_3], \partial_l) + \delta_2 J(\partial_l, \delta_3) - \delta_3 J(\partial_l, \delta_2)) \mod T\\
&\quad \equiv 0 \mod (T).
\end{aligned}$$

Here in the calculation, we used freely the facts (9.9.4), (9.8) Assertion iv), (9.9.2), (9.8.1) and (9.10.3). □

Remark. Note that in the formula (9.11.1), $\nabla^*_{\partial_l}(\omega) \in \Omega^1_{S^W}(\log \Theta^2_A)$ but is not contained in $\Omega^1_{S^W}$ except when $\omega = 0$. This fact is interpretated as that $\nabla^*_{\partial_l}(\omega)$ carries a "pure" logarithmic pole.

Corollary. *Let ∇ be the logarithmic connection defined on* Der_{S^W} *in* §6. *Then for* $^\forall\theta \in \mathcal{G}$,

(9.11.2) $$\nabla_{\partial_l}(w(\delta)) \in \mathcal{G}.$$

Proof. Since ∇ and ∇^* are metric connection w.r.t. $\tilde I_W$ and $\tilde I_W^*$, we have $\nabla_{\partial_l}(w(\delta)) = \nabla_{\partial_l}(\tilde I_W \circ J(\delta)) = \tilde I_W(\nabla^*_{\partial_l}(J(\delta))) \in \mathcal{G}$. □

(9.12) **Definition.** *A T-endomorphism $N: \mathcal{G} \to \mathcal{G}$ is defined by*

(9.12.1) $$N(\delta) := \nabla_{\partial_l} w(\delta).$$

Assertion. *Let $N^*: \mathcal{G} \to \mathcal{G}$ be the adjoint endomorphism of N w.r.t. the metric J. Then we have the following*:

(9.12.2) $$N + N^* = \mathrm{id}_{\mathcal{G}}.$$

Proof. The formula is equivalent to

(9.12.3) $$J(N\delta_1, \delta_2) + J(\delta_1, N\delta_2) = J(\delta_1, \delta_2)$$

for $\delta_1, \delta_2 \in \mathcal{G}$. The left hand of (9.12.3) is equal to

$$\begin{aligned}
&J(\nabla_{\partial_l} w(\delta_1), \delta_2) + J(\delta_1, \nabla_{\partial_l} w(\delta_2))\\
&\quad = \tilde I_W^*(\nabla_{\partial_l} w(\delta_1), w(\delta_2)) + \tilde I_W^*(w(\delta_1), \nabla_{\partial_l} w(\delta_2))\\
&\quad = \partial_l \tilde I_W^*(w(\delta_1), w(\delta_2))\\
&\quad = \partial_l J(\delta_1, w(\delta_2))\\
&\quad = \partial_l(\theta_l J(\delta_1, \delta_2) + J(\delta_1, \theta_l * \delta_2))\\
&\quad = J(\delta_1, \delta_2).
\end{aligned}$$

□

§ 10. A Graduation on Der_{S^W}

We introduce an infinite direct sum decomposition

$$\mathrm{Der}_{S^W} \simeq \bigoplus_{k=0}^{\infty} \mathscr{G}_k ,$$

$$\mathrm{Der}_{S^W}(-\log \Theta_A^2) \simeq \bigoplus_{k=1}^{\infty} \mathscr{G}_k ,$$

by a sequence of T-free modules $\mathscr{G}_k$ $(k = 0, 1, 2, \ldots)$. (For notations, recall (5.3.2), (9.4.3).)

(10.1) **Assertion.** *There exists an infinite sequence*

$$\mathscr{G}_k \qquad (k = 0, 1, 2, \ldots)$$

of T-submodules of Der_{S^W} *satisfying the following conditions.*

i) $\mathscr{G}_i$ *is a T-free module of rank* $l+2$ $(i = 0, 1, 2, \ldots)$.

ii) $\mathscr{G} = \mathscr{G}_0$,

iii) $\mathrm{Der}_{S^W} = \bigoplus_{k=0}^{\infty} \mathscr{G}_k$,

iv) $\mathrm{Der}_{S^W}(-\log \Theta_A^2) = \bigoplus_{i=1}^{\infty} \mathscr{G}_i$,

v) *The multiplication of* θ_l *induces a map,*

$$\theta_l \colon \mathscr{G}_i \longrightarrow \mathscr{G}_i \oplus \mathscr{G}_{i+1} \qquad (i = 0, 1, 2, \ldots).$$

vi) *The bracket product with* ∂_l *induces a T-isomorphism,*

$$[\partial_l, \,] \colon \mathscr{G}_{i+1} \longrightarrow \mathscr{G}_i \qquad (i = 0, 1, 2, \ldots).$$

(10.2) *Proof of* (10.1) *Assertion.*

The proof is not hard but rather of technical nature. The properties ii), iv) and vi) are enough to define the sequence $\mathscr{G}_i$ uniquely by induction on i as follows.

$\mathscr{G}_0 := \mathscr{G}$,

$\mathscr{G}_{k+1} :=$ the inverse image of $\mathscr{G}_k$ by the isomorphism (9.7.2), for which the property i) is also automatic.

To show the decomposition iii) and iv), let us define,

$$T_{\leq k} := \{P \in T[\theta_l] : \deg(P) \leq k\} \tag{10.2.1}$$

where $\deg(P)$ means the degree as a polynomial in θ_l. Put

$$\mathscr{G}_{\leq k} := \mathscr{G} \otimes_T T_{\leq k} , \tag{10.2.2}$$

and

$$\mathscr{G}_{\leq k}(-\log \Theta_A^2) := \mathscr{G}_{\leq k} \cap \mathrm{Der}_{S^W}(-\log \Theta_A^2) .$$

Let us shown the following equalities:

$$\mathscr{G}_{\leq k} = \mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_k , \tag{10.2.3$_k$}$$

$$\mathscr{G}_{\leq k}(-\log \Theta_A^2) = \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_k , \tag{10.2.4$_k$}$$

for $k = 0, 1, \ldots$ by induction on k, which imply iii) and iv).

$(10.2.3)_0$ follows from $T_{\leq 0} = T$ and $(10.2.4)_0$ follows from (9.7.1). Assume $(10.2.3)_j$

and $(10.2.4)_j$ for $j \leq k$. Since we have the isomorphisms vi), the bracket product with ∂_l induces the isomorphism:

$$[\partial_l,]: \mathscr{G}_{\leq k}(-\log \Theta_A^2) \simeq \mathscr{G}_{\leq (k-1)}.$$

Since $\mathscr{G}_{\leq k} = \mathscr{G}_{\leq (k-1)} \oplus \mathscr{G}_k$ (direct sum) due to $(10.2.3)_k$, the isomorphism (9.7.2) induces $\mathscr{G}_{\leq k}(-\log \Theta_A^2) \oplus \mathscr{G}_{k+1}$ (direct sum). Hence recalling the direct sum decomposition (9.7.1), we obtain,

$$\mathscr{G}_0 \oplus \mathscr{G}_1 \oplus \cdots \oplus \mathscr{G}_{k+1} \quad \text{(direct sum)}.$$

Let us call this module $\mathscr{K}_{k+1}$ as T-submodule of Der_{S^W}. We have only to show the equality $\mathscr{K}_{k+1} = \mathscr{G}_{\leq (k+1)}$ ($(10.2.3)_{k+1}$), since $(10.2.4)_{k+1}$ follows from this, the fact $\mathscr{G}_{k+1} \subset \mathrm{Der}_{S^W}(-\log \Theta_A^2)$ and $(10.2.4)_k$. The equality is shown by induction on k, where $\mathscr{K}_0 = \mathscr{G}_0$ is definition. Recalling vi), $\delta \in \mathscr{K}_{k+1}$ if and only if $[\partial_l, \delta] \in \mathscr{K}_k$ for $k \geq 0$. On the other hand, for $\delta = \sum_{j=-1}^{l} g_j(\theta) \frac{\partial}{\partial \theta_j}$, we have $[\partial_l, \delta] = \sum_{j=-1}^{l} (\partial_l g_j(\theta)) \frac{\partial}{\partial \theta_j}$. Hence $\delta \in \mathscr{G}_{\leq (k+1)}$ if and only if $[\partial_l, \delta] \in \mathscr{G}_{\leq k}$ for $k \geq 0$. These together with $\mathscr{K}_0 = \mathscr{G}_0$ imply the result.

v) This is also shown by induction on i, where the case $i = 0$ is trivially true due to the fact $\mathscr{G}_0 \oplus \mathscr{G}_1 = \mathscr{K}_1$ shown above. Assume that the case $i = k \geq 0$ is true. In general the formula $[\partial_l, \theta_l] = 1$ implies the relation: $[\partial_l, \theta_l \delta] = \delta + \theta_l[\partial_l, \delta]$. If $\delta \in \mathscr{G}_{k+1}$, then $[\partial_l, \delta] \in \mathscr{G}_k$ so that by the induction hypothesis the right hand of the relation belongs to $\mathscr{G}_k \oplus \mathscr{G}_{k+1}$. On the other hand since $\delta \in \mathscr{G}_{k+1} \subset \mathrm{Der}_{S^W}(-\log \Theta_A^2)$, we also have $\theta_l \delta \in \theta_l \mathscr{G}_{k+1} \subset \mathrm{Der}_{S^W}(-\log \Theta_A^2)$. Hence applying the isomorphisms (9.7.2) and vi), we conclude that $\theta_l \delta \in \mathscr{G}_{k+1} \oplus \mathscr{G}_{k+2}$. □

(10.3) Extend naturally the T-linear map w (9.8.2) S^W-linearly to a map, denoted by the same w:

$$w: \mathrm{Der}_{S^W} \simeq \mathscr{G} \otimes_T S^W \longrightarrow \mathrm{Der}_{S^W}(-\log \Theta_A^2). \tag{10.3.1}$$

This is an S^W-isomorphism, since a T-free basis of $\mathscr{G}$ is send to an S^W-free basis. The map w is "the main part of the product by θ_l" in (10.1) *Assertion* v).

Assertion. *The map* w (10.3.1) *preserves the decomposition* (10.1) iii), *inducing* T-*isomorphisms*: $\mathscr{G}_{i-1} \simeq \mathscr{G}_i$ *for* $i = 1, 2, 3, \ldots$

Proof. Let us denote by v the T-isomorphism (9.7.2). It is enough to show the following formulae.

$$[v, w] = \mathrm{id}_{\mathrm{Der}_{S^W}}. \tag{10.3.2}$$

$$v|_{\mathscr{G}_i} = i(w|_{\mathscr{G}_{i-1}})^{-1}. \tag{10.3.3}$$

Proof of (10.3.2). For $\delta \in \mathscr{G}$ and $\theta \in S^W$, recalling (9.8) **Assertion** ii), we calculate:

$$\begin{aligned}
[v, w](\theta\delta) &:= [\partial_l, w(\theta\delta)] - w([\partial_l, \theta\delta]) \\
&= [\partial_l, \theta w(\delta)] - w((\partial_l \theta)\delta + \theta[\partial_l, \delta]) \\
&= (\partial_l \theta) w(\delta) + \theta[\partial_l, w(\delta)] - (\partial_l \theta) w(\delta) \\
&= \theta\delta = \mathrm{id}_{\mathrm{Der}_{S^W}}(\theta\delta).
\end{aligned}$$

This is enough, since $\mathcal{G}$ generates Der_{S^W} as an S^W-module.

Proof of (10.3.3). For $i = 1$, (10.2.3) is true due to (9.8) **Assertion** ii). Assume (10.2.3) for i. For $\delta \in \mathcal{G}_i$, applying (10.2.2) and noting that $v(\delta) := [\partial_l, \delta] \in \mathcal{G}_{i-1}$,

$$v \circ w(\delta) = w \circ v(\delta) + [v, w](\delta) = iv^{-1}(v(\delta)) + \delta = (i+1)\delta. \qquad \square$$

The formula (10.3.3) implies the standard isomorphisms:

$$\text{(10.3.4)} \qquad \mathcal{G}_0 \underset{(1/i!)v^i}{\overset{w^i}{\rightleftarrows}} \mathcal{G}_i .$$

(10.4) The relationship between the direct sum decomposition of (10.1) and the connection ∇ introduced in §6 is the following.

Assertion. *The restriction of the connection* ∇ *induces the map*:

i) $\nabla: \mathcal{G}_i \times \mathcal{G}_j \to \mathcal{G}_{\le i+j}$ *for* $i, j \ge 0$ *with* $i + j \ge 1$.
ii) $\nabla_{\partial_l}: \mathcal{G}_j \to \mathcal{G}_{\le j-1}$ *for* $j \ge 1$.

Proof. i) It is enough to show cases $i = 0$, $j = 1$ and $i = 1$, $j = 0$, since the other cases are reduced to these cases, as

$$\nabla(T_i \otimes \mathcal{G}_0 \times T_{j-1} \otimes \mathcal{G}_1) \subset T_i(T_{j-1}\nabla(\mathcal{G}_0 \times \mathcal{G}_1) + \mathcal{G}_0(T_{j-1}) \circ \mathcal{G}_1) \subset \mathcal{G}_{\le i+j},$$

$$\nabla(T_{i-1} \otimes \mathcal{G}_1 \times T_j \otimes \mathcal{G}_0) \subset T_{i-1}(T_j\nabla(\mathcal{G}_1 \times \mathcal{G}_0) + \mathcal{G}_1(T_j) \circ \mathcal{G}_0) \subset \mathcal{G}_{\le i+j}.$$

Case $i = 0$, $j = 1$: Substitute $\delta_2 = w(\delta)$ and $\delta_3 = w(\xi)$ for δ and $\xi \in \mathcal{G}$ in the formula (6.2.1). Applying (9.9.2) repeatedly,

$$\begin{aligned} J(\nabla_{\delta_1} w(\delta), \xi) = (&\delta_1(\theta_l J(\delta, \xi) - J(\theta_l * \delta, \xi)) + J(\delta, [w(\xi), \delta_1]) - w(\xi)J(\delta, \delta_1) \\ &- \tilde{I}_W^*([w(\delta), w(\xi)], \delta_1) + w(\delta)J(\xi, \delta_1) + J(\xi, [\delta_1, w(\delta)]))/2 . \end{aligned}$$

Applying (9.8) **Assertion** iv), we rewrite

$$\begin{aligned} &= (\delta_1(\theta_l J(\delta, \xi) - J(\theta_l * \delta, \xi)) + J(\delta, [w(\xi), \delta_1]) - w(\xi)J(\delta, \delta_1) \\ &\quad - \theta_l J([\delta, \xi], \delta_1) - J(* \delta_1) + w(\delta)J(\xi, \delta_1) + J(\xi, [\delta_1, w(\delta)]))/2 \\ &= (\theta_l(\delta_1 J(\delta, \xi) - \xi J(\delta, \delta_1) + \delta J(\xi, \delta_1) - J([\delta, \xi], \delta_1) + J(\delta, [\xi, \delta_1]) \\ &\quad + J(\xi, [\delta_1, \delta])) + **)/2 \in T_{\le 1}, \qquad \text{where} \quad * \in \mathcal{G} \text{ and } ** \in T. \end{aligned}$$

Then due to the non-degeneracy of J (9.6) **Assertion**, $\nabla_{\delta_1} w(\delta)$ belongs to $\mathcal{G}_{\le 1} = \mathcal{G}_0 \oplus \mathcal{G}_1$.

If $\delta_1 = \partial_l$, applying (9.10.1), we do the same calculation,

$$\begin{aligned} J(\nabla_{\partial_l} w(\delta), \xi) &= \left(\theta_l \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}}\left(-\xi\delta\tau + \delta\xi\tau - [\delta, \xi]\tau\right) + \text{terms}\right)\Big/2 \\ &= \text{terms}/2 \in T. \end{aligned}$$

This implies that $\nabla_{\partial_l} w(\delta) \in \mathcal{G}$.

A more detailed calculation shows the equality:

$$J(\nabla_{\partial_l} w(\delta), \xi)$$

$$= \left(J(\delta, \xi) + \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}} \left(-w(\xi)\delta + w(\delta)\xi - \theta_l[\delta, \xi] - \theta_l * [\delta, \xi] \right.\right.$$

$$\left.\left. -(\delta\theta_l)\xi + (\xi\theta_l)\delta + [\delta, \theta_l * \xi] + [\theta_l * \delta, \xi] \right)\tau \right) \Big/ 2$$

$$= \left(J(\delta, \xi) + \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}} \left(-\theta_l * [\delta, \xi] - (\delta\theta_l)\xi + (\xi\theta_l)\delta + \delta(\theta_l * \xi) - \xi(\theta_l * \delta) \right)\tau \right) \Big/ 2$$

So we obtain a formula:

(10.4.1)

$$J(\nabla_{\partial_l} w(\delta), \xi)$$

$$= \left(J(\delta, \xi) + \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}} \left(-\theta_l * [\delta, \xi] - (\delta\theta_l)\xi + (\xi\theta_l)\delta + \delta(\theta_l * \xi) - \xi(\theta_l * \delta) \right)\tau \right) \Big/ 2$$

(10.5) *Remark.* If one put

$$\mathscr{H}^k := \bigoplus_{i=k}^{\infty} \mathscr{G}_i \qquad k = 0, 1, 2, \dots .$$

Then due to (10.1) **Assertion** v), this is a decreasing sequence of S^W-submodule of Der_{S^W} free of rank $l + 2$. We do not know whether this gives the Hodge filtration coming from the period mapping theory. For the purpose, one needs to show the following:

Question. Does the restriction of ∇ induce a map $\nabla: \mathscr{G}_0 \times \mathscr{G}_i \to \mathscr{G}_{i-1} \oplus \mathscr{G}_i$?

§ 11. The Flat Structure on S^W

As the goal of the present paper, we formulate the flat structure on S^W in (11.5) **Theorem.**

(11.1) We recall some technical notations.

0) $S^W \simeq \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}})[\theta_0, \dots, \theta_l] :=$ the $\tilde{W}_R$-invariant ring (4.3.3).

i) $\nabla: \mathrm{Der}_{S^W} \times \mathrm{Der}_{S^W}(\log \Theta_A^2) \to \mathrm{Der}_{S^W} :=$ the metric connection (6.1).

ii) $\partial_l :=$ a normalized derivation of S^W (9.4.1).

iii) $T := \{\theta \in S^W : \partial_l \theta = 0\}$ is a subalgebra of S^W with $S^W = T[\theta_l]$ (8.1.1).

iv) $\mathscr{G} := \{\delta \in \mathrm{Der}_{S^W} : [\partial_l, \delta] = 0\}$ is a T-free module with $\mathrm{Der}_{S^W} \simeq \mathscr{G} \otimes_T S^W$ (9.5.2).

v) $w: \mathrm{Der}_{S^W} \to \mathrm{Der}_{S^W}(-\log \Theta_A^2) :=$ an S^W-isomorphism (9.8.2), (10.3.1).

vi) $J: \mathscr{G} \times \mathscr{G} \to T := a$ nondegenerate T-symmetric bilinear form (9.9.1).

vii) $N :=$ a T-endomorphism of $\mathscr{G}$ (9.12.1).

(11.2) **Lemma.** *There exists uniquely a connection $\nabla^{\#}$ on $\mathscr{G}$* as a T-module,

(11.2.1) $$\nabla^{\#}: \mathrm{Der}_T \times \mathscr{G} \to \mathscr{G},$$

with the properties i)–vi), *which are explained below.*

i) *$\nabla^{\#}$ is integrable.*

ii) $\nabla^{\#}$ *is torsion free in an extended sense.*
iii) $\nabla^{\#}$ *is metric with respect to* J.
iv) $\nabla^{\#}N=0$.
v) $\partial_l \in \mathscr{G}$ *is horizontal.*
vi) $d\tau \in \mathscr{F}$ *is horizontal.*

These mean explicitly the following:

0) *The map* $\nabla^{\#}_{\delta} v$ (*for* $\delta \in \mathrm{Der}_T$ *and* $v \in \mathscr{G}$) *is* T-*linear in* δ *and satisfies the Leibniz rule:*

$$\nabla^{\#}_{\delta}(fv) = \delta(f)v + f\nabla^{\#}_{\delta} v \qquad \textit{for} \quad f \in T.$$

i) *For* $^{\forall}\delta,\ \xi \in \mathrm{Der}_T$

$$[\nabla^{\#}_{\delta}, \nabla^{\#}_{\xi}] = \nabla^{\#}_{[\delta,\xi]}\,. \tag{11.2.2}$$

ii) *For* $u, v \in \mathscr{G}$

$$\nabla^{\#}_{\ddot{u}} v - \nabla^{\#}_{\ddot{v}} u = [u, v]\,, \tag{11.2.3}$$

where $\ddot{u}$ *and* $\ddot{v}$ *are the images of* u *and* v *in* Der_T *by the projection map* (9.5.4).

iii) *For* $\delta \in \mathrm{Der}_T$ *and* $u, v \in \mathscr{G}$

$$\delta J(u, v) = J(\nabla^{\#}_{\delta} u, v) + J(u, \nabla^{\#}_{\delta} v)\,. \tag{11.2.4}$$

iv) *For* $\delta \in \mathrm{Der}_T$ *and* $\xi \in \mathscr{G}$

$$\nabla^{\#}_{\delta}(N(\xi)) = N(\nabla^{\#}_{\delta}(\xi))\,, \tag{11.2.5}$$

v) *For* $\delta \in \mathrm{Der}_T$

$$\nabla^{\#}_{\delta}\partial_l = 0\,. \tag{11.2.6}$$

vi) *For* $u, v \in \mathscr{G}$

$$(\nabla^{\#}_{\ddot{u}} v)^{\ddot{}}\tau = \ddot{u}\ddot{v}\tau\,. \tag{11.2.7}$$

Proof. The existence and uniqueness of connection $\nabla^{\#}$ satisfying ii) and iii) can be shown by a slight extension of the Levi-Civita connection for the metric J on $\mathscr{G}$, which will be given in (11.3). By a use of the explicit formula for $\nabla^{\#}$, we give a proof of v) and vi) in (11.3).

An essential feature in the **Lemma** is that this Levi-Civita connection satisfies further i) and iv). This is shown in (11.4) by a comparison of $\nabla^{\#}$ with ∇.

(11.3) *Existence of* $\nabla^{\#}$: First, let us define a map:

$$\nabla^{\#\#}: \mathscr{G} \times \mathscr{G} \to \mathscr{G}\,, \tag{11.3.1}$$

by the formula:

$$\begin{aligned} J(\nabla^{\#\#}_{\delta}\xi, \eta) = (&\delta J(\xi, \eta) + J(\xi, [\eta, \delta]) - \eta J(\xi, \delta) \\ &-J([\xi, \eta], \delta) + \xi J(\eta, \delta) + J(\eta, [\delta, \xi]))/2\,. \end{aligned} \tag{11.3.2}$$

This definition has a meaning, since the right hand of (11.3.2) is shown to be T-linear in the variable η and that J is a non-degenerate T-bilinear form. It is easy to check that the map $\nabla^{\#\#}$ is T-linear in the first variable δ and that it satisfies the Leibniz rule: $\nabla_\delta^{\#\#}(f\xi) = \delta(f)\xi + f\nabla_\delta^{\#\#}\xi$ for $f \in T$ and $\delta, \xi \in \mathscr{G}$. Also from the formula (11.3.2), one checks easily the relations:

$$\delta J(\xi, \eta) = J(\nabla_\delta^{\#\#}\xi, \eta) + J(\xi, \nabla_\delta^{\#\#}\eta),$$

and

$$\nabla_\delta^{\#\#}\xi - \nabla_\xi^{\#\#}\delta = [\delta, \xi].$$

Let us show:

(11.3.3) $$\nabla_{\partial_l}^{\#\#}\delta = 0 \qquad \text{for } {}^\forall\delta \in \mathscr{G}.$$

(*Proof.* Substituting ∂_l in (11.1.7) and applying (9.10.1),

$$\begin{aligned} J(\nabla_{\partial_l}^{\#\#}\xi, \eta) &= (\partial_l J(\xi, \eta) + J(\xi, [\eta, \partial_l]) - \eta J(\xi, \partial_l) \\ &\quad - J([\xi, \eta], \partial_l) + \xi J(\eta, \partial_l) + J(\eta[\partial_1, \xi]))/2 \\ &= (-\eta J(\xi, \partial_l) - J([\xi, \eta], \partial_l) + \xi J(\eta, \partial_l))/2 \\ &= \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}}(-\eta(\xi\tau) - [\xi, \eta]\tau + \xi(\eta\tau))/2 \\ &= 0. \end{aligned}$$

□)

The facts (11.3.3) and that $\nabla^{\#\#}$ is T-linear in the first variable imply that $\nabla^{\#\#}$ is factored through a connection on $\mathscr{G}$, denoted by $\nabla^{\#}$ (cf. (9.5.4)):

$$\nabla_{\bar{u}}^{\#} v := \nabla_u^{\#\#} v.$$

The induced map $\nabla^{\#}$ has the required properties ii) and iii) due to the relations for $\nabla^{\#\#}$.

Uniqueness of $\nabla^{\#}$: Using the short exact sequence (9.5.4), let us lift the map $\nabla^{\#}$ (11.2.1) as a map $\nabla^{\#\#}$ (11.3.1). Noting that $\mathscr{G}$ is closed under the bracket product (9.5) ii), a similar calculation of the uniqueness of ∇ in (6.2) 1. can be used to deduce the formula (11.3.2). Details are omitted.

Proof of v) *and* vi). Substituting ∂_l in the formula (11.3.2) and recalling the fact (9.10.1), we obtain:

v) $$\begin{aligned} J(\nabla_\delta^{\#\#}\partial_l, \eta) &= (\delta J(\partial_l, \eta) + J(\partial_l, [\eta, \delta]) - \eta J(\partial_l, \delta) \\ &\quad - J([\partial_l, \eta], \delta) + \partial_l J(\eta, \delta) + J(\eta, [\delta, \partial_l]))/2 \\ &= (\delta J(\partial_l, \eta) + J(\partial_l, [\eta, \delta]) - \eta J(\partial_l, \delta))/2 \\ &= 0. \end{aligned}$$

vi) $$\begin{aligned} J(\nabla_\delta^{\#\#}\xi, \partial_l) &= (\delta J(\xi, \partial_l) + J(\xi, [\partial_l, \delta]) - \partial_l J(\xi, \delta) \\ &\quad - J([\xi, \partial_l], \delta) + \xi J(\partial_l, \delta) + J(\partial_l, [\delta, \xi]))/2 \end{aligned}$$

$$= (\delta J(\xi, \partial_l) + \xi J(\partial_l, \delta) + J(\partial_l, [\delta, \xi]))/2$$

$$= \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}}(\delta(\xi\tau) + \xi(\delta\tau) + [\delta, \xi]\tau)/2$$

$$= \frac{(I_R : I)}{2\pi\sqrt{-1}m_{\max}}\delta(\xi\tau). \qquad \square$$

Notations. From now on, we shall confuse the maps $\nabla^{\#}$ (11.2.1) and $\nabla^{\#\#}$ (11.3.1). Both maps will be denoted by the same $\nabla^{\#}$.

(11.4) To proceed further the proof of (11.2) **Lemma**, let us give another expression of $\nabla^{\#}$ in terms of ∇ in the following.

Assertion. *Let the notations be as in* §9–10.

1) $\nabla_\delta w^i(\xi) \equiv w^i(\nabla^{\#}_\delta \xi) \mod \mathcal{G}_{\leq i-1}$ *for* $i \geq 1, \delta, \xi \in \mathcal{G}$.

2) $\nabla_{\partial_l} w^i(\xi) \equiv w^{i-1}((N + i - 1)\xi) \mod \mathcal{G}_{\leq i-2}$ *for* $i \geq 1, \delta, \xi \in \mathcal{G}$.

Proof. 1) Since $\nabla: \mathcal{G}_0 \times \mathcal{G}_1 \to \mathcal{G}_0 \oplus \mathcal{G}_1$ (10.4) **Assertion** i), one defines a map $F(\delta, \xi) \in \mathcal{G}$ by the relation $\nabla_\delta w(\xi) \equiv w(F(\delta, \xi)) \mod \mathcal{G}_0$. Let us show by induction on $i \geq 1$ that

$$\nabla_\delta w^i(\xi) \equiv w^i(F(\delta, \xi)) \mod \mathcal{G}_{\leq i-1} \qquad \text{for} \quad i \geq 1, \delta, \xi \in \mathcal{G}.$$

Assuming this for i, let us apply ∇_δ on $w^{i+1}(\xi) = w^i(\theta_l \xi - \theta_l * \xi)$:

$$\begin{aligned}\nabla_\delta w^{i+1}(\xi) &= \nabla_\delta(\theta_l w^i(\xi) - w^i(\theta_l * \xi)) \\ &= \theta_l \nabla_\delta w^i(\xi) + \delta(\theta_l) w^i(\xi) - \nabla_\delta w^i(\theta_l * \xi) \\ &= \theta_l(w^i(F(\delta, \xi)) + *) + \delta(\theta_l) w^i(\xi) - \nabla_\delta w^i(\theta_l * \xi) \\ &\equiv \theta_l w^i(F(\delta, \xi)) \mod \mathcal{G}_{\leq i} \\ &\equiv w^{i+1}(F(\delta, \xi)) \mod \mathcal{G}_{\leq i}. \qquad //\end{aligned}$$

It is a routine work to check that F is in fact a connection on the T-module $\mathcal{G}$, whose detailed verifications are omitted. Furthermore, if we have shown that

a) the connection is torsion free in the extended sense,

b) the connection has the metric property w.r.t. J,

(cf. **Lemma** ii) and iii)), then the uniqueness shown in (11.3) implies that $F(\delta, \xi) = \nabla^{\#}_\delta \xi$.

a) $w(F(\delta, \xi) - F(\xi, \delta)) \equiv \nabla_\delta w(\xi) - \nabla_\xi w(\delta) \mod \mathcal{G}_{\leq 0}$

$$\begin{aligned}&= \theta_l(\nabla_\delta \xi - \nabla_\xi \delta) + (\delta\theta_l)\xi - (\xi\theta_l)\delta + \nabla_\delta \theta_l * \xi - \nabla_\xi \theta_l * \delta \\ &\equiv \theta_l[\delta, \xi] \mod \mathcal{G}_{\leq 0} \\ &\equiv w([\delta, \xi]) \mod \mathcal{G}_{\leq 0}. \qquad //\end{aligned}$$

b) Recall the metric property of ∇ w.r.t. $\tilde{I}^*_W$.

$$\delta \tilde{I}^*_W(w(\xi), w(\eta)) = \tilde{I}^*_W(\nabla_\delta w(\xi), w(\eta)) + \tilde{I}^*_W(w(\xi), \nabla_\delta w(\eta)).$$

Hence, by taking the derivatives by ∂_l, we have

$$\text{The left hand} = \partial_l(\delta(\theta_l J(\xi, \eta) - J(\xi, \theta * \eta))) = \delta J(\xi, \eta).$$

$$\begin{aligned}\text{The right hand} &= \partial_l(J(\nabla_\delta w(\xi), \eta) + J(\xi, \nabla_\delta w(\eta)))\\ &= \partial_l(J(w(F(\delta, \xi)) + *, \eta) + J(\xi, w(F(\delta, \eta) + *))\\ &= J(F(\delta, \xi), \eta) + J(\xi, F(\delta, \eta)).\end{aligned}$$

(Here $*$ implies an element of $\mathscr{G}$.) / /

2) The case $i = 1$ is the definition (9.12.1).
Assuming the formula for i, let us calculate

$$\begin{aligned}\nabla_{\partial_l} w^{i+1}(\xi) &= \nabla_{\partial_l}(\theta_l w^i(\xi) - w^i(\theta_l * \xi))\\ &= \theta_l \nabla_{\partial_l} w^i(\xi) + w^i(\xi) - \nabla_{\partial_l} w^i(\theta_l * \xi)\\ &= \theta_l(w^{i-1}((N + i - 1)\xi) + *) + w^i(\xi) - \nabla_{\partial_l} w^i(\theta_l * \xi)\\ &= \theta_l(w^{i-1}((N + i - 1)\xi)) + w^i(\xi) \mod \mathscr{G}_{\leq i-1}\\ &= w^i((N + i)\xi) \mod \mathscr{G}_{\leq i-1}.\end{aligned}$$ / /

These complete the proof of the **Assertion.** □

Using the expressions 1) and 2) of the **Assertion**, let us show the remaining i) and iv) of the (11.2) **Lemma**, which are now a straightforward calculations as follows.

i) *Integrability of* $\nabla^{\#}$. For $i \geq 2$, let us calculate,

$$\begin{aligned}w^i(\nabla_\delta^{\#}\nabla_\xi^{\#}\eta) &\equiv \nabla_\delta w^i(\nabla_\xi^{\#}\eta) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_\delta \nabla_\xi w^i(\eta) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_\xi \nabla_\delta w^i(\eta) - \nabla_{[\delta,\xi]} w^i(\eta) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_\xi w^i(\nabla_\delta^{\#}\eta) - w^i(\nabla_{[\delta,\xi]}^{\#}\eta) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv w^i(\nabla_\xi^{\#}\nabla_\delta^{\#}\eta) - w^i(\nabla_{[\delta,\xi]}^{\#}\eta) && \mod \mathscr{G}_{\leq i-1}\end{aligned}$$ / /

ii) For

$$\begin{aligned}w^i(\nabla_\delta^{\#} N(\xi)) &\equiv \nabla_\delta w^i(N(\xi)) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_\delta w^i(N(\xi)) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_\delta(\nabla_{\partial_l} w^{i+1}(\xi) + * - w^i(i\xi)) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_{\partial_l}\nabla_\delta w^{i+1}(\xi) - \nabla_\delta w^i(i\xi) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv \nabla_{\partial_l} w^{i+1}(\nabla_\delta^{\#}\xi) - w^i(i\nabla_\delta^{\#}\xi) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv w^i((N + i)\nabla_\delta^{\#}\xi) - w^i(i\nabla_\delta^{\#}\xi) && \mod \mathscr{G}_{\leq i-1}\\ &\equiv w^i(N(\nabla_\delta^{\#}\xi)) && \mod \mathscr{G}_{\leq i-1}.\end{aligned}$$ / /

These altogether prove the (11.2) **Lemma**. □

(11.5) Let us give the consequence of the (11.2) **Lemma**. Let the notations be as in (11.1) and recall some more notations of (2.4).

Theorem (*The flat structure on* S^W)

Let (R, G) *be a marked EARS such that* $\mathrm{cod}(R, G) = 1$ (2.4.2).

1. *There exists a quintuplet* $(\Omega, \mathbb{J}, \mathbb{N}, \tau, P)$, *where*

i) Ω *is a complex graded vector space of rank* $l+2$, *whose weights are*: $0 := \tilde{m}_{-1}$, $-\tilde{m}_0, \ldots, -\tilde{m}_l\ (= -(l_{\max}+1))$ (*cf.* (2.4.3)). *I.e.*

$$\Omega = \bigoplus_{m_i} \Omega_{-\tilde{m}_i}, \qquad \mathrm{rank}_{\mathbf{C}}\, \Omega_{-\tilde{m}_i} = \textit{multiplicity of } \tilde{m}_i\,,$$

ii) $\mathbb{J}$ *is a nondegenerate symmetric bilinear form on* Ω, *which is homogeneous of degree* $\tilde{m}_{\max}$. *Hence* $\mathbb{J}$ *induces the perfect paring*:

$$\mathbb{J}\colon \Omega_{-\tilde{m}_i} \times \Omega_{\tilde{m}_i - \tilde{m}_{\max}} \longrightarrow \mathbf{C}$$

for any $\tilde{m}_i$,

iii) $\mathbb{N}$ *is a graded endomorphism of* Ω *given by*

$$\mathbb{N} = \sum_{m_i} (m_i/m_{\max}) \cdot \mathrm{id}_{\Omega_{-\tilde{m}_i}}\,,$$

such that $\mathbb{N} + \mathbb{N}^* = \mathrm{id}_\Omega$ *for* $\mathbb{N}^* :=$ *the adjoint of* $\mathbb{N}$ *with respect to* $\mathbb{J}$,

iv) τ *is a homogeneous* $\mathbf{C}$*-linear coordinate of* Ω *of degree* 0,

v) P *is a graded* $\mathbf{C}$*-linear embedding of the* $\mathbf{C}$*-dual space* Ω^* *of* Ω *into the invariants* S^W (4.3.3), *such that*

a) P *maps the* $\tau \in \Omega^*$ *to the coordinate* $\tau \in S^W$ (3.2.1) *and a* $\mathbf{C}$*-linear functional on* $\Omega_{-\tilde{m}_{\max}}$ *to the normalized generator* $\theta_l \in S^W$ *in* (9.3).

b) P *induces the algebra isomorphisms*:

$$S^W \simeq \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}) \underset{\mathbf{C}[\tau]}{\otimes} S[\Omega^*], \quad \textit{and} \quad T \simeq \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}) \underset{\mathbf{C}[\tau]}{\otimes} S\Big[\bigoplus_{m_i < m_{\max}} \Omega^*_{-\tilde{m}_i}\Big]\,,$$

where $S[V]$ *means the symmetric tensor algebra of* V *over* $\mathbf{C}$. *Hence an element of* Ω *can be extended to a derivation of* S^W, *inducing a map* $P^*\colon \Omega \to \mathrm{Der}_{S^W}$ *and an isomorphism for the small tangent module* $\mathcal{G}$ (9.5.2):

$$P^*\colon \Omega \underset{\mathbf{C}}{\otimes} T \simeq \mathcal{G}\,.$$

c) *By the identification in* b), *the* $\mathbb{J}$ *on* Ω *is identified with the metric* J *on the small tangent module* $\mathcal{G}$ ((9.9.1), (9.6.2)). *I.e.*

$$\mathbb{J}^*(\omega_1, \omega_2) = \partial_l \tilde{I}_W(dP(\omega_1), dP(\omega_2))$$

for $\omega_1, \omega_2 \in \Omega^*$. (*Here* $\tilde{I}_W$ *is a metric on* Der_{S^W} *induced from the Killing form* (§ 5 (5.2.3)).)

d) *By the identification in* b), *the* $\mathbb{N}$ *on* Ω *is identified with the endomorphism* N *on the small tangent module* $\mathcal{G}$ ((9.12.1)). *I.e.*

$$P^*(\mathbb{N}(\delta)) = \nabla_{\partial_l}(\tilde{I}_W(dP(\mathbb{J}(\delta))))$$

for $\delta \in \Omega$. (*Here* ∇ *is the Gauss Manin connection on* Der_{S^W} (§ 6 (6.1)).)

2. *Such quintuplet* $(\Omega, \mathbb{J}, \mathbb{N}, \tau, P)$ *is unique up to a graded affine linear isometry of* Ω *leaving* τ, $\mathbb{J}$ *and* $\mathbb{N}$ *invariant. Particularly the image set* $P(\Omega^*)$ *in* S^W *is unique.*

Note. The above construction induces an identification of the space $\mathrm{Spec}(S^W)$ *with the flat affine half space* $\{x \in \Omega: \mathrm{Im}(\tau(x)) > 0\} = \mathbb{H} \oplus \bigoplus_{0<m_i} \Omega_{-\tilde{m}_i}$. *This is the reason for the name "flat structure".*

Proof. The **Theorem** is essentialy a reformulation of the (11.2) **Lemma**. Put

$$\Omega := \{\xi \in \mathscr{G} : \nabla_\delta^{\#} \xi = 0 \text{ for } {}^\forall\delta \in \mathrm{Der}_T\}$$

for which let us show the following i)–iv).

i) Ω is a graded complex vector space of rank $l+2$ with the prescribed degrees.

ii) For ${}^\forall\delta, \xi \in \Omega$, $[\delta, \xi] = 0$.

iii) The restriction of J on Ω defines a complex constant valued form, which we shall denote by $\mathbb{J}$.

iv) The restriction of N on Ω induces an endomorphism of Ω, which we shall denote by $\mathbb{N}$.

Proof.

i) Since $\nabla^{\#}$ is an integrable connection, one has $\mathscr{G} \simeq \Omega \otimes_{\mathbb{C}} T$. This implies $\mathrm{rank}_{\mathbb{C}} \Omega = \mathrm{rank}_T \mathscr{G} = l + 2$, and the graded module structure of $\mathscr{G}$ over T, generated by elements $\partial/\partial\theta_i$ of degree $-\tilde{m}_i$, is inherited to Ω over $\mathbb{C}$.

ii) Torsion-freeness of $\nabla^{\#}$ implies this.

$$[\delta, \xi] = \nabla_\delta^{\#} \xi - \nabla_\xi^{\#} \delta = 0 \qquad \text{for } \delta, \xi \in \Omega .$$

iii) Metric property of $\nabla^{\#}$ implies this.

$$dJ(\delta, \xi) = J(\nabla^{\#}\delta, \xi) + J(\delta, \nabla^{\#}\xi) = 0 \qquad \text{for } \delta, \xi \in \Omega .$$

iv) That N is horizontal implies this. If $\delta \in \Omega$ then $\nabla^{\#} N(\delta) = N(\nabla^{\#}\delta) = N(0) = 0$ implies $N(\delta) \in \Omega$. For the formula for $\mathbb{N}$, recall (9.11.2) and (9.9.4).

The dual vector space of Ω is realized as the horizontal space,

$$\Omega^* := \{\omega \in \mathscr{F} : \nabla^{\#*}\omega = 0\}$$

where $\mathscr{F}$ (defined in (9.5.1)) is a dual T-module of $\mathscr{G}$ and $\nabla^{\#*}$ is the dual connection of $\nabla^{\#}$. Of course Ω^* is a graded vector space with additional structures. In the following I $\sim$ III, we embed Ω^* into S^W:

$$P^* : \Omega^* \longrightarrow S^W .$$

I) Let us show that any element ω of Ω^* is closed as a differential form. Substitute $\delta, \xi \in \Omega$ in the formula

$$2\langle \delta \wedge \xi, d\omega \rangle = \delta\langle \xi, \omega \rangle - \xi\langle \delta, \omega \rangle - \langle [\delta, \xi], \omega \rangle .$$

Then each term of the right hand vanishes. This implies $d\omega = 0$, since $\mathrm{Der}_{S^W} \simeq \Omega \otimes_{\mathbb{C}} S^W$. On the other hand, since the sequence $0 \to \mathbb{C} \to S^W \xrightarrow{d} \Omega^1_{S^W} \xrightarrow{d} \Omega^2_{S^W} \to \cdots$ is exact (From the analytic view point, this is trivial, for $\mathrm{Spec}(S^W) \simeq \mathbb{H} \times \mathbb{C}^{l+1}$ is simply connected.), for any $\omega \in \Omega^*$ there exists some $\theta \in S^W$ such that $\omega = d\theta$.

II) If ω is homogeneous of positive degree, one may choose θ homogeneous of the same degree and this choice eliminates the ambiguity of the constant term of θ and so we put $\theta := P^*(\omega)$.

III) Let us show $d\tau \in \Omega^*$. Recall (11.2) **Lemma** vi). Then for ${}^\forall\delta$, $\xi \in \mathrm{Der}_T$

$$\langle \nabla_\delta^{\#*} d\tau, \xi \rangle = \delta\langle d\tau, \xi \rangle - \langle d\tau, \nabla_\delta^{\#} \xi \rangle = \delta(\xi\tau) - (\nabla_\delta^{\#} \xi)\tau = 0 .$$

We shall call the element $\tau := d\tau \in \Omega^*$ and define $P^*(\tau) := \tau$.

These I), II) and III) together define the inclusion map P^*.

Proof of a). Remark that $\partial_l \in \Omega$, due to the fact (11.2) **Lemma** v). Hence for any $\omega \in \Omega^*$, we have $\partial_l P^*(\omega) \in \mathbb{C}$.

Proof of b). Since Ω^* is the set of linear functionals on Ω, the inclusion map $P^* : \Omega^* \subset S^W$ induces a $\Gamma(\mathbb{H}, \mathcal{O})$-algebra homomorphism,

$$P^*: \Gamma(\mathbb{H}, \mathcal{O}_{\mathbb{H}}) \underset{\mathbb{C}[\tau]}{\otimes} S[\Omega^*] \longrightarrow S^W$$

which is obviously an isomorphism of graded algebra, since both are polynomial algebras over $\Gamma(\mathbb{H}, \mathcal{O})$ with equally graded generators.

Proof of c) *and* d). By the definition of P^*, for any linear coordinate $\omega \in \Omega^*$ of Ω, the differential $dP^*(\omega)$ coincides with ω. So $J^*(dP^*(\omega), dP^*(\omega')) = J^*(\omega, \omega') = \mathbf{J}^*(\omega, \omega')$. A similar proof is valid also for the case d).

2. Obvious from the unicity of $\nabla^{\#}$.

These complete the proof of the **Theorem**. □

Notation. For a linear basis ω_i $(i = 0, \ldots, l)$ and $d\tau$ of Ω^*, the system of functions $P^*(\omega_i) \in S^W$ $(i = 0, \ldots, l)$ and τ will be called the flat generator system of S^W.

Problem. Determine explicitely as θ-functions the flat generator system for S^W. Compare also a flat generator system with the fundamental characters χ_i $(i = 0, \ldots, l)$.

For a suitable arithmetic subgroup $\Gamma \subset SL(2, \mathbb{Z})$, one should study the quotient Ω/Γ by extending the action of $\tilde{W}_R$ on $\tilde{\mathbb{E}}$ by the group Γ.

Appendix. Families of Line Bundles over Elliptic Curves

This appendix gives an elementary invariant theoretic construction of families of line bundles over elliptic curves with an Hermitian metric on it. This leads to a construction of simple elliptic singularities [21], which is an attempt toward a construction of a hyperkähler structure on smoothings of a simple elliptic singularity, while such structure is constructed for a simple singularity by P. Kronheimer [34].

Line bundles over elliptic curves is a classic and there may be many different approaches to the objects. The approach in this appendix is a Heisenberg group invariant theoretic one, which is parallel to this paper and we shall use often the same notations to indicate an analogy. The detailed verifications of the statements are left to the reader.

(A.1) *The semi-positive quadratic form I and its extention* $\tilde{I}$.

Let F be a real 3 dimensional vector space with a positive semi-definite symmetric bilinear form:

(A.1.1) $$I: F \times F \longrightarrow \mathbb{R},$$

whose radical $\mathrm{rad}(I) := F^{\perp}$ has rank 2 over $\mathbb{R}$.

Assume that I is defined over $\mathbb{Z}$ in the sense that

i) $^{\exists}Q$: a free abelian group of rank 3, s.t. $Q \otimes_{\mathbb{Z}} \mathbb{R} = F$.

ii) $I|Q \otimes Q$ is integral valued and $I(Q, Q) = \mathbb{Z}$.

Under the assumption, $\mathrm{rad}(I) \cap Q$ is a free abelian group of rank 2.

A 1-dimensional $\mathbb{R}$-subspace G of $\mathrm{rad}(I)$ defined over $\mathbb{Z}$ is fixed and will be called *the marking* for I as in this paper (2.1) Def. A generator of $G \cap Q \simeq \mathbb{Z}$, denoted by a, is unique up to a sign.

(A.1.3) $$G \cap Q = \mathbb{Z}a.$$

Depending on G, there exists uniquely a pair $(\tilde{F}, \tilde{I})$ of a real vector space $\tilde{F}$ of rank 4 and a symmetric bilinear form $\tilde{I}$ on $\tilde{F}$ (up to a linear isomorphism) such that

i) There is an injective linear map: $F \subset \tilde{F}$, regard as inclusion.

ii) $\tilde{I}|_F = I$.

iii) $\mathrm{rad}(\tilde{I}) = G$.

Once and for all in this Appendix, we fix basis z, b, and a s.t.

$$Q = \mathbb{Z}z + \mathbb{Z}b + \mathbb{Z}a, \quad \text{and} \quad \mathrm{rad}(I) \cap Q = \mathbb{Z}b + \mathbb{Z}a.$$

The intersection matrix of I with respect to the basis (z, b, a) is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By a suitable choice of a base $\tilde{\lambda} \in \tilde{F} \backslash F$, the intersection matrix for $\tilde{I}$ with respect to the basis $(\tilde{\lambda}, z, b, a)$ becomes:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The choice of the basis $\tilde{\lambda}$, z and b are done only for the sake of explicitness and that the choice does not affect the results of the appendix.

(A2) *The Eichler Siegel transformations E* and $\tilde{E}$.

Let the notations be as in (A.1). Let us define a semi-group homomorphisms, called an Eichler Siegel transformation for I,

(A.2.1) $$E: F \otimes_{\mathbb{R}} F/\mathrm{rad}(I) \longrightarrow \mathrm{End}(F),$$

as follows (cf. §2 (2.3) or [25, §1]): For $u \in F$ *and* $\sum_i p_i \otimes q_i \in F \otimes_{\mathbb{R}} F/\mathrm{rad}(I)$, (A.2.2) $$E\Big(\sum_i p_i \otimes q_i\Big)(u) := u - \sum_i p_i I(q_i, u)\,.$$

Here the semi-group structure on $F \otimes_{\mathbf{R}} F/\mathrm{rad}\,(I)$ is defined as

(A.2.3) $$\Big(\sum_i p_i \otimes q_i\Big) \circ \Big(\sum_j r_j \otimes s_j\Big) := \sum_i p_i \otimes q_i + \sum_j r_j \otimes s_j - \sum_i \sum_j p_i \otimes s_j I(q_i, r_j)\,.$$

Similarly, one defines the Eichler-Siegel transformation for $\tilde{I}$

(A.2.4) $$\tilde{E}\colon F \otimes_{\mathbf{R}} F/\mathrm{rad}\,(\tilde{I}) \longrightarrow \mathrm{End}\,(\tilde{F})\,,$$

by replacing I by $\tilde{I}$ in the expressions (A.2.2) and (A.2.3).

(A.3) *The Heisenberg group $\tilde{H}$.*

Let $O(F, I)$ (resp. $O(\tilde{F}, \tilde{I})$) be the set of linear isomorphism of F (resp. $\tilde{F}$), which preserves I (resp. $\tilde{I}$).

For $\xi = xz \otimes z + pa \otimes z + qb \otimes z \in F \otimes F/\mathrm{rad}\,(I)$ with $x, p, q \in \mathbf{R}$, we have,

$$E(\xi)\begin{bmatrix} z \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1-x & -q & -p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z \\ b \\ a \end{bmatrix}.$$

Hence $E(\xi) \in O(F, I)$ if and only if $x = 1 - \varepsilon$ for $\varepsilon \in \{\pm 1\}$. This implies the following short exact sequence:

(A.3.1) $$0 \longrightarrow H \longrightarrow W \xrightarrow{\det} \{\pm 1\} \longrightarrow 1\,,$$

where $$W := (F \otimes F/\mathrm{rad}\,(I)) \cap E^{-1}(O(F, I)),$$

$$H := (F \otimes F/\mathrm{rad}\,(I)) \cap E^{-1}(SO(F, I)) = \mathrm{rad}\,(I) \otimes_{\mathbf{R}} F/\mathrm{rad}\,(I).$$

The group structure $\circ$ (A.2.3) on H coincides with its vector space structure of rank 2 generated by $a \otimes z$ and $b \otimes z$. So the group W is an extension of $\{\pm 1\}$ by $H \simeq \mathbf{R}^2$, where the extension of (A.3.1) is given by the action of $-1 \in \{\pm 1\}$ on the vector space by $-\mathrm{id}_{\mathbf{R}^2}$.

For $\xi = \begin{matrix} xz \otimes z + pa \otimes z + qb \otimes z \\ + yz \otimes b + zb \otimes b + ra \otimes b \end{matrix} \in F \otimes F/\mathrm{rad}\,(\tilde{I})$ with $x, y, z, p, q, r \in \mathbf{R}$, we have

$$\tilde{E}(\xi)\begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & -y & -z & -r \\ 0 & 1-x & -q & -p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix}.$$

Hence $\tilde{E}(\xi) \in O(\tilde{F}, \tilde{I})$, iff $x = 1 - \varepsilon$ for $\varepsilon \in \{\pm 1\}$, $y = -\varepsilon q$ and $z = q^2/2$. So

(A.3.2) $$\xi = (1-\varepsilon)z \otimes z + pa \otimes z + qb \otimes z - \varepsilon qz \otimes b + \tfrac{1}{2}q^2 b \otimes b + ra \otimes b\,.$$

We obtain a short exact sequence,

(A.3.3) $$0 \longrightarrow \tilde{K} \longrightarrow \tilde{W} \xrightarrow{P_*} W \longrightarrow 1$$

where

i) $\tilde{W} := (F \otimes F/\mathrm{rad}\,(\tilde{I})) \cap \tilde{E}^{-1}(O(\tilde{F}, \tilde{I}))$,

ii) $\tilde{K} := \mathrm{rad}\,(\tilde{I}) \otimes (\mathrm{rad}\,(I)/\mathrm{rad}\,(\tilde{I}) \simeq \mathbb{R}a \otimes b$,

iii) The map P_* is induced from the natural projection:

$$F \otimes F/\mathrm{rad}\,(\tilde{I}) \longrightarrow F \otimes F/\mathrm{rad}\,(I)\,.$$

iv) Let us define index 2 subgroup $\tilde{H}$ of $\tilde{W}$ *by*

$$\tilde{H} := (F \otimes F/\mathrm{rad}\,(\tilde{I})) \cap \tilde{E}^{-1}(SO(\tilde{F}, \tilde{I})) = P_*^{-1}(H)\,.$$

$\tilde{H}$ is a Heisenberg group, for which the commutative diagram holds:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{K} & \longrightarrow & \tilde{H} & \xrightarrow[P_*]{} & H & \longrightarrow & 1 \\
 & & \| & & \cap & & \cap & & \\
0 & \longrightarrow & \tilde{K} & \longrightarrow & \tilde{W} & \xrightarrow[P_*]{} & W & \longrightarrow & 1 \\
 & & & & \downarrow \mathrm{det} & & \downarrow \mathrm{det} & & \\
 & & & & \{\pm 1\} & = = & \{\pm 1\} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

As a set, $\tilde{H}$ is parametrized by $(p, q, r) \in \mathbb{R}^3$ as follows.

(A.3.4) $$\xi_{pqr} := pa \otimes z + qb \otimes z - qz \otimes b + \tfrac{1}{2}q^2 b \otimes b + ra \otimes b\,.$$

The multiplication rule for ξ's and its action on $\tilde{F}$ are given by:

(A.3.5) $$\xi_{pqr} \circ \xi_{p'q'r'} = \xi_{p+p', q+q', r+r'+p'q}\,,$$

(A.3.6) $$\tilde{E}(\xi_{pqr}) \begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & q & -q^2/2 & -r \\ 0 & 1 & -q & -p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix}.$$

Remark. The following 1, 2 together with 3 show that.

An $\tilde{H}$-invariant symmetric bilinear form on $\tilde{F}$, which is not zero on F, equals to a constant multiple of $\tilde{I}$ up to an automorphism of $\tilde{F}$ by a central element of $F \otimes F/\mathrm{rad}\,(\tilde{I})$.

1. *The space of $\tilde{H}$-invariant forms on $\tilde{F}$ is given by* (A.3.7) $\{\tilde{H}$*-invariant symmetric bilinear forms on* $\tilde{F}\} = \mathbb{R}\tilde{I} + \mathbb{R}\tilde{I}_\infty$. *Here $\tilde{I}_\infty$ is a symmetric form on $\tilde{\mathbb{F}}$ characterized by* $\tilde{I}_\infty(\tilde{\lambda}, \tilde{\lambda}) = 1$ *and* $\tilde{I}_\infty | (F \times \tilde{F}) \cup (\tilde{F} \times F) = 0$. *The intersection matrix for $\tilde{I}_\infty$ w.r.t. the basis* $(\tilde{\lambda}, z, b, a)$ *is:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. *The center of the following group is given by*

$$Z((F\otimes F/\mathrm{rad}\,(\tilde I))\cap \tilde E^{-1}(GL(\tilde F)))$$
$$= \mathrm{rad}\,(I)\otimes(\mathrm{rad}\,(I)/\mathrm{rad}\,(\tilde I)) \simeq \mathbf{R}b\otimes b+\mathbf{R}a\otimes b\,.$$

where the group structure $\circ$ *in the second line coincides with the addition as the vectors. Additionally, we notice that,*

$$(\mathbf{R}b\otimes b+\mathbf{R}a\otimes b)\cap \tilde E^{-1}(O(\tilde F,\tilde I)) = \mathbf{R}a\otimes b\,.$$

3. *Let* I_1 *be an* $\tilde H$*-invariant symmetric forms on* $\tilde F$*, which is not zero on* F*. Then there exists a constants* $c,\ d\in\mathbf{R}$ *such that for* $g:=\tilde E(cb\otimes b)\in \mathrm{SL}(\tilde F)$*, we have* $I_1(g(x),g(y)) = d\tilde I(x,y)$ *for* $x,\ y\in\tilde F$*.*

Proof.
1. Use the expression ξ_{pqr} (A.3.6) for an element of $\tilde H$.
2. $\tilde E(\xi)$ $(\xi\in F\otimes F/\mathrm{rad}\,(\tilde I))$ belongs to the center, iff $\xi\in F^{\perp}\otimes F^{\perp}$.
3. Due to 1., $I_1 = d(\tilde I+2c\tilde I_\infty)$ for $c,\ d\in\mathbf{R}$ with $d\neq 0$. Put $g:=\tilde E(cb\otimes b)$. Then $g^{-1}(x) = x+c\tilde I(x,b)b$, etc., and hence

$$\begin{aligned}\tilde I(g^{-1}(x),g^{-1}(y)) &= \tilde I(x+c\tilde I(x,b)b,\ y+c\tilde I(y,b)b)\\ &= \tilde I(x,y)+2c\tilde I(x,b)\tilde I(y,b)\\ &= (\tilde I+2c\tilde I_\infty)(x,y)\,.\end{aligned}$$ □

(A.4) *Discrete subgroups* $\tilde H_d$ *of the group* $\tilde H$.

Let us consider the lattice $H_{\mathbf{Z}}$ of rank 2 in the group H.

(A.4.1) $$H_{\mathbf{Z}} := (F\otimes F/\mathrm{rad}\,(I))\cap E^{-1}(SO(Q,I)) = \mathbf{Z}a\otimes z\oplus\mathbf{Z}b\otimes z\,.$$

For an integer $d\in\mathbf{Z}$ with $d\neq 0$, let us consider subgroups H_d of $P_*^{-1}(H_{\mathbf{Z}})\subset\tilde H$, satisfying the short exact sequence

(A.4.2) $$0\longrightarrow \mathbf{Z}\frac{1}{d}a\otimes b\longrightarrow \tilde H_d\xrightarrow{P_*} H_{\mathbf{Z}}\longrightarrow 0\,.$$

Such subgroups $\tilde H_d$'s are parametrized by the real 2-torus

(A.4.3) $$T_d := \mathrm{Hom}_{\mathbf{Z}}\left(H_{\mathbf{Z}},\ \mathbf{R}\Big/\left(\frac{1}{d}\mathbf{Z}\right)\right)$$

as follows. Namely put

(A.4.4) $$\tilde H_d^{tot} := \left\{(\xi_{pqr},l)\in P_*^{-1}(H_{\mathbf{Z}})\times T_d : r\equiv l(p,q)\ \mathrm{mod}\frac{1}{d}\mathbf{Z}\right\},$$

where $l(p,q):=l(P_*(\xi_{pqr}))$. $\tilde H_d^{tot}$ is a 2-dimensional manifold. The natural projection to T_d induces a covering map:

(A.4.5) $$\pi\colon \tilde H_d^{tot}\longrightarrow T_d\,.$$

The fiber $\tilde H_d(l):=\pi^{-1}(l)$ over $l\in T_d$ is closed under the product $\circ$, for $r_1\equiv l(p_1,q_1)$ and $r_2\equiv l(p_2,q_2)$ implies $r_1+r_2+p_2q_1\equiv l(p_1+p_2,q_1+q_2)$. (Note that $\tilde H_d^{tot}$ is not a group and π is not a homomorphism in $\circ$.)

Explicitly the group $\tilde{H}_d(l)$ is given as follows.

For $(u, v) \in \mathbb{R}^2 \simeq \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{R})$ = the universal covering of T_d, put

$$\tilde{H}_d(u, v) := \{\xi_{p,q,pu+qv+r/d} : (p, q, r) \in \mathbb{Z}^3\} . \tag{A.4.6}$$

This depends only on the image $l \in T_d$ of $(u, v) \in \mathbb{R}^2$, which is $\tilde{H}_d(l)$.

It is a straight forward to see that all groups $\tilde{H}_d(l)$ $(l \in T_d)$ are isomorphic each other, denoted as $\tilde{H}_d$ as an abstract group, and satisfy the short exact sequence (A.4.2). The monodromy action $\frac{1}{d}H_{\mathbb{Z}}^{\sharp} := \pi_1(T_d) \to \mathrm{Aut}(\tilde{H}_d)$ of (A.4.5) will be given in (A.9.2).

To see that $\tilde{H}_d(l)$ $(l \in T_d)$ are the only subgroups of $\tilde{H}$ satisfying (A.4.2), it is enough to choose and fix an element of $P_*^{-1}(a \otimes z)$ and $P_*^{-1}(b \otimes z)$ as ξ_{10u} and ξ_{01v} for a $u, v \in \mathbb{R}$ respectively.

The extension class of (A.4.2) is calculated as follows. Let

$$J: H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z} \tag{A.4.7}$$

be the unique skew symmetric form with $J(a \otimes z, b \otimes z) = 1$, so that

$$\mathrm{Ext}_{\mathbb{Z}}^2(H_{\mathbb{Z}}, \mathbb{Z}) \simeq \bigwedge^2 \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{Z}) = \mathbb{Z}J \simeq \mathbb{Z} .$$

Since ξ_{pq0} gives a section $H_{\mathbb{Z}} \to \tilde{H}_d(0)$ for P_* (A.4.2), the extension class c of the sequence (A.4.2) is calculated by

$$\begin{aligned} c(pa \otimes z + qb \otimes z, p'a \otimes z + q'b \otimes z) &:= \xi_{pq0} \circ \xi_{p'q'0} - \xi_{p'q'0} \circ \xi_{pq0} \\ &= \xi_{00(pq'-qp')} = (pq' - qp')d(a \otimes b/d) \\ &\simeq (pq' - qp')d \\ &= dJ(pa \otimes z + qb \otimes z, p'a \otimes z + q'b \otimes z) . \end{aligned}$$

Assertion. *The extension class of* (A.4.2) *is* $dJ \in \mathrm{Ext}_{\mathbb{Z}}^2(H_{\mathbb{Z}}, \mathbb{Z})$.

Remark. It is easy to see that $\tilde{H}_d(l)$ is Zariski dence in the group $\tilde{H}$. As a consequence, any $\tilde{H}_d(l)$-invariant symmetric bilinear form on $\tilde{F}$ is automatically $\tilde{H}$-invariant, and therefore, it is a linear combination of $\tilde{I}$ and $\tilde{I}_\infty$ due to (A.3) *Remark.*

(A.5) *Conjugacy relation among* $\tilde{H}_d(l)$.

Recalling the product rule (A.3.5), for any $p, q, r, s, t, u \in \mathbb{R}$,

$$\xi_{stu} \circ \xi_{p,q,r-J} = \xi_{pqr} \circ \xi_{stu} , \tag{A.5.1}$$

where
$$J := J(P_*(\xi_{pqr}), P_*(\xi_{stu})) = pt - qs$$

is the $\mathbb{R}$-linear extention of (A.4.7), denoted by the same J.

Assertion. *Groups* $\tilde{H}_d(l)$ *and* $\tilde{H}_d(l')$ *are conjugate by a* ξ_{stu}, *where* $l' - l \equiv (-t, s)$ $\left(= J(P_*(\xi_{stu}))\right) \bmod \frac{1}{d}\mathbb{Z}$.

Proof. Comparing (A.4.6) with (A.3.5), we have

$$\xi_{stu}^{-1} \circ \tilde{H}_d(l) \circ \xi_{st0} = \tilde{H}_d(l + (-t, s)) . \qquad \square$$

This is the reformulation of the fact that an orthogonal base change:

(A.5.2)
$$\begin{bmatrix} \tilde{\lambda}' \\ z' \\ b \\ a \end{bmatrix} := \tilde{E}(\xi_{stu}) \begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & t & -t^2/2 & -u \\ 0 & 1 & -t & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix},$$

transforms the matrix representation of $\tilde{E}(\xi_{pqr})$ (A.3.6) to

(A.5.3)
$$E(\xi_{pqr}) \begin{bmatrix} \tilde{\lambda}' \\ z' \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & q & -\frac{1}{2}q^2 & -r + pt - qs \\ 0 & 1 & -q & -p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\lambda}' \\ z' \\ b \\ a \end{bmatrix},$$

whose matrix is the same as that of

(A.5.4)
$$\tilde{E}(\xi_{p,q,r-pt+qs}) \begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix} = \begin{bmatrix} 1 & q & -\frac{1}{2}q^2 & -r + pt - qs \\ 0 & 1 & -q & -p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{\lambda} \\ z \\ b \\ a \end{bmatrix}.$$

(A.6) *The complex half spaces $\tilde{\mathbb{E}}$ and $\mathbb{E}$.*
Let us define complex half spaces.

$$\tilde{\mathbb{E}} := \{x \in \mathrm{Hom}_{\mathbb{R}}(\tilde{F}, \mathbb{C}) : a(x) = 1 \text{ and } \mathrm{Im}(b(x)) > 0\} .$$

$$\mathbb{E} := \{x \in \mathrm{Hom}_{\mathbb{R}}(F, \mathbb{C}) : a(x) = 1 \text{ and } \mathrm{Im}(b(x)) > 0\} .$$

$$\mathbb{H} := \{x \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{rad}\,(I), \mathbb{C}) : a(x) = 1 \text{ and } \mathrm{Im}(b(x)) > 0\} .$$

Here $a, b \in \mathrm{rad}\,(I)$ are regarded as complex valued linear functionals on the dual spaces of $\tilde{F}$, F and $\mathrm{rad}(I)$. Put

(A.6.1)
$$\tau := b/a ,$$

by which $\mathbb{H}$ is identified with the complex upper half plane.

Notational remark. For convenience, the same notation τ is used for different meanings as in (3.2) *Notational remark.*

The inclusions: $\tilde{F} \supset F \supset \mathrm{rad}\,(I)$ induces natural projections:

(A.6.2)
$$\tilde{\mathbb{E}} \longrightarrow \mathbb{E} \longrightarrow \mathbb{H} .$$

The fibers $\tilde{\mathbb{E}}_\tau$ and $\mathbb{E}_\tau$ over $\tau \in \mathbb{H}$ are affine spaces of dim. 2 and 1.

(A.7) *Principal $\mathbb{C}^*$-bundle L_a^* over a family X of elliptic curves.*
The left action of $E(g)$ for $g \in W$ on F (resp. $\tilde{E}(g)$ for $g \in \tilde{W}$ on $\tilde{F}$) fixes $\mathrm{rad}\,(I)$ pointwisely (cf. (A.3.6)). So its dual action $E(g)^*$ induces a right action on $\mathbb{E}$ (resp.

$\tilde{\mathbb{E}}$). The homomorphism P_* (A.3.3) is equivariant with the projection $\tilde{\mathbb{E}} \to \mathbb{E}$ (A.6.2). Let us describe the quotient spaces $\mathbb{E}/H_{\mathbb{Z}}$ and $\tilde{\mathbb{E}}/\tilde{H}_d(l)$ by the discrete subgroups $H_{\mathbb{Z}}$ and $\tilde{H}_d(l)$ $(l \in T_d)$.

Assertion. i) *The action of* $H_{\mathbb{Z}}$ *on* $\mathbb{E}$ *is properly discontinuous and fixed point free. The quotient manifold is denoted by* $X := \mathbb{E}/H_{\mathbb{Z}}$.

ii) *The projection map,*

$$X \longrightarrow \mathbb{H}, \tag{A.7.1}$$

induced from (A.6.2) *is a family of smooth elliptic curves. The fiber over* $\tau \in \mathbb{H}$ *is an elliptic curve of modulous* τ.

iii) *The action of* $\tilde{H}_d(l)$ $\left(\textit{for } l \in \mathrm{Hom}_{\mathbb{Z}}\left(H_{\mathbb{Z}}, \mathbb{R}\middle/\left(\frac{1}{d}\mathbb{Z}\right)\right)\right)$ *on* $\tilde{\mathbb{E}}$ *is properly discontinuous and fixed point free. The quotient manifold is denoted by* $L_d^*(l) := \tilde{\mathbb{E}}/\tilde{H}_d(l)$.

iv) *The projection map,*

$$L_d^*(l) \longrightarrow X, \tag{A.7.2}$$

induced from (A.6.2) *defines a principal* $\mathbb{C}^*$*-bundle over* X, *whose Chern class in* $H^2(X, \mathbb{Z}) \simeq \mathbb{Z}$ *is equal to* d.

v) *For any two* $l, l' \in T_d$, *there exists a pair* $(\varphi_{stu}, \psi_{st})$ *of complex analytic isomorphisms, making the following diagram commutative.*

$$\begin{array}{ccccc} L_d^*(l) & \longrightarrow & X & \longrightarrow & \mathbb{H} \\ \downarrow{\scriptstyle \varphi_{stu}} & & \downarrow{\scriptstyle \psi_{st}} & & \| \\ L_d^*(l') & \longrightarrow & X & \longrightarrow & \mathbb{H}, \end{array} \tag{A.7.3}$$

where $s, t, u \in \mathbb{R}$ are real parameters satisfying a relation:

$$l' - l \equiv (-t, s) \mod \frac{1}{d} H_{\mathbb{Z}}^*.$$

Notation. The complex analytic isomorphism class of $L_d^*(l)$ *is denoted by* L_d^*, *which will be represented by* $L_d^*(0)$.

Proof. We consider $(\tilde{\lambda}, z, \tau = b/a)$ (resp. $(z, \tau = b/a)$) as complex coordinates for $\tilde{\mathbb{E}}$ (resp. $\mathbb{E}$). The action of an element $\xi \in \tilde{H}_d(l)$ is given in (A.3.6). Then i) and ii) may be obvious from the description: For $t_{pq} := pa \otimes z + qb \otimes z \in H_{\mathbb{Z}}$ $(p, q \in \mathbb{Z})$,

$$t_{pq}^*(z) = z - q\tau - p. \tag{A.7.4}$$

Due to (A.5) **Assertion**, one has the commutative diagram:

$$\begin{array}{ccc} \tilde{\mathbb{E}} & \xrightarrow{\tilde{E}(\xi_{pqr})^*} & \tilde{\mathbb{E}} \\ {\scriptstyle {}^*\tilde{E}(\xi_{stu})^*}\downarrow & & \downarrow{\scriptstyle \tilde{E}(\xi_{stu})^*} \\ \tilde{\mathbb{E}} & \xrightarrow[\tilde{E}(\xi_{pq(r-pt+qs)})^*]{} & \tilde{\mathbb{E}} \end{array}$$

Hence iii) and iv) are reduced to the case where $\tilde{H}_d(0)$ and the isomorphisms of v) are given by

$$\varphi_{stu} := \text{the isomorphism induced from } \tilde{E}(\xi_{stu})^* ,$$

$$\psi_{st} := \text{the isomorphism induced from } E(P_*(\xi_{stu}))^* .$$

Let us explain the Assertion for the case $\tilde{H}_d(0)$ explicitly. The generator $\frac{1}{d}a \otimes b$ of $\tilde{K}$ acts on the coordinate system of $\tilde{\mathbb{E}}$ by

$$\tilde{E}\left(\frac{1}{d}a \otimes b\right)\begin{bmatrix} \tilde{\lambda} \\ z \\ \tau \end{bmatrix} = \begin{bmatrix} \tilde{\lambda} \\ z \\ \tau \end{bmatrix} + \begin{bmatrix} 1/d \\ 0 \\ 0 \end{bmatrix}. \tag{A.7.5}$$

Hence by introducing a new coordinate λ

$$\lambda := \exp(2\pi\sqrt{-1}d\tilde{\lambda}) \tag{A.7.6}$$

the quotient variety $\tilde{\mathbb{E}}/\mathbf{Z}\left(\frac{1}{d}a \otimes b\right)$ is a $\mathbf{C}^*$-bundle over $\mathbb{E}$.

Recalling again (A.3.6), the action of a translation $t_{pq} := pa \otimes z + qb \otimes z \in H_{\mathbf{Z}} \simeq H_d(l)/\mathbf{Z}\left(\frac{1}{d}a \otimes b\right)$ (for $p, q \in \mathbf{Z}$) on the coordinates λ of $\tilde{\mathbb{E}}/\mathbf{Z}\left(\frac{1}{d}a \otimes b\right)$ is given by,

$$t_{pq}^*(\lambda) = e_{t_{pq}}\lambda , \tag{A.7.7}$$

where

$$e_{t_{pq}} := \exp(2\pi\sqrt{-1}d(-qz + \tfrac{1}{2}q^2\tau)) .$$

As is well known [14], this gives a line bundle of the Chern class $d \in H^2(X, \mathbf{Z})$ over the elliptic curve X.

This completes a proof of the Assertion. □

(A.8) *Indefinite metric on* $L_d^*(l)$.

In the spirit of the present paper, we consider the metric on the space $L_d^*(l)$. Some readers may be suggested to skip to (A.10).

By construction, the (co-)tangent space of $L_d^*(l)$ at a point x is canonically isomorphic to that of $\tilde{\mathbb{E}}$.

$$T_x(L_d^*(l)) \simeq \mathbf{C} \otimes_{\mathbf{R}} (\tilde{F}/G)^* , \qquad T_x^*(L_d^*(l)) \simeq \mathbf{C} \otimes_{\mathbf{R}} (\tilde{F}/G) .$$

Hence we have cannonical isomorphisms:

$$\mathrm{Der}_{L_d^*(l)} \simeq \mathcal{O}_{L_d^*(l)} \otimes_{\mathbf{R}} (\tilde{F}/G)^* , \qquad \Omega^1_{L_d^*(l)} \simeq \mathcal{O}_{L_d^*(l)} \otimes_{\mathbf{R}} F/G ,$$

where $\tilde{\mathcal{O}}_{L_d^*(l)}$, $\mathrm{Der}_{L_d^*(l)}$, and $\Omega^1_{L_d^*(l)}$ are the sheaves of germs of holomorphic functions, vector fields, and 1-forms on $L_d^*(l)$ respectively.

Consider now the dual of the bilinear form $\tilde{I}: \tilde{F}/G \times \tilde{F}/G \to \mathbf{R}$.

$$\tilde{I}^* : (\tilde{F}/G)^* \times (\tilde{F}/G)^* \to \mathbf{R} ,$$

By extending $\tilde{I}^*$ complex linearly to the tangent spaces, one obtains a non-degenerate $\mathcal{O}$-bilinear form:

$$\text{(A.8.1)} \qquad \tilde{I}^*\colon \mathrm{Der}_{L_d^*(l)} \times \mathrm{Der}_{L_d^*(l)} \to \mathcal{O}_{L_d^*(l)}\,.$$

As a tensor, this is expressed by a form:

$$\text{(A.8.2)} \qquad \omega := dz \otimes dz + \frac{1}{2d\pi\sqrt{-1}}\left(\frac{d\lambda}{\lambda} \otimes d\tau + d\tau \otimes \frac{d\lambda}{\lambda}\right).$$

The $\mathcal{O}$-bilinear form (A.8.1) induces the $\mathcal{O}$-isomorphism:

$$\tilde{I}^* : \mathrm{Der}_{L_d(l)}(-\log X) \simeq \Omega^1_{L_d(l)}(\log X) : \tilde{I}$$

$$\frac{\partial}{\partial\tau} \leftrightarrow d\tilde{\lambda} = \frac{1}{2d\pi\sqrt{-1}}\frac{d\lambda}{\lambda}$$

$$\frac{\partial}{\partial z} \leftrightarrow dz$$

$$2\pi d\sqrt{-1}\,\lambda\frac{\partial}{\partial\lambda} \leftrightarrow d\tau\,.$$

By extending the real bilinear form $\tilde{I}^*$ sesqui-linearly to the tangent spaces, one obtains in the same way a Hermitian form:

$$\text{(A.8.3)} \qquad g := dz \otimes d\bar{z} + \frac{1}{2d\pi\sqrt{-1}}\left(\frac{d\lambda}{\lambda} \otimes d\bar{\tau} - d\tau \otimes \frac{d\bar{\lambda}}{\bar{\lambda}}\right).$$

The g is not positive definite but indefinite of the sign (2, 1). It is a trivial calculation to show that

i) $g = \partial\bar{\partial}(|z|^2 + 2\mathrm{Re}\,(\tilde{\lambda}\bar{\tau})) = \partial\bar{\partial}\left(|z|^2 + \frac{1}{d\pi}\mathrm{Im}\,(\log(\lambda)\bar{\tau})\right)$,

ii) $\mathrm{Ricc}\,(g) := \partial\bar{\partial}\left(\log\left(\det\begin{bmatrix} 0 & 0 & 1/\lambda \\ 0 & 1 & 0 \\ 1/\lambda & 0 & 0 \end{bmatrix}\right)\right) \equiv 0\,.$

The fact that $\tilde{E}(\xi_{stu}) \in O(\tilde{F}, \tilde{I})$ implies the following.

Assertion. *The isomorphism* $\varphi_{stu}\colon L_d^*(l) \to L_d^*(l')$ *of* (A.7) **Assertion** v) *leaves the forms* ω *and* g *invariant.*

(A.9) *Real deformation of* L_d^* *over the torus* T_d.

In this §, we consider a group $\hat{H}_d$ satisfying the extension:

$$\text{(A.9.1)} \qquad 0 \longrightarrow \tilde{H}_d \longrightarrow \hat{H}_d \longrightarrow \frac{1}{d}H_{\mathbb{Z}}^* \longrightarrow 0$$

and its action on $\hat{\mathbb{E}} := \tilde{\mathbb{E}} \times \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, \mathbb{R})$. (cf. *Remark* 2).

First we define $\hat{H}_d$ by a topological argument as follows.

Consider a manifold $\mathscr{H}$, which is a quotient of $\tilde{H} \times T_d$ by the relation: $(x, l) \sim (y, l') \Leftrightarrow l = l'$ and $^{\exists}\xi \in \tilde{H}_d(l)$ s.t. $\tilde{E}(\xi) \circ x = y$. By definition, there is a locally trivial fibration $\mathscr{H} \to T_d$, whose fiber over $l \in T_d$ is the quotient manifold $\tilde{H}_d(l) \backslash \tilde{H}$. Put

$$\hat{H}_d := \pi_1(\mathscr{H}, (0, 0)) .$$

Recalling that $\tilde{H} \simeq \mathbb{R}^3$ and $T_d \simeq \mathrm{Hom}_{\mathbf{Z}}(H_{\mathbf{Z}}, \mathbb{R}) / \frac{1}{d} H^*_{\mathbf{Z}}$ and applying the homotopy exact sequence for the fibration, we obtain (A.9.1).

The action of $\frac{1}{d} H^*_{\mathbf{Z}}$ on $\tilde{H}_d$ induced from the extension (A.9.1) is the same as the monodromy action of the covering (A.4.5), given

$$\text{(A.9.2)} \qquad \frac{1}{d}(m, n) \in \frac{1}{d} H^*_{\mathbf{Z}} \longmapsto (\xi_{p,q,r/d} \mapsto \xi_{p,q,(pm+qn+r)/d}) \in \mathrm{Aut}\,(\tilde{H}_d(0)) .$$

The group $\hat{H}_d$ consists of the symbols:

$$\text{(A.9.3)} \qquad \hat{H}_d = \{\Xi_{p,q,r/d,m/d,n/d} : p, q, r, u, v \in \mathbf{Z}\} ,$$

such that the multiplication rule is given by

$$\text{(A.9.4)} \quad \Xi_{p,q,r/d,m/d,n/d} \circ \Xi_{p',q',r'/d,m'/d,n'/d} = \Xi_{p+p',q+q',(r+r'+pm'+qn')/d+pq',(m+m')/d,(n+n')'d} .$$

As is easily seen from (A.9.4), we have the exact sequences:

$$\text{(A.9.5)} \qquad 0 \longrightarrow \mathbf{Z}\left(\frac{1}{d} a \otimes b\right) \longrightarrow \hat{H}_d \longrightarrow H_{\mathbf{Z}} \times \frac{1}{d} H^*_{\mathbf{Z}} \longrightarrow 0 .$$

$$0 \longrightarrow \mathbf{Z}\left(\frac{1}{d} a \otimes b\right) \times \frac{1}{d} H^*_{\mathbf{Z}} \longrightarrow \hat{H}_d \longrightarrow H_{\mathbf{Z}} \longrightarrow 0 .$$

The action of $\Xi_{p,q,r/d,m/d,n/d} \in \hat{H}_d$ on the space,

$$\text{(A.9.6)} \qquad \hat{\mathbb{E}} := \tilde{\mathbb{E}} \times \mathrm{Hom}_{\mathbf{Z}}(H_{\mathbf{Z}}, \mathbb{R})$$

is defined by

$$\text{(A.9.7)} \qquad \Xi_{p,q,r/d,m/d,n/d}(x, u, v) := \left(E^*(\xi_{p,q,pu+qv+r/d})x,\ u + \frac{1}{d} m, v + \frac{1}{d} n\right) .$$

The action of $\hat{H}_d$ is properly discontinuous and fixed point free so that we define the quotient (real) manifold:

$$\text{(A.9.8)} \qquad \hat{L}^*_d := \hat{\mathbb{E}} / \hat{H}_d .$$

Natural projection $\hat{\mathbf{E}} \to \mathbf{E} \times \mathrm{Hom}_{\mathbf{Z}}(H_{\mathbf{Z}}, \mathbb{R})$ induces a map

$$\text{(A.9.9)} \qquad \hat{L}^*_d \longrightarrow X \times T_d$$

due to the sequence (A.9.5). The map (A.9.9) is a principal $\mathbf{C}^*$-bundle, whose fiber coordinate is given by λ (A.7.6) and the action of $T := T_{p,q,m/d,n/d} \in H_{\mathbf{Z}} \times \frac{1}{d} H^*_{\mathbf{Z}}$ on λ is

$$\text{(A.9.10)} \qquad T^*(\lambda) = e_T \lambda ,$$

where

(A.9.11) $$e_T := \exp(2\pi\sqrt{-1}d(-qz + \tfrac{1}{2}q^2\tau + pu + qv)).$$

By forgetting the first factor in (A.9.9), one obtains a map:

(A.9.12) $$\hat{L}_d^* \longrightarrow T_d.$$

The fiber over l by this map (A.9.12) is the space $L_d^*(l)$ in §7, and the restriction of the $\mathbb{C}^*$-bundle (A.9.9) to this subspace is the $\mathbb{C}^*$-bundle (A.7.2).

Remark 1. The family (A.9.12) is topologically non-trivial, since the monodromy action of $\frac{1}{d}H_{\mathbb{Z}}^* \simeq \pi_1(T_d, 0)$ on the homotopy group of the fiber is given by (A.9.2), which is nonzero.

Remark 2. It was shown that the fibers $L_d^*(l)$ of (A.9.12) are complex analytically isomorphic preserving the forms ω and g (§7 Assertion v) and §8 Assertion). One may ask, which differential geometric structure in the frame bundle of L_d^* is deformed in (9.12).

Remark 3. In a sense, the construction in this Appendix is incomplete. To obtain a complete picture, one should consider an extension of Aut $(H_{\mathbb{Z}})$ by the group $\hat{H}_d$. I.e. instead of (A.9.1), one should consider an extension of the group Aut $(\tilde{H}_d)$ by $\tilde{H}_d$, where

$$0 \longrightarrow \frac{1}{d}H_{\mathbb{Z}}^* \longrightarrow \mathrm{Aut}\,(\tilde{H}_d) \longrightarrow \mathrm{Aut}\,(H_{\mathbb{Z}}) \longrightarrow 0.$$

Since we have fixed the fibration map $L_d^*(l) \to \mathbb{H}$ (cf. (A.6) and (A.7)), we have neglected the term Aut $(H_{\mathbb{Z}})$ in our consideration. Then describe $\tilde{L}_d^*/\Gamma$ for $\Gamma \subset \mathrm{Aut}\,(H_{\mathbb{Z}})$.

(A.10) *The family* $\mathbb{L}_d \to \mathbb{H}$ *of simple elliptic singularities.*

Let L_d be the line bundle over X associated to the principal bundle L_d^*, which is set-theoretically obtained by adding the zero section $\simeq X$ to the $\mathbb{C}^*$-bundle L_d^*.

(A.10.1) $$L_d = L_d^* \cup X.$$

As a consequence of the (A.7) Assertion iii), if $d < 0$ then the line bundle L_d is negative (relative to $\mathbb{H}$), so that the zero section can be blown down to $\mathbb{H}$. Let us denote by $\mathbb{L}_d$ the blow-down space,

(A.10.2) $$\mathbb{L}_d = L_d^* \cup \mathbb{H}.$$

There is a natural projection map $\mathbb{L}_d \to \mathbb{H}$ so that the fiber over a point $\tau \in \mathbb{H}$ is a normal two dimensional affine variety, having an isolated singular point at the zero section $\tau \in \mathbb{H} \subset \mathbb{L}_d$, which is called the simple elliptic singularity of modulous τ.

(A.11) *The invariant ring S.*

For an integer k with $k \geq 0$, put,

(A.11.1) $$S_k := \{\theta(z, \tau)\text{: holomorphic on } \mathbb{E} \text{ s.t. } t^*(\theta) = e_t^{-k}\theta \text{ for } t \in H_{\mathbb{Z}}\},$$

where e_t $(t \in H_{\mathbb{Z}})$ is defined in (A.7.7). (Here we choose $l = 0 \in T_d$.)

S_k is a free module of finite rank over $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$. Put

(A.11.2) $$S := \bigoplus_{k=0}^{\infty} S_k$$

which is an algebra finitely generated over $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$. This is the coordinate ring for the space $\mathbb{L}_d$ (A.10.2). That is,

(A.11.3) $$\mathbb{L}_d \simeq \mathrm{Spec}\,(S)\,.$$

Generators of the algebra S can be explicitly written down by the Weierstrauss p-function, as done in [21]. For instance

$$d = -1 \qquad S = \Gamma(\mathcal{O}_{\mathbf{H}})[\lambda, \lambda^2 p, \lambda^3 p']\,,$$

$$d = -2 \qquad S = \Gamma(\mathcal{O}_{\mathbf{H}})[\lambda, \lambda p, \lambda^2 p']\,,$$

$$d = -3 \qquad S = \Gamma(\mathcal{O}_{\mathbf{H}})[\lambda, \lambda p, \lambda p']\,,$$

$$d = -4 \qquad S = \Gamma(\mathcal{O}_{\mathbf{H}})[\lambda, \lambda p, \lambda p', \lambda p p']\,.$$

Thus for $d = -1, -2, -3$, $\mathbb{L}_d$ is a hypersurface defined by a weighted homogeneous polynomial, and for $d = -4$, $\mathbb{L}_d$ is a complete intersection of two homogeneous polynomials, whose weights and associated exponents are as follows.

d	Name	Weight $(a, b, c : h)$	Exponents
-1	$\tilde{E}_8$	$(1, 2, 3; 6)$	$0, 1, 2, 2, 3, 3, 4, 4, 5, 6.$
-2	$\tilde{E}_7$	$(1, 1, 2; 4)$	$0, 1, 1, 2, 2, 2, 3, 3, 4.$
-3	$\tilde{E}_6$	$(1, 1, 1; 3)$	$0, 1, 1, 1, 2, 2, 2, 3.$
-4	$\tilde{D}_5$	$(1, 1, 1, 1; 2, 2)$	$0, 1, 1, 1, 1, 2, 2.$
-5	$\tilde{A}_4$	$(**; 1, 1, 1, 1, 1)$	$0, 1, 1, 1, 1, 1.$

(The case $\tilde{A}_4$ is included from [13], for the sake of completeness.)

As is mentioned in the introduction, the period mappings for these singularities identify the universal unfolding spaces for these singularities with the spaces $\mathrm{Spec}\,(S^W)$ for the extended affine root systems of types $E_8^{(1,1)}$, $E_7^{(1,1)}$, $E_6^{(1,1)}$ and $D_5^{(1,1)}$.

The details will appear elsewhere.

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