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Extended coherent states and path integrals with auxiliary variables

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Abstract. The usual construction of coherent states allows a wider interpretation in which the number of distinguishing state labels is no longer minimal; the label measure determining the required resolution of unity is then no longer unique and may even be concentrated on manifolds with positive co-dimension. Paying particular attention to the residual restrictions on the measure, we choose to capitalize on this inherent freedom and in formally distinct ways, systematically construct suitable sets of extended coherent states which, in a minimal sense, are characterized by auxiliary labels. Interestingly, we find these states lead to path integral constructions containing auxiliary (essentially unconstrained) path-space variables. The impact of both standard and extended coherent state formulations on the content of classical theories is briefly examined, the latter showing the existence of new, and generally constrained, classical variables. Some implications for the handling of constrained classical systems are given, with a complete analysis awaiting further study.

1. Introduction

Traditionally, coherent-state path integrals have been constructed using a minimum number of quantum operators and classical variables. But many problems in classical physics are often described in terms of more variables than a minimal formulation would entail. In particular, we have in mind constraint variables and gauge degrees of freedom. Although such additional variables are frequently convenient in a classical description, they invariably become a nuisance in quantization. In this paper we introduce auxiliary quantum operators in order to create extended coherent states bearing additional labels, and incorporate them into new path integrals. We note especially the subsequent emergence of classical constraints, and are able to comment on their elimination as part of some general quantization scheme.

Of course, there exist in the literature several widely accepted methods for path integral quantization subject to constraints [1–3], especially for systems with purely gauge degrees of freedom [4]. In addition, there seems to be in the physics community a certain measure of ambivalence towards genuine (i.e. second class—in the language of Dirac) constraints, ranging from not treating them, to eliminating them completely in principle, to turning them into first class constraints and then proceeding by one of the methods referred to above. From the most general point of view these various

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methods all have one feature in common which is less than satisfactory, namely they too liberally allow rather unrestricted canonical tranformations within formal phase-space path integrals, where the validity of their use often cannot be substantiated [5].

In this paper we imagine beginning with the purely quantum description of a physical system, from which we extract a coherent-state path integral for the propagator as a step *en route* to the classical description. By this procedure we can assess the characteristics of a classical theory deduced in this way and the role of auxiliary variables, both in modifying the realization of the path integral without changing its value, and then in allowing an extension of the coherent-state description of the entire quantum system. Finally, we are able to investigate the impact of extended coherent states at the purely classical level, and the subsequent appearance of constraints.

Section 2 contains a formal treatment of coherent states for auxiliary variables. To provide a concrete demonstration of the features emerging from this exposition, section 3 presents a number of simple examples in which we construct a variety of extended coherent states from canonical and affine coherent states for one degree of freedom. Armed with these examples, and having identified an appropriate expression to play the role of a 'classical' action, we then consider, in section 4, the properties of classical theories stemming from specific quantum Hamiltonians. We examine when and how auxiliary variables lead to classical constraints, as well as the nature of other effects they have at the classical level.

Readers who, by reason of interest or inclination, would prefer to avoid a somewhat formal treatment of extended coherent states, could skip ahead to section 3 without serious loss on first reading.

2. General case of extended coherent states

Although much of the following lies scattered or implied throughout a very wide literature, for the convenience of the reader we gather and present it here in a way unified with our approach. This also enables us to incorporate it into a single cohesive development of extended coherent states, and path integrals with auxiliary variables, as new objects of study.

2.1. Background

A quantum action principle can be derived as follows. (We normally set $\hbar = 1$ although occassionally we also discuss explicit dependence on $\hbar$, including the limit where $\hbar \to 0$.) First write

$$I_{qm} = \int \left[\mathrm{i}\langle\psi|\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle - \langle\psi|\mathcal{H}|\psi\rangle \right] \mathrm{d}t \tag{2.1}$$

in which $|\psi\rangle$ represents the quantum state of a system as a vector in an abstract Hilbert space, and $\mathcal{H}$ is the quantum Hamiltonian operator, which generates the time evolution of the quantum state by its action in the Hilbert space. Variation of this quantum action leads to the familiar Schrödinger equation. For concrete purposes it is necessary to introduce some specific representation of the abstract Hilbert space; e.g. the Schrödinger (coordinate) representation or, as we employ throughout this paper, a coherent-state representation. We now pass to a description of coherent states from a general viewpoint [6].

2.2. General coherent-state review

Coherent states are a set of non-zero vectors $\{|\ell\rangle : \ell = (\ell^1, \ell^2, \ldots, \ell^L) \in \boldsymbol{L}\}$ that are continuously labelled, so that $\langle\ell|\ell'\rangle$ is jointly continuous, and which admit a resolution of unity in the form

$$\boldsymbol{I} = \int |\ell\rangle\langle\ell|\delta\ell \tag{2.2}$$

integrated over $\boldsymbol{L}$, where $\delta\ell$ denotes a positive measure. Usually, the labels ℓ are chosen to be a minimal set for the problem at hand. With $\langle\ell|\ell\rangle > 0$ for all ℓ we can, without loss of generality, rescale $\delta\ell$ so that $\langle\ell|\ell\rangle = 1$ for all ℓ. Although not mandatory, it is often especially convenient to choose $|\ell\rangle = U(\ell)\,|\eta\rangle$, where the operators $U(\ell)$ form a (not necessarily irreducible) unitary representation of some continuous group, and where $|\eta\rangle$ is a fixed, normalized fiducial vector. In the group case, the measure $\delta\ell$ is typically a suitably normalized left-invariant group measure. Even when the $U(\ell)$ do not form a group it is useful to embed them, at least conceptually, in the group they would generate. For convenience we suppose that the ℓ variables are real. A great variety of different kinds of coherent states have been defined over the years, and we refer the reader to the literature for many examples [6–8].

Every set of coherent states leads to a representation of abstract Hilbert space by bounded, continuous functions $\psi(\ell) \equiv \langle\ell|\psi\rangle$ in which the inner product reads

$$\langle\varphi|\psi\rangle = \int \varphi^*(\ell)\psi(\ell)\delta\ell. \tag{2.3}$$

Each such function satisfies the integral equation

$$\psi(\ell') = \int \mathcal{K}(\ell';\ell)\,\psi(\ell)\delta\ell \tag{2.4}$$

where $\mathcal{K}(\ell';\ell) = \langle\ell'|\ell\rangle$ denotes the reproducing kernel. In turn, $\mathcal{K}(\ell';\ell)^* = \mathcal{K}(\ell;\ell')$ and

$$\mathcal{K}(\ell'';\ell') = \int \mathcal{K}(\ell'';\ell)\,\mathcal{K}(\ell;\ell')\,\delta\ell \tag{2.5}$$

demonstrating that $\mathcal{K}$ is the representation of a projection operator on $L^2(\delta\ell)$.

In the context of coherent states, operators, such as $\mathcal{H}$, generally act through integral kernels defined by $\langle\ell'|\mathcal{H}|\ell\rangle$. However, two 'symbols' associated with $\mathcal{H}$ are important to identify. One, the upper symbol, on which we shall concentrate mainly for convenience, is defined by

$$H(\ell) = \langle\ell|\mathcal{H}|\ell\rangle \tag{2.6}$$

the other, the lower symbol, is defined through the equation

$$\mathcal{H} = \int h(\ell)|\ell\rangle\langle\ell|\delta\ell \tag{2.7}$$

and is of some advantage particularly in alternative constructions of the path integral. The upper symbol, H, exists (modulo domain issues) quite generally and is unique; the lower symbol, h, may exist and be unique, may exist but not be unique, or may not exist at all, depending on the operator in question and the specific set of coherent states [6]. If both symbols exist they are related by the integral equation

$$H(\ell') = \int h(\ell)|\langle\ell|\ell'\rangle|^2\delta\ell. \tag{2.8}$$

As we shall see, each of them might be regarded as a candidate for the classical Hamiltonian.

Two alternative formulations of coherent-state path integrals are standard [6]. In the first definition, repeated application of the resolution of unity is used to re-express the integral kernel

$$\begin{aligned}\langle\ell''|\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\ell'\rangle &= \langle\ell''|\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{H}}\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{H}}\cdots\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{H}}|\ell'\rangle \\ &= \int\cdots\int\prod_{n=0}^{N}\langle\ell_{n+1}|\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{H}}|\ell_n\rangle\prod_{n=1}^{N}\delta\ell_n\end{aligned} \tag{2.9}$$

where $\ell' = \ell_0$ and $\ell'' = \ell_{N+1}$, and $\varepsilon = T/(N+1)$. For small ε

$$\langle\ell_{n+1}|\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{H}}|\ell_n\rangle \simeq \langle\ell_{n+1}|(\mathbb{I} - \mathrm{i}\varepsilon\mathcal{H})|\ell_n\rangle = \langle\ell_{n+1}|\ell_n\rangle[1 - \mathrm{i}\varepsilon\langle\ell_{n+1}|\mathcal{H}|\ell_n\rangle/\langle\ell_{n+1}|\ell_n\rangle] \tag{2.10}$$

leading to

$$\langle\ell''|\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\ell'\rangle = \lim_{N\to\infty}\int\cdots\int\prod_{n=0}^{N}\langle\ell_{n+1}|\ell_n\rangle[1 - \mathrm{i}\varepsilon\langle\ell_{n+1}|\mathcal{H}|\ell_n\rangle/\langle\ell_{n+1}|\ell_n\rangle]\prod_{n=1}^{N}\delta\ell_n. \tag{2.11}$$

It is customary to interchange the order of the limit and the integration in a formal way, and to write for the integral the form it assumes for continuous and differentiable paths. Since

$$\begin{aligned}\prod\langle\ell_{n+1}|\ell_n\rangle &= \prod[1 - \langle\ell_{n+1}|(|\ell_{n+1}\rangle - |\ell_n\rangle)] \\ &\simeq \mathrm{e}^{-\Sigma\langle\ell_{n+1}|(|\ell_{n+1}\rangle - |\ell_n\rangle)}\end{aligned} \tag{2.12}$$

it follows that the propagator admits the formal path integral representation

$$\langle\ell''|\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\ell'\rangle = \int\exp\left\{\mathrm{i}\int\left[\mathrm{i}\langle\ell|\frac{\mathrm{d}}{\mathrm{d}t}|\ell\rangle - \langle\ell|\mathcal{H}|\ell\rangle\right]\mathrm{d}t\right\}\mathcal{D}\ell \tag{2.13}$$

an expression that involves the upper symbol $H(\ell) = \langle\ell|\mathcal{H}|\ell\rangle$. Here and elsewhere, formal expressions such as that represented by this path integral do not admit a rigorous mathematical formulation, but they may be expected to inherit a well defined meaning in terms of a lattice limit such as that used here in this construction.

The second coherent-state path integral makes use of the lower symbol associated with the Hamiltonian, i.e. $h(\ell)$, which, for the present, we assume exists. It follows that

$$I - \mathrm{i}\varepsilon\mathcal{H} = \int [1 - \mathrm{i}\varepsilon h(\ell)]|\ell\rangle\langle\ell|\delta\ell \tag{2.14}$$

and therefore that

$$\mathrm{e}^{-\mathrm{i}\varepsilon\mathcal{H}} = \int \mathrm{e}^{-\mathrm{i}\varepsilon h(\ell)}|\ell\rangle\langle\ell|\delta\ell + \mathrm{O}(\varepsilon^2). \tag{2.15}$$

Repeated use of this expression leads to

$$\langle\ell''|\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\ell'\rangle = \lim_{N\to\infty} \int \cdots \int \prod_{n=0}^{N} \langle\ell_{n+1}|\ell_n\rangle \prod_{n=1}^{N} \mathrm{e}^{-\mathrm{i}\varepsilon h(\ell_n)}\delta\ell_n \tag{2.16}$$

which is a perfectly acceptable prescription. When the integrations and the limit are interchanged, we arrive at an alternative formal path integral expression for the propagator

$$\langle\ell''|\mathrm{e}^{-\mathrm{i}T\mathcal{H}}|\ell'\rangle = \int \exp\left\{\mathrm{i}\int \left[\mathrm{i}\langle\ell|\frac{\mathrm{d}}{\mathrm{d}t}|\ell\rangle - h(\ell)\right] \mathrm{d}t\right\}\mathcal{D}\ell \tag{2.17}$$

which now involves the lower symbol.

Generally $H(\ell) \neq h(\ell)$. To make firm contact with classical physics it is desirable to have

$$\lim_{\hbar\to 0} H(\ell) = \lim_{\hbar\to 0} h(\ell) \equiv H_c(\ell) \tag{2.18}$$

in which the limit is what normally becomes identified as *the* classical Hamiltonian. When appropriate, the coincidence of these limits can be achieved by choosing $|\ell\rangle$ (and especially $|\eta\rangle$) so that, roughly speaking,

$$\lim_{\hbar\to 0} |\langle\ell|\ell'\rangle|^2 \propto \delta(\ell - \ell'). \tag{2.19}$$

(When such a limit is inappropriate because H_c vanishes, e.g. as in the case of a spin s system, $s < \infty$, then we might choose either H or h to act as the classical Hamiltonian. That case is not of particular interest in this paper.)

In one of the above formulations, it will be seen that the expression

$$\int \left[\mathrm{i}\langle\ell|\frac{\mathrm{d}}{\mathrm{d}t}|\ell\rangle - \langle\ell|\mathcal{H}|\ell\rangle\right] \mathrm{d}t \tag{2.20}$$

enters in the same way that the classical action does in the standard representation of the path integral. However it is important to realize that here $\hbar$ has *not* been set to zero. Thus, we will wish to use the adjective 'classical' in two distinct senses, for which the meaning will generally be clear from the context. In particular, $H(\ell) = \langle\ell|\mathcal{H}|\ell\rangle$, where $\hbar \neq 0$, will frequently be referred to as the classical Hamiltonian, as will $H_c(\ell) = \lim_{\hbar\to 0} H(\ell)$, which we will generally try to distinguish as the strictly classical Hamiltonian. A similar, potentially ambiguous, use of classical will apply to action functionals as well.

2.3. Case of extended coherent states

Extended coherent states are coherent states for which the measure associated with the resolution of unity is not unique, including examples where the measure is concentrated on manifolds of co-dimension one or higher. Thus, in principle, all their properties and consequences may be obtained from the foregoing simply by treating cases where $\delta\ell$ has the proper non-uniqueness and support. However, we prefer to use a less implicit scheme and instead will assume that extended coherent states, denoted by vectors such as $|\ell;\lambda\rangle$, where $\ell \in L$, $\lambda \in \Lambda$, have the property that

$$\mathbb{I} = \int |\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell \tag{2.21}$$

holds for all λ. In consequence, we will have

$$\mathbb{I} = \int |\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell\,\mathrm{d}\sigma(\lambda) = \int |\ell;\lambda\rangle\langle\ell;\lambda|\,\mathrm{d}\mu(\ell;\lambda) \tag{2.22}$$

for any σ such that $\int \mathrm{d}\sigma(\lambda) = 1$. There is no restriction on the λ dependence of the states $|\ell;\lambda\rangle$ in this construction, although our primary interest lies in cases where each value of $(\ell;\lambda)$ labels a distinct ray. The measure $\mathrm{d}\mu(\ell;\lambda)$ is not unique because of the freedom in $\mathrm{d}\sigma(\lambda)$.

For the sake of definiteness we now consider in detail two generally distinct prescriptions to define extended coherent states.

2.4. Outer extended coherent states

As a starting point, we assume that

$$\mathbb{I} = \int |\ell\rangle\langle\ell|\delta\ell \tag{2.23}$$

in which, for purposes of exposition, $\delta\ell$ is now definitely taken to be unique, and we introduce a family of unitary operators $V(\lambda)$, where $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^\Lambda) \in \Lambda$, so that

$$\mathbb{I} = V(\lambda)\mathbb{I}V^\dagger(\lambda) = \int |\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell \tag{2.24}$$

holds for any λ, where $|\ell;\lambda\rangle \equiv V(\lambda)|\ell\rangle$. Supposing that $|\ell\rangle = U(\ell)|\eta\rangle$ as in the previous consideration, we now call the states $V(\lambda)U(\ell)|\eta\rangle$ *outer* extended coherent states. Consequently,

$$\mathbb{I} = \int |\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell\,\mathrm{d}\sigma(\lambda) = \int |\ell;\lambda\rangle\langle\ell;\lambda|\,\mathrm{d}\mu(\ell;\lambda) \tag{2.25}$$

for any σ such that $\int \mathrm{d}\sigma(\lambda) = 1$. In addition we introduce the upper symbol by

$$H(\ell;\lambda) = \langle\ell;\lambda|\mathcal{H}|\ell;\lambda\rangle. \tag{2.26}$$

For the lower symbol we first define

$$V^\dagger(\lambda)\mathcal{H}V(\lambda) = \int h(\ell;\lambda)|\ell\rangle\langle\ell|\delta\ell \tag{2.27}$$

from which it follows that

$$\mathcal{H} = \int h(\ell;\lambda)|\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell. \tag{2.28}$$

These two symbols are then related by

$$H(\ell';\lambda') = \int h(\ell;\lambda)|\langle\ell;\lambda|\ell';\lambda'\rangle|^2\delta\ell \tag{2.29}$$

or when $\lambda' = \lambda$, by

$$H(\ell';\lambda) = \int h(\ell;\lambda)|\langle\ell|\ell'\rangle|^2\delta\ell \tag{2.30}$$

just as before.

Equipped with the quantities defined above, it follows that we can then write

$$\langle\ell'';\lambda''|\mathrm{e}^{-\mathrm{i}\mathcal{H}T}|\ell';\lambda'\rangle = \int \exp\left\{\mathrm{i}\int\left[\mathrm{i}\langle\ell;\lambda|\frac{\mathrm{d}}{\mathrm{d}t}|\ell;\lambda\rangle - H(\ell;\lambda)\right]\mathrm{d}t\right\}\mathcal{D}\mu(\ell;\lambda) \tag{2.31}$$

where it must be emphasized that the formal measure $\mathcal{D}\mu(\ell;\lambda)$ *may depend on time* in an essentially free manner, so long as the paths are pinned to the desired boundary conditions $\ell(0);\lambda(0) = \ell';\lambda'$ and $\ell(T);\lambda(T) = \ell'';\lambda''$. A similar formal path integral expression exists, just as before, with $h(\ell;\lambda)$ replacing $H(\ell;\lambda)$.

When it does not vanish, we may adopt

$$I_{\mathrm{c}} = \lim_{\hbar\to 0}\int\left[\mathrm{i}\langle\ell;\lambda|\frac{\mathrm{d}}{\mathrm{d}t}|\ell;\lambda\rangle - \langle\ell;\lambda|\mathcal{H}|\ell;\lambda\rangle\right]\mathrm{d}t \tag{2.32}$$

as the classical action appropriate to the system under consideration.

2.5. *Inner extended coherent states*

We next proceed to introduce a potentially different form of extended coherent states which, as will become evident, have virtues of their own. Let $\{U(\ell)\}$ denote a set of unitary operators that have the property that

$$\boldsymbol{I} = \int U(\ell)|\xi\rangle\langle\xi|U^{\dagger}(\ell)\delta\ell \tag{2.33}$$

holds for a wide class of unit vectors $|\xi\rangle$. Next assume that the vectors

$$|\lambda\rangle = V(\lambda)|\eta\rangle \tag{2.34}$$

for a fixed unit vector $|\eta\rangle$ and a set of unitary transformations $\{V(\lambda)\}$ are all vectors of the form $|\xi\rangle$ for which equation (2.33) holds. In that case we find that

$$\boldsymbol{I} = \int |\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell \tag{2.35}$$

holds for all λ, where, adopting the same notation, we now set

$$|\ell;\lambda\rangle = U(\ell)V(\lambda)|\eta\rangle. \tag{2.36}$$

We call the vectors in equation (2.36) *inner* extended coherent states (V is inside) in contrast to the vectors appearing in equation (2.25), the *outer* extended coherent states (V is outside). Of course, even these two cases are by no means exhaustive.

Whichever the prescription, inner or outer, the same expansion of the resolution of unity applies:

$$\mathbb{I} = \int |\ell;\lambda\rangle\langle\ell;\lambda|\delta\ell\,\mathrm{d}\sigma(\lambda) = \int |\ell;\lambda\rangle\langle\ell;\lambda|\,\mathrm{d}\mu(\ell;\lambda) \tag{2.37}$$

where $\int \mathrm{d}\sigma(\lambda) = 1$; and in either case, equation (2.26) gives the upper symbol while equation (2.31) corresponds to the representation of the formal extended coherent-state path integral.

What is the advantage of inner compared with outer extended coherent states? If $U(\ell)V(\lambda)$ make an $L+\Lambda$ parameter continuous group there is no real difference between the inner and outer states, only a coordinate change. A difference is apparent, however, when the operators $U(\ell)V(\lambda)$ taken together do *not* form a group.

For the rest of this section we focus on inner extended coherent states. Consider the case where $\{V(\lambda)\}$ form an Abelian set of transformations. Then it is straightforward to see that the kinematical one-form has the structure

$$\mathrm{i}\langle\ell;\lambda|\mathrm{d}|\ell;\lambda\rangle = y_a(\ell;\lambda)\mathrm{d}\ell^a + \mathrm{d}\zeta(\lambda) \tag{2.38}$$

(summation convention implied) where [9]

$$y_a(\ell;\lambda) = \mathrm{i}\langle\eta|V^\dagger(\lambda)U^\dagger(\ell)\frac{\partial}{\partial\ell^a}U(\ell)V(\lambda)|\eta\rangle. \tag{2.39}$$

No matter how many λs, the maximum number of independent ys is determined by the number, L, of labels occurring in $U(\ell)$. Examples in section 4 will show a case where this maximum is realized, as well as cases where it is not. It is clear that whenever $\Lambda > L$ there will effectively be fewer variables represented in the one-form with components denoted by equation (2.39) than occur in the extended coherent states, and this will eventuate in there being constraints among the classical equations of motion. Of course, if not all the ys are independent as functions of the λs (or equivalently, if some are constant), there will be additional constraints.

Up to surface terms the classical action now reads

$$I = \int [y_a(\ell;\lambda)\dot{\ell}^a - H(\ell;\lambda)]\,\mathrm{d}t. \tag{2.40}$$

For convenience let us assume that the operators $\{U(\ell)\}$ generate a group, and let $\{X_b\}$ denote generators of that group. Furthermore, let us assume that the quantum Hamiltonian $\mathcal{H} = \mathcal{H}(X_1, X_2, \ldots, X_L) \equiv \mathcal{H}(X_b)$. In that case

$$\begin{aligned} H(\ell;\lambda) &= \langle\ell;\lambda|\mathcal{H}(X_b)|\ell;\lambda\rangle = \langle\eta|\mathcal{H}(A_b^c(\ell)X_c(\lambda))|\eta\rangle \\ &= \langle\eta|\mathcal{H}(A_b^c(\ell)\sigma_c(\lambda)\mathbb{I} + A_b^c(\ell)\delta X_c(\lambda))|\eta\rangle \end{aligned} \tag{2.41}$$

in which the $A_b^c(\ell)$ are determined entirely by the group generated by the $U(\ell)$, and where $X_c(\lambda) = V^\dagger(\lambda)X_cV(\lambda)$ and $\sigma_c(\lambda) = \langle\eta|X_c(\lambda)|\eta\rangle$. The quantum corrections are all contained in $\delta X_c(\lambda) \equiv X_c(\lambda) - \sigma_c(\lambda)\mathbb{I}$, but the auxiliary variables also appear tangled up in the classical Hamiltonian in a quite complicated way.

3. Basic examples of extended coherent states

3.1. Canonical coherent states

We start with conventional canonical coherent states based on an irreducible, self-adjoint representation of a Heisenberg pair of variables Q and P that satisfy $[Q, P] = i\mathbb{1}$. If $|\eta\rangle$ denotes an arbitrary, normalized fiducial vector, then

$$|p, q\rangle \equiv e^{-iqP} e^{ipQ} |\eta\rangle \tag{3.1}$$

defined for all $(p, q) \in \mathbb{R}^2$, constitute the set of canonical coherent states. For any $|\eta\rangle$ these states satisfy

$$\mathbb{1} = \int |p, q\rangle\langle p, q| \, dp \, dq/2\pi \tag{3.2}$$

yielding a resolution of unity in terms of an equally weighted integral of the one-dimensional projection operators these states make. If $\mathcal{H} = \mathcal{H}(P, Q)$ denotes the self-adjoint Hamiltonian for some quantum mechanical system then, following section 2, the propagator expressed in a canonical coherent-state representation admits a formal path integral expression in the form

$$\langle p'', q''|e^{-i\mathcal{H}T}|p', q'\rangle = \mathcal{M} \int \exp\left\{i \int \left[i\langle p, q|\frac{d}{dt}|p, q\rangle - \langle p, q|\mathcal{H}|p, q\rangle\right] dt\right\} \mathcal{D}p\mathcal{D}q \tag{3.3}$$

where $p(0), q(0) = p', q'$ and $p(T), q(T) = p'', q''$. This prescription certainly suggests the interpretation of

$$I = \int \left[i\langle p, q|\frac{d}{dt}|p, q\rangle - \langle p, q|\mathcal{H}|p, q\rangle\right] dt \tag{3.4}$$

as a classical action for this system. Indeed, if $\langle\eta|P|\eta\rangle = 0 = \langle\eta|Q|\eta\rangle$, then

$$I = \int [p\dot{q} - H(p, q)] \, dt \tag{3.5}$$

where

$$H(p, q) \equiv \langle p, q|\mathcal{H}(P, Q)|p, q\rangle = \langle\eta|\mathcal{H}(P + p\mathbb{1}, Q + q\mathbb{1})|\eta\rangle. \tag{3.6}$$

Whenever the dispersions $\langle\eta|P^2|\eta\rangle$ and $\langle\eta|Q^2|\eta\rangle$, and the higher moments, vanish as $\hbar \to 0$, it then follows that

$$H(p, q) = H_c(p, q) + O(\hbar; p, q) \tag{3.7}$$

showing, as previously indicated, that apart from $\hbar$-dependent corrections $H(p, q) = H_c(p, q)$ the strictly classical Hamiltonian for theories with $\hbar = 0$. We often encounter situations for which both dispersion terms depend on $\hbar$ in the same way (to leading order). These will be of particular interest to us in what follows.

3.2. Dilation extended canonical coherent states

For our first example of extension, let $D = (QP + PQ)/2$ denote the self-adjoint dilation generator that satisfies $[Q, D] = \mathrm{i}Q$ and $[P, D] = -\mathrm{i}P$. Together with $[Q, P] = \mathrm{i}\boldsymbol{I}$, it follows that P, Q and D make a three-parameter Lie algebra in the same sense that P and Q make a two-parameter Lie algebra (namely, as factor representations of groups). Define

$$V(r) = \mathrm{e}^{\mathrm{i}(\ln r)D} \qquad r > 0 \tag{3.8}$$

and observe that

$$\begin{aligned} \boldsymbol{I} = V(r)\boldsymbol{I}V^{\dagger}(r) &= \int V(r)|p, q\rangle\langle p, q|V^{\dagger}(r)\,\mathrm{d}p\,\mathrm{d}q/2\pi \\ &= \int |p, q; r\rangle\langle p, q; r|\,\mathrm{d}p\,\mathrm{d}q/2\pi \end{aligned} \tag{3.9}$$

holds for any r, where

$$|p, q; r\rangle \equiv V(r)|p, q\rangle \tag{3.10}$$

denotes an (outer) extended coherent state. If σ is a measure on r such that $\int \mathrm{d}\sigma(r) = 1$, then it also follows that

$$\boldsymbol{I} = \int |p, q; r\rangle\langle p, q; r|\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}\sigma(r)/2\pi. \tag{3.11}$$

Among such possible measures we mention $\mathrm{d}\sigma(r) = \delta(r-\xi)\,\mathrm{d}r$, even with ξ different in every time slice which arises in the construction of a path integral.

This particular example serves to illustrate an additional feature of certain types of extended coherent states: the same set of extended coherent states may be obtained by the extension of several inequivalent sets of coherent states. For suppose we consider the set of states

$$|p, r\rangle \equiv \mathrm{e}^{\mathrm{i}pQ}\mathrm{e}^{\mathrm{i}\ln|r|D}\Pi(r)|\eta\rangle \tag{3.12}$$

defined for all $p \in \boldsymbol{R}$ and $r \in \boldsymbol{R}^* \equiv \boldsymbol{R}\backslash\{0\}$. Here

$$\Pi(r) \equiv \theta(r)\boldsymbol{I} + \theta(-r)\Pi \tag{3.13}$$

where Π denotes the *parity operator*. Then provided $c_\eta \equiv \langle\eta|\,|Q|^{-1}|\eta\rangle < \infty$, these states also lead to a resolution of unity in the form (see appendix)

$$\boldsymbol{I} = \int |p, r\rangle\langle p, r|r^{-2}\,\mathrm{d}p\,\mathrm{d}r/2\pi c_\eta \tag{3.14}$$

which is one form of the well known resolutions of unity associated with the *affine coherent states* [10, 11]. In analogy with the previous discussion we observe that

$$\boldsymbol{I} = \int |p, r; q\rangle\langle p, r; q|r^{-2}\,\mathrm{d}p\,\mathrm{d}r\,\mathrm{d}\sigma(q)/2\pi c_\eta \tag{3.15}$$

where $\int d\sigma(q) = 1$ and where

$$|p, r; q\rangle \equiv e^{-iqP}|p, r\rangle \tag{3.16}$$

denotes an element of an 'alternative' set of extended coherent states. As a matter of fact the $|p, r; q\rangle$ states are closely related to the $|p, q; r\rangle$ states thanks to the group structure involved; specifically

$$|p, r; q\rangle = |rp, r^{-1}q; |r|\rangle \tag{3.17}$$

a relation that holds for all $r \in \boldsymbol{R}^*$.

Additionally, we may introduce the states

$$|q, r\rangle \equiv e^{-iqP}e^{i\ln|r|D}\Pi(r)|\eta\rangle \tag{3.18}$$

defined for all $q \in \boldsymbol{R}$ and $r \in \boldsymbol{R}^*$, for which (see appendix)

$$\boldsymbol{I} = \int |q, r\rangle\langle q, r| \, dr \, dq/2\pi\bar{c}_\eta \tag{3.19}$$

where $\bar{c}_\eta = \langle\eta||P|^{-1}|\eta\rangle < \infty$ restricts the choice of $|\eta\rangle$. As before

$$\boldsymbol{I} = \int |q, r; p\rangle\langle q, r; p| \, dr \, dq \, d\sigma(p)/2\pi\bar{c}_\eta \tag{3.20}$$

provided $\int d\sigma(p) = 1$, where

$$|q, r; p\rangle \equiv e^{ipQ}|q, r\rangle. \tag{3.21}$$

These states are phase related to the others by

$$|q, r; p\rangle = e^{ipq}|p, r; q\rangle. \tag{3.22}$$

3.3. Generalization

Now, from the more general point of view adopted in the first part of section 2, we may equally well consider the extended coherent states

$$|p, q, r\rangle \equiv |p, r; q\rangle \tag{3.23}$$

as a set of coherent states in their own right (where $(p, q, r) \in \boldsymbol{R}^2 \times \boldsymbol{R}^*$) for which a resolution of unity exists in the form

$$\boldsymbol{I} = \int |p, q, r\rangle\langle p, q, r| \, d\mu(p, q, r) \tag{3.24}$$

for any of an infinite class of suitable measures μ [12]. This measure may even just pick out certain *sections* of the coherent states, signifying that only two of the variables are required, the remaining one being auxiliary. Based on our previous discussion three such sectional measures are given by

$$\begin{aligned} \mathrm{d}\mu_r(p,q,r) &= \delta(r-\xi)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r/2\pi \\ \mathrm{d}\mu_q(p,q,r) &= \delta(q-\xi)r^{-2}\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r/2\pi c_\eta \\ \mathrm{d}\mu_p(p,q,r) &= \delta(p-\xi)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r/2\pi \bar{c}_\eta \end{aligned} \tag{3.25}$$

clearly demonstrating the non-uniqueness, and the support on a manifold of co-dimension one, as discussed in section 2. Alternatively, μ may be a *non-sectional* measure, such as

$$\mathrm{d}\mu(p,q,r) = f(r)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r/2\pi \tag{3.26}$$

where $f(r) \geqslant 0$ and $\int f(r)\,\mathrm{d}r = 1$; for example, $f(r) = \chi(r)$, where $\chi(r) = 1$ if $2 < r < 3$ and $\chi = 0$ otherwise. Observe that a restriction or weighting of some sort *must* apply to the measure $\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r$ or $\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r/r^2$, since a totally unrestricted integration over $\boldsymbol{R}^2 \times \boldsymbol{R}^*$ necessarily leads to a divergence.

Let us choose μ as any of the suitable measures appropriate to the coherent states $\{|p,q,r\rangle\}$. In that case $\psi(p,q,r) = \langle p,q,r|\psi\rangle$ leads to a (μ-independent) representation of Hilbert space in the usual way. Additionally, a formal path integral expression for the propagator exists in the form (see section 2)

$$\begin{aligned} &\langle p'',q'',r''|\mathrm{e}^{-\mathrm{i}\mathcal{H}T}|p',q',r'\rangle \\ &\quad = \int \exp\left\{\mathrm{i}\int\left[\mathrm{i}\langle p,q,r|\frac{\mathrm{d}}{\mathrm{d}t}|p,q,r\rangle\right.\right. \\ &\qquad \left.\left. -\langle p,q,r|\mathcal{H}|p,q,r\rangle\right]\mathrm{d}t\right\}\mathcal{D}\mu(p,q,r) \end{aligned} \tag{3.27}$$

where, we emphasize, the choice of $\mu(p,q,r)$ *can depend on time*. To illustrate just one case covered by this general formulation we observe that

$$\begin{aligned} &\langle p'',q'',r''|\mathrm{e}^{-\mathrm{i}\mathcal{H}T}|p',q',r'\rangle \\ &\quad = \mathcal{M}\int \exp\left\{\mathrm{i}\int\left[\mathrm{i}\langle p,q,r|\frac{\mathrm{d}}{\mathrm{d}t}|p,q,r\rangle - \langle p,q,r|\mathcal{H}|p,q,r\rangle\right]\mathrm{d}t\right\} \\ &\qquad \times \delta\{r(\cdot)-\xi(\cdot)\}\mathcal{D}p\mathcal{D}q\mathcal{D}r \end{aligned} \tag{3.28}$$

where $\xi(t)$ is *any* smooth path that connects $\xi(0) = r'$ and $\xi(T) = r''$. Thus, here is a formal path integral representation for a propagator where $r(t)$ can assume any smooth path one chooses consistent with the boundary data, yet the propagator in no way depends on the chosen path. Alternatively, similar δ-measures on q or p could equally be chosen.

As we have seen on several occasions now, from a path integral point of view the expression

$$I = \int\left[\mathrm{i}\langle p,q,r|\frac{\mathrm{d}}{\mathrm{d}t}|p,q,r\rangle - \langle p,q,r|\mathcal{H}|p,q,r\rangle\right]\mathrm{d}t \tag{3.29}$$

assumes the role of the classical action, and it is clearly different from the expression considered previously in equation (3.4), where, in effect, $r(t) \equiv 1$. In this new expression for the action there is no trace of which variables are 'fundamental' and which are 'auxiliary'; this division—artificial but nevertheless necessary—is embodied in the choice of μ and arises only in the path integral quantization as we have described it.

3.4. Additional extensions of canonical coherent states

The example we have just introduced is of particular interest since elimination of any one of the three generators P, Q or D will still lead to a closed subalgebra. In the next set of examples we consider, this will not be the case. For each further example we will specify the (inner) unitary operator chosen for the extension, and from it obtain the resultant path integral action. And we will see that, in general, it is judicious to introduce some new classical variables, shifted from the original canonical label values by amounts which depend on $\hbar$. In what follows the basic generators are P and Q supplemented by those arising from the inner extension.

To deal collectively with the various particular cases we are about to consider, it will be helpful to introduce some additional convenient notation. Thus, let

$$V(v) = \exp(\mathrm{i}v^1 Y_1)\exp(\mathrm{i}v^2 Y_2)\cdots\exp(\mathrm{i}v^k Y_k)\cdots\exp(\mathrm{i}v^\Lambda Y_\Lambda) \tag{3.30}$$

where the Y_k have been included as further generators of the extended Lie algebra. We will consider only the case where all Y_k commute among themselves. Then with $\langle\cdot\rangle \equiv \langle\eta|\cdot|\eta\rangle$ we define

$$\begin{aligned}
&P(v) = V^\dagger(v) P V(v) \qquad Q(v) = V^\dagger(v) Q V(v) \\
&\delta P(v) = P - \mathbb{I}\langle P(v)\rangle \qquad \delta Q(v) = Q - \mathbb{I}\langle Q(v)\rangle \\
&\overline{p} = p + \langle P(v)\rangle \qquad \overline{q} = q + \langle Q(v)\rangle .
\end{aligned} \tag{3.31}$$

Expressed in these variables the path integral action becomes

$$I = \int \left[\overline{p}\dot{\overline{q}} - (p\langle Q(v)\rangle)^{\cdot} - \langle P(v)\rangle\langle Q(v)\rangle^{\cdot} - \sum_{k=1}^{\Lambda}\langle Y_k\rangle\dot{v}^k - H(\overline{p},\overline{q},v)\right]\mathrm{d}t \tag{3.32}$$

in which $^{\cdot}$ after a quantity represents the time derivative of that preceding quantity, and where

$$H(\overline{p},\overline{q},v) = \langle\eta|\mathcal{H}(\overline{p}\mathbb{I} + \delta P(v), \overline{q}\mathbb{I} + \delta Q(v))|\eta\rangle \tag{3.33}$$

which depends on the labels $\{v^k\}$ through the implicit dependence of $\delta P(v), \delta Q(v)$.

We now choose two distinct classes of extended coherent states for further consideration: (i) $Y_k = Q^{k+1}$ and (ii) $Y_k = D^k$; and we select two particular examples for examination in the next section. For $V_Q = \mathrm{e}^{\mathrm{i}\beta Q^2}\mathrm{e}^{\mathrm{i}\gamma Q^3}$ (i.e. $\beta = v_1$, $\gamma = v_2$) we find

$$Q(\beta,\gamma) = Q \qquad P(\beta,\gamma) = P + 2\beta Q + 3\gamma Q^2$$

with a straightforward generalization. It follows that $\langle Q(\beta,\gamma)\rangle^{\cdot} = 0$ in this case, which will lead to constraints in the classical theory, even when both β and γ are considered time dependent. Alternatively, for $V_D = \mathrm{e}^{\mathrm{i}\ln|r|D}\mathrm{e}^{\mathrm{i}uD^2}\Pi(r)$, we have

$$Q(r,u) = r^{-1}\mathrm{e}^{\mathrm{i}u}Q\mathrm{e}^{-2uD} \qquad P(r,u) = r\mathrm{e}^{\mathrm{i}u}P\mathrm{e}^{2uD} \tag{3.34}$$

with generalization being not so straightforward. It is clear that, for r and u independent, $\langle P(r,u)\rangle\langle Q(r,u)\rangle^{\cdot}$ is not a total derivative (unless u or r is some fixed constant). Thus there will be no classical constraints here in general.

As a final remark in this section we note first that, quite generally, there will be no constraints when the rank of the totally antisymmetric matrix M_{ab} given by

$$M_{ab} = \frac{1}{2}\left(\frac{\partial\langle P(v)\rangle}{\partial v^a}\frac{\partial\langle Q(v)\rangle}{\partial v^b} - \frac{\partial\langle Q(v)\rangle}{\partial v^a}\frac{\partial\langle P(v)\rangle}{\partial v^b}\right) \tag{3.35}$$

is equal to its dimension, Λ. In examples based on canonical coherent states for one degree of freedom, this is never possible unless $\Lambda = 2$. Our example with V_D is one such occurrence from which constraints are absent, while for V_Q all additional classical equations will be constraints.

The motivation behind the development in section 2 was that auxiliary variables can lead to a more complete description of a quantum state than is afforded by the standard coherent states constructed with a minimal label set. The introduction of extended coherent states also leaves its imprint on the extended version of the classical equations of motion which they generate. Just such equations and their interpretation are the subject of the following section.

4. Quantum to classical limit

Simple examples are enough to demonstrate several of the various situations which arise. We shall use the free particle, the harmonic oscillator and the quartic potential, each of which yields non-trivial results. The following list relates individual terms in $\mathcal{H}$ to their corresponding contributions in H (D is added to this list for later reference):

$$\begin{aligned}
&P \to \overline{p}\\
&Q \to \overline{q}\\
&D \to \overline{p}\,\overline{q} + \langle D(v)\rangle - \langle P(v)\rangle\langle Q(v)\rangle\\
&P^2 \to \overline{p}^2 + \langle \delta P(v)^2\rangle\\
&Q^2 \to \overline{q}^2 + \langle \delta Q(v)^2\rangle\\
&Q^4 \to \overline{q}^4 + 6\overline{q}^2\langle \delta Q(v)^2\rangle + 4\overline{q}\langle \delta Q(v)^3\rangle + \langle \delta Q(v)^4\rangle .
\end{aligned} \tag{4.1}$$

It is worth realizing to begin with that even for canonical coherent states quantum corrections to the path integral action already arise. Thus, for $2\mathcal{H} = P^2 + Q^2$, we find

$$2H = \overline{p}^2 + \overline{q}^2 + \langle(\Delta P)^2\rangle + \langle(\Delta Q)^2\rangle \tag{4.2}$$

where $\Delta Q = Q - \langle Q\rangle$ and $\Delta P = P - \langle P\rangle$. In the case of the extended coherent states generated by P, Q and D, the classical Hamiltonian satisfies

$$2H = \overline{p}^2 + \overline{q}^2 + r^2\langle(\Delta P)^2\rangle + r^{-2}\langle(\Delta Q)^2\rangle \tag{4.3}$$

and since r enters into the first-order (path integral derived) Lagrangian density only through H and as part of a total derivative, it is clear that stationary variation of the action with respect to r will lead to a constraint,

$$\frac{\partial H}{\partial r} = r\langle(\Delta P)^2\rangle - r^{-3}\langle(\Delta Q)^2\rangle = 0. \tag{4.4}$$

Now, although the quantum corrections are proportional to $\hbar$ (since we assume this property of the two dispersion terms) and, in fact, the whole constraint is thus an equation of nominal order $\hbar$, nevertheless its solution for r from

$$r^4 = \langle(\Delta Q)^2\rangle / \langle(\Delta P)^2\rangle \tag{4.5}$$

will be some number effectively independent of $\hbar$, which will exist and equally could be required to hold even in the limit $\hbar \to 0$. Had we considered just the free particle, it would have been impossible to impose the constraint without going to the classical limit so that the dispersion could vanish (recall $r \neq 0$ from equation (3.8)).

With extended coherent states generated by P, Q and Q^2, we find $\langle V^\dagger(\beta) P V(\beta)\rangle = \langle P\rangle + 2\beta\langle Q\rangle$, so that $\delta P(\beta)$ becomes $\Delta P + 2\beta\Delta Q$ while δQ remains simply ΔQ. Then the harmonic oscillator Hamiltonian becomes

$$2H = \overline{p}^2 + \overline{q}^2 + \langle(\Delta P)^2\rangle + \langle(\Delta Q)^2\rangle + 2\beta\langle\Delta P\Delta Q + \Delta Q\Delta P\rangle + 4\beta^2\langle(\Delta Q)^2\rangle \tag{4.6}$$

and there similarly arises the constraint (of order $\hbar$)

$$\frac{\partial H}{\partial\beta} = \langle\Delta P\Delta Q + \Delta Q\Delta P\rangle + 4\beta\langle(\Delta Q)^2\rangle = 0 \tag{4.7}$$

with its classical solution (which, again, will remain well defined, even for $\hbar \to 0$)

$$\beta = -\frac{\langle\Delta P\Delta Q + \Delta Q\Delta P\rangle}{4\langle(\Delta Q)^2\rangle}. \tag{4.8}$$

For both of these examples, and in a number of specific respects, the $\hbar \to 0$ limit has an effect somewhat resembling that of the $m \to 0$ limit in classical electromagnetism. Specifically, we have an analogue of the decoupling of the longitudinal degree of freedom but, of course, we are without an analogue of the covariant conservation of sources.

As explained in the previous section, the addition of a Q^3 generator leads to an additional constraint. With this extension, $\delta P(\beta,\gamma)$ becomes $\Delta P + 2\beta\Delta Q + 3\beta\Delta Q^2$, in which $\Delta Q^2 = Q^2 - \langle Q^2\rangle$, and the constraint equations which follow give

$$\begin{aligned}\frac{\partial H}{\partial\beta} &= \langle\Delta P\Delta Q + \Delta Q\Delta P\rangle + 4\beta\langle(\Delta Q)^2\rangle + 6\gamma\langle\Delta Q\Delta Q^2\rangle = 0\\ \frac{2}{3}\frac{\partial H}{\partial\gamma} &= \langle\Delta P\Delta Q^2 + \Delta Q^2\Delta P\rangle + 4\beta\langle\Delta Q\Delta Q^2\rangle + 6\gamma\langle(\Delta Q^2)^2\rangle = 0\end{aligned} \tag{4.9}$$

which generally will have non-trivial solutions. A solution in which γ is also independent of $\hbar$ to leading order will coincide with a contribution in the next-to-leading order dependence of β on $\hbar$. We note that in all these cases considered so far, we could satisfy the constraints within the path integral by a suitable choice of the measure, μ, in the resolution of unity, simply because the constraints for r, or β, or β and γ were constants.

In the last example we shall consider related to the harmonic oscillator, we use the full extension given by V_D. It will be convenient to introduce

$$P(u) = \mathrm{e}^{-\mathrm{i}uD^2} P \mathrm{e}^{\mathrm{i}uD^2} \qquad \text{and} \qquad Q(u) = \mathrm{e}^{-\mathrm{i}uD^2} Q \mathrm{e}^{\mathrm{i}uD^2} \tag{4.10}$$

and to define a new quantity:

$$\bar{\delta} D(u) \equiv D - \langle P(u)\rangle\langle Q(u)\rangle \mathbb{I} \tag{4.11}$$

while $\delta P(u)$ and $\delta Q(u)$ are defined analogously to equation (3.31). The r, u degrees of freedom again decouple from the $\overline{p}$, $\overline{q}$ degrees of freedom, and the equations of motion they satisfy are now dynamical:

$$\frac{\partial\langle\bar{\delta} D(u)\rangle}{\partial u}\frac{\dot{r}}{r} = \frac{\partial H}{\partial u} \qquad \text{and} \qquad -\frac{\partial\langle\bar{\delta} D(u)\rangle}{\partial u}\frac{\dot{u}}{r} = \frac{\partial H}{\partial r}. \tag{4.12}$$

We thus have two new classical variables with no constraints.

The final example we consider is obtained from the quantum Hamiltonian operator

$$2\mathcal{H} = P^2 + Q^4 \tag{4.13}$$

and we use the P, Q and D extended coherent states, for which the classical Hamiltonian becomes

$$2H = \overline{p}^2 + \overline{q}^4 + r^2\langle\Delta P^2\rangle + 6\overline{q}^2 r^{-2}\langle\Delta Q^2\rangle + 4\overline{q} r^{-3}\langle\Delta Q^3\rangle + r^{-4}\langle\Delta Q^4\rangle. \tag{4.14}$$

In this case there is again a constraint, $\partial H/\partial r = 0$, with solution $r = r(\overline{q})$, where now the auxiliary and the original (shifted) canonical variables are completely intercoupled. Elimination of this constraint will not change the classical dynamics, but it will clearly break contact with the original quantum theory, because there is no reason to suppose that any measure exists which can preserve the resolution of unity (heavily used in the construction of the path integral, hence the classical action), while being compatible with the constraint. This is perhaps a very simple, yet striking, example of how elimination of classical constraints can affect the transition back to a quantum theory.

5. Discussion and conclusions

The formalism surrounding the general theory of coherent states is exceptionally rich. Not only does it provide a bridge between the quantum action principle of equation (2.1) and a classical action principle such as equation (3.5) that arises for canonical classical variables, but, by allowing for suitably supported measures in the resolution of unity, equation (2.2), it also encompasses a description of extended coherent states containing auxiliary variables which are accompanied by new path integral representations of the propagator, for example equation (2.31). These same path integrals identify classical actions with additional, and often constrained, degrees of freedom. Thus, the choice of coherent-state extension can strongly affect what becomes *the* classical theory. One thing which has become evident in our examination of specific examples in sections 3 and 4, and which we would like to stress, is that the relation between the resultant classical theories and the initial quantum theories is far from being trivially transparent. This is especially so when the classical equations contain new dynamical degrees of freedom, but is still the case when constrained variables couple to the genuinely dynamical variables. In particular, only when constraint variables completely decouple does there generally exist a measure for which elimination of the constraint will remain compatible with a coherent state resolution of unity.

In some elementary examples, which we did not discuss above at all, such as

$$\mathcal{H} = aP + bQ + cD \tag{5.1}$$

with extended coherent states generated by P, Q and D, the classical action does not contain *any* reference to the auxiliary variable (up to a total time derivative). This behaviour is reminiscent of a *gauge* variable, which should normally be accommodated by maps for which $(\ell;\lambda)$ is many-to-one onto rays rather than one-to-one as we have exclusively used in section 3. In the few examples we have considered in section 4 it is clear that the constraints were always second class. We have not yet shown a general method of relating the existence of constraints of a particular class to the presence of identifiable structures in the representative of the quantum theory. However, it is certainly clear that the process of eliminating the constraints by hand at the classical level may be actually further separating the resultant classical theory from its quantum parent. Although elimination of constraints is a widespread practice, it does not appear to present itself as a way of narrowing the gap between classical and quantum physics.

In this paper we have incorporated a coherent-state representation of quantum systems into a wider description based on extended coherent states. In this context we have examined the impact of auxiliary variables both on the path integral representation of the quantum propagator and on the subsequent equations of motion for purely classical variables. Constraints have emerged as a frequent but non-mandatory outcome of the introduction of auxiliary variables.

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Appendix

Here we give additional facts regarding the group(s) discussed in section 3 as well as demonstrate the relevant resolutions of unity needed there.

Choose P, Q and $D = (QP + PQ)/2$ as self-adjoint operators, and introduce Π, the parity operator, which satisfies $\Pi^2 = \mathbb{I}$, $\Pi^\dagger = \Pi$, and $\Pi Q \Pi = -Q$, $\Pi P \Pi = -P$. We then set

$$U(p,q,r) = \mathrm{e}^{-\mathrm{i}qP}\mathrm{e}^{\mathrm{i}pQ}\mathrm{e}^{\mathrm{i}\ln|r|D}\Pi(r)$$

where $(p,q) \in \mathbb{R}^2$ and $r \in \mathbb{R}\backslash\{0\}$, and with $\theta(r) = (1 + r/|r|)/Z$,

$$\Pi(r) = \theta(r)\mathbb{I} + \theta(-r)\Pi .$$

It follows that U is a unitary operator for all p, q and r, and that the combination law reads

$$U(p,q,r)U(p',q',r') = \mathrm{e}^{\mathrm{i}prq'}U(p + r^{-1}p', q + rq', rr').$$

Subgroups of interest are those defined by

$$U_r(p,q) = U(p,q,1) \qquad U_p(q,r) = U(0,q,r) \qquad U_q(p,r) = U(p,0,r)$$

which satisfy combination laws that follow directly from the combination law for the unrestricted U operators. We note further that the operators $\{U_r(p,q)\}$ are irreducible by assumption, and, in addition, the operators $\{U_p(q,r)\}$ and $\{U_q(p,r)\}$ are also irreducible. (If we had omitted the reflections by restricting r to be positive, $r > 0$, then neither $\{U_p\}$ nor $\{U_q\}$ would have been irreducible. Although this case could be treated satisfactorily we omit it from our discussion.)

We next take up the question of the resolution of unity expressed in terms of projection operators onto the extended coherent states. As in section 3 we define

$$|p,q,r\rangle = \mathrm{e}^{-\mathrm{i}qP}\mathrm{e}^{\mathrm{i}pQ}\mathrm{e}^{\mathrm{i}\ln|r|D}\Pi(r)|\eta\rangle$$

which assumes the form

$$\langle x|p,q,r\rangle = \sqrt{|r|}\mathrm{e}^{\mathrm{i}p(x-q)}\eta(r(x-q))$$

when expressed in the x-representation ($Q|x\rangle = x|x\rangle$) where $\int|\eta(x)|^2\,\mathrm{d}x = 1$.

First, we examine

$$\begin{aligned}\int\langle x|p,q,r\rangle\langle p,q,r|y\rangle\,\mathrm{d}p\,\mathrm{d}q &= |r|\int \mathrm{e}^{\mathrm{i}p(x-y)}\eta(r(x-q))\eta^*(r(y-q))\,\mathrm{d}p\,\mathrm{d}q\\ &= 2\pi\delta(x-y)|r|\int|\eta(r(x-q))|^2\,\mathrm{d}q\\ &= 2\pi\delta(x-y)\end{aligned}$$

establishing the stated resolution of unity, equation (3.9), for any $r(r\neq 0)$.

Next, we examine

$$\begin{aligned}\int\langle x|p,q,r\rangle\langle p,q,r|y\rangle\,\mathrm{d}p\,\mathrm{d}r/r^2 &= \int \mathrm{e}^{\mathrm{i}p(x-y)}\eta(r(x-q))\eta^*(r(y-q))\,\mathrm{d}p\,\mathrm{d}r/|r|\\ &= 2\pi\delta(x-y)\int|\eta(r(x-q))|^2\,\mathrm{d}r/|r|\\ &= 2\pi\delta(x-y)\int|\eta(r)|^2\,\mathrm{d}r/|r|\end{aligned}$$

establishing the appropriate resolution of unity, equation (3.15).

Finally, we examine

$$\begin{aligned}&\int\langle x|p,q,r\rangle\langle p,q,r|y\rangle\,\mathrm{d}q\,\mathrm{d}r\\ &\quad= \int|r|\mathrm{e}^{\mathrm{i}p(x-y)}\eta(r(x-q))\eta^*(r(y-q))\,\mathrm{d}q\,\mathrm{d}r\\ &\quad= (2\pi)^{-1}\int|r|\mathrm{e}^{\mathrm{i}p(x-y)}\,\mathrm{d}q\,\mathrm{d}r\int \mathrm{e}^{-\mathrm{i}ur(x-q)+\mathrm{i}vr(y-q)}\tilde{\eta}(u)\tilde{\eta}^*(v)\,\mathrm{d}u\,\mathrm{d}v\\ &\quad= \int|r|\mathrm{e}^{\mathrm{i}p(x-y)+\mathrm{i}r(vy-ux)}\delta(vr-ur)\tilde{\eta}(u)\tilde{\eta}^*(v)\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\\ &\quad= \int \mathrm{e}^{\mathrm{i}p(x-y)-\mathrm{i}ru(x-y)}|\tilde{\eta}(u)|^2\,\mathrm{d}r\,\mathrm{d}u\\ &\quad= 2\pi\delta(x-y)\int|\tilde{\eta}(u)|^2\,\mathrm{d}u/|u|\end{aligned}$$

establishing the required resolution of unity, equation (3.20).

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