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Extended elliptic-type integrals with associated properties and Turán-type inequalities

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Abstract

Our aim is to study and investigate the family of (p, q) -extended (*incomplete and complete*) elliptic-type integrals for which the usual properties and representations of various known results of the (*classical*) elliptic integrals are extended in a simple manner. This family of elliptic-type integrals involves a number of special cases and has a connection with (p, q) -extended Gauss' hypergeometric function and (p, q) -extended Appell's double hypergeometric function F_1 . Turán-type inequalities including log-convexity properties are proved for these (p, q) -extended complete elliptic-type integrals. Further, we establish various Mellin transform formulas and obtain certain infinite series representations containing Laguerre polynomials. We also obtain some relationship between these (p, q) -extended elliptic-type integrals and Meijer G -function of two variables. Moreover, we obtain several connections with (p, q) -extended beta function as special values and deduce numerous differential and integral formulas. In conclusion, we introduce (p, q) -extension of the Epstein–Hubbell (E-H) elliptic-type integral.

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1 Introduction, preliminaries, and motivation

Elliptic-type integrals such as (*classical*) Legendre elliptic integrals, generalized complete elliptic integrals of the first and second kind (see, for example, [5, 29, 30, 33–35]), and symmetric elliptic integrals [7] and also several definite integrals of such families are known to play a prominent role in special functions in terms of their modulus or complementary modulus in the theory of conformal mappings [2], studies of crystallographic minimal surfaces, radiation physics problems [3], nuclear technology, fracture mechanics studies of elliptical crack problems, the study of electromagnetic or acoustic waves being scattered by an elliptic disk [15], astronomy, geometry, physics, and engineering mechanics [4]. In recent years (2009), Lin *et al.* [16, p. 1178, Eq. (1.12)] introduced and extensively

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investigated the following generalized family of elliptic integrals (*incomplete*) $H(\psi, k, \gamma)$:

$$H(\psi, k, \gamma) := \int_0^\psi (1 - k^2 \sin^2 \theta)^{\gamma - \frac{1}{2}} d\theta = \int_0^{\sin \psi} \frac{(1 - k^2 t^2)^{\gamma - \frac{1}{2}}}{\sqrt{1 - t^2}} dt, \tag{1.1}$$

$$\left(|k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2}; \gamma \in \mathbb{C} \right).$$

Obviously, their special cases are

$$H\left(\frac{\pi}{2}, k, \gamma\right) := H(k, \gamma),$$

$$H(\psi, k, 0) =: F(\psi, k), \quad \text{and} \quad H(\psi, k, 1) =: E(\psi, k),$$

and

$$H\left(\frac{\pi}{2}, k, 0\right) =: K(k) \quad \text{and} \quad H\left(\frac{\pi}{2}, k, 1\right) =: E(k),$$

and they respectively yield the families of *complete* elliptic integrals $H(k, \gamma)$ due to the Bushell [5, p. 2, Eq. (2.2)]

$$H(k, \gamma) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \theta)^{\gamma - \frac{1}{2}} d\theta = \int_0^1 \frac{(1 - k^2 t^2)^{\gamma - \frac{1}{2}}}{\sqrt{1 - t^2}} dt \tag{1.2}$$

$$\left(|k^2| < 1; \gamma \in \mathbb{C} \right)$$

and well-known (canonical) Legendre *incomplete* elliptic integrals $F(\psi, k)$ and $E(\psi, k)$ and *complete* elliptic integrals $K(k)$ and $E(k)$ of the first and second kind (in terms of modulus $|k|$ and amplitude ψ) [6]:

$$F(\psi, k) = \int_0^\psi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^{\sin \psi} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \tag{1.3}$$

$$\left(|k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2} \right),$$

$$E(\psi, k) = \int_0^\psi \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^{\sin \psi} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt \tag{1.4}$$

$$\left(|k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2} \right),$$

and

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \quad (|k^2| < 1), \tag{1.5}$$

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} dt \quad (|k^2| < 1). \tag{1.6}$$

In recent years, extensions of a number of well-known special functions have been investigated and studied the (p, q) -variant, and in turn, when $p = q$ the p -variant together

with the set of related higher transcendental hypergeometric type special functions (see, for details, [8–10, 17, 18, 20, 22–24]). In what follows we shall use the following recently defined (p, q) -extensions of the *classical* beta function $B(x, y)$ and *classical* Gauss’s hypergeometric function $F(\lambda, \mu; \nu; \mathfrak{Z})$ [11, p. 360, Eq. (1.14)]:

$$B_{p,q}(\delta, \sigma) = B(\delta, \sigma; p, q) = \int_0^1 t^{\delta-1} (1-t)^{\sigma-1} e^{-\frac{p}{t} - \frac{q}{1-t}} dt \tag{1.7}$$

$$(\min\{\Re(p), \Re(q); \min\{\Re(\delta), \Re(\sigma)\} > 0\} \geq 0),$$

and [11, p. 371, Eq. (7.1)]

$$F_{p,q}(\lambda, \mu; \nu; \mathfrak{Z}) = \sum_{n \geq 0} (\lambda)_n \frac{B(\mu + n, \nu - \mu; p, q)}{B(\mu, \nu - \mu)} \frac{\mathfrak{Z}^n}{n!} \tag{1.8}$$

$$(p, q \geq 0; \Re(\nu) > \Re(\mu) > 0; |\mathfrak{Z}| < 1).$$

Also, we shall need a (p, q) -extension of Appell’s (first) hypergeometric function F_1 [21, Eq. (1.9)]:

$$F_1(\lambda, \mu, \mu'; \nu; \mathfrak{X}, \mathfrak{Y}; p, q) = \sum_{m, n \geq 0} (\mu)_m (\mu')_n \frac{B(\lambda + m + n, \nu - \lambda; p, q)}{B(\lambda, \nu - \lambda)} \frac{\mathfrak{X}^m \mathfrak{Y}^n}{m! n!} \tag{1.9}$$

$$(\min\{\Re(p), \Re(q); \max\{|\mathfrak{X}|, |\mathfrak{Y}|\} < 1\} \geq 0).$$

The goal of this paper is to introduce and investigate the family of (p, q) -extended (*incomplete*) elliptic-type integrals and (*complete*) elliptic-type integrals, which are analogous on the basis of definition (1.7) of the (p, q) -extended beta function $B(\delta, \sigma; p, q)$ so that many of the known properties of the elliptic-type integrals carry over naturally. In Sect. 2, we introduce a family of (p, q) -extended elliptic-type integrals. The (p, q) -extension proposed in this section provides connections with (p, q) -extended Gauss’ hypergeometric function $F_{p,q}$ and (p, q) -extended Appell’s double hypergeometric function F_1 . In Sect. 3, Turán-type inequalities including the log-convexity property are proved for these (p, q) -extended (*complete*) elliptic-type integrals. Furthermore, in Sect. 4, Mellin transform formulas and some infinite series representations containing the Laguerre polynomials are derived for these (p, q) -extended elliptic-type integrals. Moreover, in Sect. 5, we provide certain connections with the (p, q) -extended beta function and Meijer’s G -function of two variables as new representations for the parameter (special) values and differential and integral properties of the (p, q) -extended (*complete*) elliptic-type integrals. Finally, in conclusion Sect. 6, we introduce a (p, q) -extension of the Epstein–Hubbell (E-H) elliptic-type integral.

2 A family of (p, q) -extended elliptic-type integrals

We begin by introducing a family of (p, q) -extended (*incomplete*) elliptic-type integrals and (*complete*) elliptic-type integrals (in terms of modulus $|k|$ with amplitude ψ), given by (1.1)

$$\begin{aligned} \mathcal{H}_{p,q}(\psi, k, \gamma) &:= \int_0^\psi (1 - k^2 \sin^2 \theta)^{\gamma - \frac{1}{2}} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta \\ &= \int_0^{\sin \psi} \frac{(1 - k^2 t^2)^{\gamma - \frac{1}{2}}}{\sqrt{1 - t^2}} e^{-\frac{p}{t^2} - \frac{q}{1 - t^2}} dt \end{aligned} \tag{2.1}$$

$$\left(\min\{\Re(p), \Re(q)\} > 0; |k^2| < 1 \text{ with } 0 \leq \psi \leq \frac{\pi}{2} \text{ when } p = q = 0 \right).$$

It is clear that from (2.1), in the particular cases $\gamma = 0$ and $\gamma = 1$, respectively, we have

$$\begin{aligned} F_{p,q}(\psi, k) &:= \int_0^\psi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta \\ &= \int_0^{\sin \psi} \frac{1}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} e^{-\frac{p}{t^2} - \frac{q}{1 - t^2}} dt \end{aligned} \tag{2.2}$$

$$\left(\min\{\Re(p), \Re(q)\} > 0; |k^2| < 1 \text{ with } 0 \leq \psi \leq \frac{\pi}{2} \text{ when } p = q = 0; \right)$$

and

$$\begin{aligned} E_{p,q}(\psi, k) &:= \int_0^\psi \sqrt{1 - k^2 \sin^2 \theta} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta \\ &= \int_0^{\sin \psi} \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} e^{-\frac{p}{t^2} - \frac{q}{1 - t^2}} dt \end{aligned} \tag{2.3}$$

$$\left(\min\{\Re(p), \Re(q)\} > 0; |k^2| < 1 \text{ with } 0 \leq \psi \leq \frac{\pi}{2} \text{ when } p = q = 0 \right).$$

Furthermore, when $\psi = \frac{\pi}{2}$, equations (2.1), (2.2), and (2.3) reduce to the corresponding (p, q) -extended (*complete*) elliptic-type integrals given respectively by

$$\begin{aligned} \mathcal{H}_{p,q}(k, \gamma) &:= \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \theta)^{\gamma - \frac{1}{2}} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta \\ &= \int_0^1 \frac{(1 - k^2 t^2)^{\gamma - \frac{1}{2}}}{\sqrt{1 - t^2}} e^{-\frac{p}{t^2} - \frac{q}{1 - t^2}} dt, \end{aligned} \tag{2.4}$$

$$\begin{aligned} K_{p,q}(k) &:= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta \\ &= \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} e^{-\frac{p}{t^2} - \frac{q}{1 - t^2}} dt \end{aligned} \tag{2.5}$$

$$\left(\min\{\Re(p), \Re(q)\} > 0; |k^2| < 1 \text{ when } p = q = 0 \right),$$

and

$$\begin{aligned}
 E_{p,q}(k) &:= \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta \\
 &= \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} e^{-\frac{p}{t^2} - \frac{q}{1-t^2}} dt
 \end{aligned} \tag{2.6}$$

$$(\min\{\Re(p), \Re(q)\} > 0; |k^2| < 1 \text{ when } p = q = 0).$$

In terms of the complementary modulus k' , the (p, q) -extended(*complete*)elliptic-type integrals are defined by

$$K'_{p,q}(k) = K_{p,q}(k') = K_{p,q}(\sqrt{1 - k^2}) \quad (k' := \sqrt{1 - k^2}) \tag{2.7}$$

and

$$E'_{p,q}(k) = E_{p,q}(k') = E_{p,q}(\sqrt{1 - k^2}) \quad (k' := \sqrt{1 - k^2}). \tag{2.8}$$

The particular cases of (2.2) to (2.8) when $p = q$ and $p = q = 0$ reduce to the generalized elliptic-type integrals studied by Srivastava *et al.* [33] and classical elliptic integrals (*incomplete and complete*) (1.1) to (1.6), respectively (see, e.g., [5, 6], and [12]).

2.1 (p, q) -Hypergeometric(extended) function representations

In this section, we express the above defined (p, q) -extensions of complete elliptic-type integrals in terms of (p, q) -extended hypergeometric functions.

Theorem 2.1 Consider (p, q) -Gauss's hypergeometric function $F_{p,q}(\lambda, \mu; \nu; \mathfrak{z})$ given by (1.8). Let $\Re(p), \Re(q) > 0$, and $|k^2| < 1$. Then

$$\mathcal{H}_{p,q}(k, \gamma) = \frac{\pi}{2} F_{p,q}\left(\frac{1}{2} - \gamma, \frac{1}{2}; 1; k^2\right), \tag{2.9}$$

$$K_{p,q}(k) = \frac{\pi}{2} F_{p,q}\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{2.10}$$

and

$$E_{p,q}(k) = \frac{\pi}{2} F_{p,q}\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \tag{2.11}$$

Proof Letting $t^2 = u$ in (2.4), (2.5), and (2.6) and using definition (1.8), we get the required representation (2.9), (2.10), and (2.11), respectively. \square

Theorem 2.2 Consider (p, q) -Appell's hypergeometric function $F_1(\lambda, \mu, \mu'; \nu; \mathfrak{X}, \mathfrak{Y}; p, q)$ defined as in (1.9). For $\Re(p), \Re(q) > 0$ and $|k^2| < 1$, we have

$$\mathcal{H}_{p,q}(k, \gamma) = \frac{\pi}{2} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} - \gamma; \frac{3}{2}; 1, k^2; p, q\right), \tag{2.12}$$

$$K_{p,q}(k) = \frac{\pi}{2} F_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1, k^2; p, q\right), \tag{2.13}$$

$$E_{p,q}(k) = \frac{\pi}{2} F_1\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}; \frac{3}{2}; 1, k^2; p, q\right). \tag{2.14}$$

Proof Setting $t^2 = u$ in equations (2.4), (2.5), and (2.6) and using the integral representation defined by Parmar and Pogány [21, Lemma 8, Eqs. (2.1)]

$$F_1(\lambda, \mu, \mu'; \nu; x, y; p, q) = \int_0^1 \frac{t^{\lambda-1}(1-t)^{\nu-\lambda-1}}{B(\lambda, \nu-\lambda)} (1-xt)^{-\mu} (1-yt)^{-\mu'} e^{-\frac{p}{t}-\frac{q}{1-t}} dt, \tag{2.15}$$

where $\max\{|\arg(1-\mathfrak{X})|, |\arg(1-\mathfrak{Y})|\} < \pi; \Re(\nu) > \Re(\lambda) > 0$ and $\min\{\Re(p), \Re(q)\} > 0$, we get the required relations (2.12), (2.13), and (2.14), respectively. \square

3 Log-convexity properties and Turán-type inequalities

In this section, we establish the Turán-type inequalities based upon log-convexity properties for the $\mathcal{H}_{p,q}(k, \gamma)$, $K_{p,q}(k)$, and $E_{p,q}(k)$ in (2.4), (2.5), and (2.6).

Theorem 3.1 *The following assertions are true for $\Re(p) > 0, \Re(q) > 0$:*

- (1) *The function $\gamma \mapsto \mathcal{H}_{p,q}(k, \gamma)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*
- (2) *The function $p \mapsto \mathcal{H}_{p,q}(k, \gamma)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*
- (3) *The function $q \mapsto \mathcal{H}_{p,q}(k, \gamma)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*

Moreover, for the same parametric range, the following Turán inequalities hold true:

$$\mathcal{H}_{p,q}^2(k, \gamma) - \mathcal{H}_{p,q}(k, \gamma - 1) \cdot \mathcal{H}_{p,q}(k, \gamma + 1) \leq 0 \quad \gamma \in (1, \infty), \tag{3.1}$$

$$\mathcal{H}_{p,q}^2(k, \gamma) - \mathcal{H}_{p-1,q}(k, \gamma) \cdot \mathcal{H}_{p+1,q}(k, \gamma) \leq 0 \quad p \in (1, \infty), \tag{3.2}$$

$$\mathcal{H}_{p,q}^2(k, \gamma) - \mathcal{H}_{p,q-1}(k, \gamma) \cdot \mathcal{H}_{p,q+1}(k, \gamma) \leq 0 \quad q \in (1, \infty). \tag{3.3}$$

Proof By using the definition of the classical Hölder–Rogers inequality for integrals in the integral representation (2.4), we have

$$\begin{aligned} &\mathcal{H}_{p,q}(k, \lambda\gamma_1 + (1-\lambda)\gamma_2) \\ &= \int_0^1 \frac{(1-k^2t^2)^{\lambda\gamma_1+(1-\lambda)\gamma_2-\frac{1}{2}}}{\sqrt{1-t^2}} e^{-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \\ &= \int_0^1 \left[\frac{(1-k^2t^2)^{\lambda\gamma_1-\frac{1}{2}}}{\sqrt{1-t^2}} e^{-\frac{p}{t^2}-\frac{q}{1-t^2}} \right]^\lambda \left[\frac{(1-k^2t^2)^{\gamma_2-\frac{1}{2}}}{\sqrt{1-t^2}} e^{-\frac{p}{t^2}-\frac{q}{1-t^2}} \right]^{1-\lambda} dt \\ &\leq \left[\int_0^1 \frac{(1-k^2t^2)^{\lambda\gamma_1-\frac{1}{2}}}{\sqrt{1-t^2}} e^{-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \right]^\lambda \left[\int_0^1 \frac{(1-k^2t^2)^{\gamma_2-\frac{1}{2}}}{\sqrt{1-t^2}} e^{-\frac{p}{t^2}-\frac{q}{1-t^2}} dt \right]^{1-\lambda}. \end{aligned}$$

This is equivalent to

$$\mathcal{H}_{p,q}(k, \lambda\gamma_1 + (1-\lambda)\gamma_2) \leq [\mathcal{H}_{p,q}(k, \gamma_1)]^\lambda [\mathcal{H}_{p,q}(k, \gamma_2)]^{1-\lambda} \tag{3.4}$$

$$\forall \gamma_1, \gamma_2, k \in (0, 1) \text{ and } \lambda \in [0, 1],$$

which proves the first assertion.

In a similar manner, by using (2.4) and using the Hölder–Rogers inequality, we get

$$\mathcal{H}_{\lambda p_1+(1-\lambda)p_2,q}(k, \gamma) \leq [\mathcal{H}_{p_1,q}(k, \gamma)]^\lambda [\mathcal{H}_{p_2,q}(k, \gamma)]^{1-\lambda} \tag{3.5}$$

$$\forall p_1, p_2 > 0, k \in (0, 1) \text{ and } \lambda \in [0, 1],$$

$$\mathcal{H}_{p,\lambda q_1+(1-\lambda)q_2}(k, \gamma) \leq [\mathcal{H}_{p,q_1}(k, \gamma)]^\lambda [\mathcal{H}_{p,q_2}(k, \gamma)]^{1-\lambda} \tag{3.6}$$

$\forall q_1, q_2 > 0, k \in (0, 1)$ and $\lambda \in [0, 1]$,

which proves the second assertion and the third assertion.

Next, choosing $\lambda = \frac{1}{2}$ in (3.4), (3.5), and (3.6), we conclude the Turán inequalities (3.1), (3.2), and (3.3), respectively. \square

Corollary 3.1 *The following assertions are true for $\Re(p) > 0, \Re(q) > 0$:*

- (1) *The function $p \mapsto K_{p,q}(k)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*
- (2) *The function $q \mapsto K_{p,q}(k)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*
- (3) *The function $p \mapsto E_{p,q}(k)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*
- (4) *The function $q \mapsto E_{p,q}(k)$ is log-convex on $(0, \infty)$ for all $k \in (0, 1)$.*

Moreover, for the same parametric range, the following Turán inequalities hold true:

$$K_{p,q}^2(k) - K_{p-1,q}(k).K_{p+1,q}(k) \leq 0 \quad p \in (1, \infty), \tag{3.7}$$

$$K_{p,q}^2(k) - K_{p,q-1}(k).K_{p,q+1}(k) \leq 0 \quad q \in (1, \infty), \tag{3.8}$$

$$E_{p,q}^2(k) - E_{p-1,q}(k).E_{p+1,q}(k) \leq 0 \quad p \in (1, \infty), \tag{3.9}$$

$$E_{p,q}^2(k) - E_{p,q-1}(k).E_{p,q+1}(k) \leq 0 \quad q \in (1, \infty). \tag{3.10}$$

Proof Furthermore, by specifying $\gamma = 0$ and $\gamma = 1$ in assertions (3.5) and (3.6) (or alternatively employing definitions (2.5) and (2.6) and using the Hölder–Rogers inequality), we can easily obtain

$$K_{\lambda p_1+(1-\lambda)p_2,q}(k) \leq [K_{p_1,q}(r)]^\lambda [K_{p_2,q}(r)]^{1-\lambda} \quad \forall p_1, p_2 > 0, k \in (0, 1) \text{ and } \lambda \in [0, 1], \tag{3.11}$$

$$K_{p,\lambda q_1+(1-\lambda)q_2}(k) \leq [K_{p,q_1}(r)]^\lambda [K_{p,q_2}(r)]^{1-\lambda} \quad \forall q_1, q_2 > 0, k \in (0, 1) \text{ and } \lambda \in [0, 1], \tag{3.12}$$

$$E_{\lambda p_1+(1-\lambda)p_2,q}(k) \leq [E_{p_1,q}(r)]^\lambda [E_{p_2,q}(r)]^{1-\lambda} \quad \forall p_1, p_2 > 0, k \in (0, 1) \text{ and } \lambda \in [0, 1], \tag{3.13}$$

and

$$E_{p,\lambda q_1+(1-\lambda)q_2}(k) \leq [E_{p,q_1}(r)]^\lambda [E_{p,q_2}(r)]^{1-\lambda} \quad \forall q_1, q_2 > 0, k \in (0, 1) \text{ and } \lambda \in [0, 1]. \tag{3.14}$$

Again, specifying $\lambda = \frac{1}{2}$ and $p_1 = p - 1, p_2 = p + 1$ in (3.11) and (3.13) and $q_1 = q - 1, q_2 = q + 1$ in (3.12) and (3.14), we obtain the Turán-type inequalities (3.7) to (3.10), respectively. \square

4 Mellin transform formulas and Laguerre polynomial representations

The Mellin transforms of the function $f(x, y)$ of two variables with respect to the indices r and s are given by [19]

$$\mathcal{M}\{f(x, y)\}(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} f(x, y) \, dx \, dy, \tag{4.1}$$

where it is assumed that the integral (improper) in (4.1) exists.

Theorem 4.1 Consider Appell's hypergeometric function $F_1(\lambda, \mu, \mu'; \nu; \mathfrak{X}, \mathfrak{Y})$. The following Mellin transformation formula for $\mathcal{H}_{p,q}(\psi, k, \gamma)$ in (2.1) holds true:

$$\begin{aligned} \mathcal{M}\{\mathcal{H}_{p,q}(\psi, k, \gamma)\}(r, s) &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2(r + \frac{1}{2})} \\ &\cdot F_1\left(r + \frac{1}{2}, \frac{1}{2} - \gamma, \frac{1}{2} - s; r + \frac{3}{2}; k^2 \sin^2 \psi, \sin^2 \psi\right) \\ &\left(\Re(r) > 0, \Re(s) > 0; |k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2}\right). \end{aligned} \tag{4.2}$$

Proof By making use of definition (4.1), we get from (2.1) that

$$\begin{aligned} &\mathcal{M}\{\mathcal{H}_{p,q}(\psi, k, \gamma)\}(r, s) \\ &= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \mathcal{H}_{p,q}(\psi, k, \gamma) dp dq \\ &= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left(\int_0^\psi (1 - k^2 \sin^2 \theta)^{\gamma - \frac{1}{2}} e^{-\frac{p}{\sin^2 \theta} - \frac{q}{\cos^2 \theta}} d\theta\right) dp dq \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left(\int_0^{\sin^2 \psi} \frac{(1 - k^2 t)^{\gamma - \frac{1}{2}}}{\sqrt{t(1-t)}} e^{-\frac{p}{t} - \frac{q}{1-t}} dt\right) dp dq, \end{aligned}$$

where the value in the inner θ -integral has also been evaluated by $\sin^2 \theta = t$ and $d\theta = \frac{dt}{2\sqrt{t(1-t)}}$. Now, by the absolute convergence of the integrations involved under the conditions mentioned in Theorem 4.1, we change the order of integrals on the right-hand side (RHS) to find

$$\begin{aligned} &\mathcal{M}\{\mathcal{H}_{p,q}(\psi, k, \gamma)\}(r, s) \\ &= \frac{1}{2} \int_0^{\sin^2 \psi} \frac{(1 - k^2 t)^{\gamma - \frac{1}{2}}}{\sqrt{t(1-t)}} \left(\int_0^\infty p^{r-1} e^{-\frac{p}{t}} dp\right) \left(\int_0^\infty q^{s-1} e^{-\frac{q}{1-t}} dq\right) dt \\ &= \frac{\Gamma(r)\Gamma(s)}{2} \int_0^{\sin^2 \psi} t^{r-\frac{1}{2}} (1-t)^{s-\frac{1}{2}} (1 - k^2 t)^{\gamma - \frac{1}{2}} dt, \end{aligned} \tag{4.3}$$

where we obviously have used the classical Euler's gamma integral

$$\Gamma(\eta)\xi^{-\eta} = \int_0^\infty e^{-\xi t} t^{\eta-1} dt, \quad \Re(\xi) > 0, \Re(\eta) > 0, \tag{4.4}$$

in the inner p -integral and q -integral. Now, setting $t = (\sin^2 \psi)\tau$ and $dt = (\sin^2 \psi) d\tau$ yields

$$\begin{aligned} &\mathcal{M}\{\mathcal{H}_{p,q}(\psi, k, \gamma)\}(r, s) \\ &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2} \int_0^1 \tau^{r-\frac{1}{2}} (1 - k^2 \sin^2 \psi \tau)^{\gamma - \frac{1}{2}} (1 - \sin^2 \psi \tau)^{s-\frac{1}{2}} d\tau. \end{aligned} \tag{4.5}$$

Finally, by using (see, e.g., [32, p. 276, Eq. 9.4(7)])

$$F_1(\lambda, \mu, \mu'; \nu; \mathfrak{X}, \mathfrak{Y}) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^1 t^{\lambda-1}(1-t)^{\nu-\lambda-1}(1-\mathfrak{X}t)^{-\mu}(1-\mathfrak{Y}t)^{-\mu'} dt \quad (4.6)$$

$$(\max\{|\arg(1-\mathfrak{X})|, |\arg(1-\mathfrak{Y})|\}) < \pi; \Re(\nu) > \Re(\lambda) > 0,$$

we obtain the required Mellin transformation formula (4.2) given by Theorem 4.1. \square

Appell's functions F_1 are represented in terms of Kampé de Fériet's hypergeometric function [32, p. 22, Eq. 1.3(2)] and the Srivastava–Daoust hypergeometric function of the two variables [32, p. 37, Eq. 1.4(21)]:

$$F_1[\alpha, \beta_1, \beta_2; \gamma; \mathfrak{X}, \mathfrak{Y}] = F_{1:0:0}^{1:1:1} \left[\begin{matrix} \alpha : \beta_1; \beta_2; \\ \gamma : -; -; \end{matrix} \mathfrak{X}, \mathfrak{Y} \right] \quad (4.7)$$

and

$$F_1[\alpha, \beta_1, \beta_2; \gamma; \mathfrak{X}, \mathfrak{Y}] = F_{1:0:0}^{1:1:1} \left[\begin{matrix} (\alpha : 1, 1) : (\beta_1, 1); (\beta_2, 1); \\ (\gamma : 1, 1) : -; -; \end{matrix} \mathfrak{X}, \mathfrak{Y} \right]. \quad (4.8)$$

We first apply representation (4.7) in (4.2) and representation (4.8) in (4.2), and afterward employing the Legendre duplication formula [32, p. 17, Eq. 1.2(14)] for the gamma function

$$\Gamma(2\eta) = \frac{2^{2\eta-1}}{\sqrt{\pi}} \Gamma(\eta) \Gamma\left(\eta + \frac{1}{2}\right),$$

we can evaluate the Mellin transform formulas for $\mathcal{H}_{p,q}(\psi, k, \gamma)$ in (2.1) as given by Corollary 4.1. In particular, if we put $\gamma = 0$ and $\gamma = 1$, we can obtain certain Mellin transformation formulas for the $F_{p,q}(\psi, k)$ and $E_{p,q}(\psi, k)$ in (2.2) and (2.3) as given by Corollary 4.2. The proofs are omitted for Corollaries 4.1 and 4.2.

Corollary 4.1 *The following Mellin transformation formulas hold true:*

$$\begin{aligned} \mathcal{M}\{\mathcal{H}_{p,q}(\psi, k, \gamma)\}(r, s) &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2(r + \frac{1}{2})} \\ &\cdot F_{1:0:0}^{1:1:1} \left[\begin{matrix} r + \frac{1}{2} : \frac{1}{2} - \gamma; \frac{1}{2} - s; \\ r + \frac{3}{2} : -; -; \end{matrix} k^2 \sin^2 \psi, \sin^2 \psi \right] \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \mathcal{M}\{\mathcal{H}_{p,q}(\psi, k, \gamma)\}(r, s) &= \Gamma(r)\Gamma(s) \sin^{2r+1} \psi \\ &\cdot F_{1:0:0}^{1:1:1} \left[\begin{matrix} (2r + 1 : 2, 2) : (\frac{1}{2} - \gamma, 1); (\frac{1}{2} - s, 1); \\ (2r + 2 : 2, 2) : -; -; \end{matrix} k^2 \sin^2 \psi, \sin^2 \psi \right]. \end{aligned} \quad (4.10)$$

Corollary 4.2 *The following Mellin transformation formulas for $F_{p,q}(\psi, k)$ and $E_{p,q}(\psi, k)$ in (2.2) and (2.3) hold true:*

$$\begin{aligned} \mathcal{M}\{F_{p,q}(\psi, k)\}(r, s) &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2(r + \frac{1}{2})} \\ &\quad \cdot F_1\left(r + \frac{1}{2}, \frac{1}{2}, \frac{1}{2} - s; r + \frac{3}{2}; k^2 \sin^2 \psi, \sin^2 \psi\right), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \mathcal{M}\{F_{p,q}(\psi, k)\}(r, s) &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2(r + \frac{1}{2})} \\ &\quad \times F_{1:0:0}^{1:1:1} \left[\begin{matrix} r + \frac{1}{2} : \frac{1}{2}; \frac{1}{2} - s; \\ r + \frac{3}{2} : \text{---}; \text{---}; \end{matrix} k^2 \sin^2 \psi, \sin^2 \psi \right], \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathcal{M}\{F_{p,q}(\psi, k)\}(r, s) &= \Gamma(r)\Gamma(s) \sin^{2r+1} \psi \\ &\quad \cdot F_{1:0:0}^{1:1:1} \left[\begin{matrix} (2r + 1 : 2, 2) : (\frac{1}{2}, 1); (\frac{1}{2} - s, 1); \\ (2r + 2 : 2, 2) : \text{---}; \text{---}; \end{matrix} k^2 \sin^2 \psi, \sin^2 \psi \right], \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathcal{M}\{E_{p,q}(\psi, k)\}(r, s) &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2(r + \frac{1}{2})} \cdot F_1\left(r + \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} - s; r + \frac{3}{2}; k^2 \sin^2 \psi, \sin^2 \psi\right), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \mathcal{M}\{E_{p,q}(\psi, k)\}(r, s) &= \frac{\Gamma(r)\Gamma(s) \sin^{2r+1} \psi}{2(r + \frac{1}{2})} F_{1:0:0}^{1:1:1} \left[\begin{matrix} r + \frac{1}{2} : -\frac{1}{2}; \frac{1}{2} - s; \\ r + \frac{3}{2} : \text{---}; \text{---}; \end{matrix} k^2 \sin^2 \psi, \sin^2 \psi \right], \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \mathcal{M}\{E_{p,q}(\psi, k)\}(r, s) &= \Gamma(r)\Gamma(s) \sin^{2r+1} \psi \\ &\quad \times F_{1:0:0}^{1:1:1} \left[\begin{matrix} (2r + 1 : 2, 2) : (-\frac{1}{2}, 1); (\frac{1}{2} - s, 1); \\ (2r + 2 : 2, 2) : \text{---}; \text{---}; \end{matrix} k^2 \sin^2 \psi, \sin^2 \psi \right]. \end{aligned} \tag{4.16}$$

Theorem 4.2 *The mentioned Mellin transformation formulas for $\mathcal{H}_{p,q}(k, \gamma)$, $K_{p,q}(k)$, and $E_{p,q}(k)$ in (2.4), (2.5), and (2.6) hold true:*

$$\begin{aligned} \mathcal{M}\{\mathcal{H}_{p,q}(k, \gamma)\}(r, s) &= \frac{\Gamma(r)\Gamma(s)B(r + \frac{1}{2}, s + \frac{1}{2})}{2} {}_2F_1\left(r + \frac{1}{2}, \frac{1}{2} - \gamma; r + s + 1; k^2\right) \\ &\quad \left(\Re(r) > 0, \Re(s) > 0; |k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2}\right), \end{aligned} \tag{4.17}$$

$$\begin{aligned} \mathcal{M}\{K_{p,q}(k)\}(r, s) &= \frac{\Gamma(r)\Gamma(s)B(r + \frac{1}{2}, s + \frac{1}{2})}{2} {}_2F_1\left(r + \frac{1}{2}, \frac{1}{2}; r + s + 1; k^2\right) \\ &\quad \left(\Re(r) > 0, \Re(s) > 0; |k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2}\right), \end{aligned} \tag{4.18}$$

and

$$\mathcal{M}\{E_{p,q}(k)\}(r,s) = \frac{\Gamma(r)\Gamma(s)B(r+\frac{1}{2},s+\frac{1}{2})}{2} {}_2F_1\left(r+\frac{1}{2},-\frac{1}{2};r+s+1;k^2\right) \tag{4.19}$$

$$\left(\Re(s) > 0; |k^2| < 1; 0 \leq \psi \leq \frac{\pi}{2}\right).$$

Proof By first substituting $\psi = \frac{\pi}{2}$ in (4.2), (4.11), and (4.14) and afterward employing the following identity (see [13, p. 239, Eq. (10)])

$$F_1[\lambda, \mu_1, \mu_2; \nu; \mathfrak{X}, 1] = \frac{\Gamma(\nu)\Gamma(\nu - \lambda - \mu_2)}{\Gamma(\nu - \lambda)\Gamma(\nu - \mu_2)} {}_2F_1(\lambda, \mu_1; \nu - \mu_2; \mathfrak{X}),$$

we obtain the required Mellin transformation formulas given by Theorem 4.2. □

If we set $r = 1$ and $s = 1$ in (4.18) and (4.19), we obtain the following connection between the classical elliptic integrals and the (p, q) -extended (*complete*) elliptic integrals:

$$\int_0^\infty \int_0^\infty K_{p,q}(k) dp dq = \frac{\pi}{16} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 3; k^2\right), \tag{4.20}$$

which, in view of the well-known representation [26, p. 473, Entry (93)], yields

$$\int_0^\infty \int_0^\infty K_{p,q}(k) dp dq = \frac{1}{3k^2} [K(k) - (2 - k^2)D(k)] \tag{4.21}$$

and

$$\int_0^\infty \int_0^\infty E_{p,q}(k) dp dq = \frac{\pi}{16} {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}; 3; k^2\right), \tag{4.22}$$

which, by means of the well-known representation [26, p. 469, Entry (20)], can be written in the following form:

$$\int_0^\infty \int_0^\infty E_{p,q}(k) dp dq = \frac{1}{15k^2} [(1 + k^2)K(k) - 2(1 - k^2 + k^4)D(k)]. \tag{4.23}$$

The Laguerre(*simple*) polynomials $L_n(x)$ are defined by (see, e.g., [25, p. 645])

$$L_n(x) := L_n^{(0)}(x) \quad \text{and} \quad L_n^{(\lambda)}(x) := \sum_{j=0}^n \binom{n+\lambda}{n-j} \frac{(-x)^j}{j!}.$$

Now we derive the Laguerre polynomial relation asserted by (4.24).

Theorem 4.3 *The following Laguerre polynomial relation holds true for $\mathcal{H}_{p,q}(k)$:*

$$\mathcal{H}_{p,q}(k) = \sum_{m,n=0}^\infty \frac{e^{-p-q} B(n+\frac{3}{2}, m+\frac{3}{2})}{2} L_m(p) L_n(q) {}_2F_1\left(\gamma - \frac{1}{2}, n + \frac{3}{2}; m + n + 3; k^2\right). \tag{4.24}$$

Proof Putting $\sin^2 \theta = t$ and $2 d\theta = dt/\sqrt{t(1-t)}$ in (2.4), we get

$$\mathcal{H}_{p,q}(k) = \frac{1}{2} \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-k^2t)^{\nu-\frac{1}{2}} e^{-\frac{p}{t}-\frac{q}{1-t}} dt. \tag{4.25}$$

By making use of the known identity in a slightly corrected form for the simple Laguerre polynomials due to Choi *et al.* [11, p. 350, Eq. (5.5)]

$$\exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) = e^{-p-q} \left\{ \sum_{m,n=0}^{\infty} L_m(p)L_n(q)t^{m+1}(1-t)^{n+1} \right\}$$

in (4.25), we have

$$\begin{aligned} \mathcal{H}_{p,q}(k) &= \frac{e^{-p-q}}{2} \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-k^2t)^{\nu-\frac{1}{2}} \left\{ \sum_{m,n=0}^{\infty} L_m(p)L_n(q)t^{m+1}(1-t)^{n+1} \right\} dt. \end{aligned} \tag{4.26}$$

Now, changing integration and summation order and using the integral representation of ${}_2F_1$ [27]

$${}_2F_1(\lambda, \mu; \nu; 3) = \frac{\Gamma(\nu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^1 t^{\mu-1}(1-t)^{\nu-\mu-1}(1-3t)^{-\lambda} dt,$$

when there holds $\Re(\nu) > \Re(\mu) > 0; |\arg(1-3)| \leq \pi - \epsilon$ ($0 < \epsilon < \pi$) in (4.26), we are led to the required result. □

Corollary 4.3 *The following Laguerre polynomial relations hold true for $K_{p,q}(k)$ and $E_{p,q}(k)$:*

$$K_{p,q}(k) = \sum_{m,n=0}^{\infty} \frac{e^{-p-q}B\left(n+\frac{3}{2}, m+\frac{3}{2}\right)}{2} L_m(p)L_n(q) {}_2F_1\left(\frac{1}{2}, n+\frac{3}{2}; m+n+3; k^2\right) \tag{4.27}$$

and

$$E_{p,q}(k) = \sum_{m,n=0}^{\infty} \frac{e^{-p-q}B\left(n+\frac{3}{2}, m+\frac{3}{2}\right)}{2} L_m(p)L_n(q) {}_2F_1\left(-\frac{1}{2}, n+\frac{3}{2}; m+n+3; k^2\right). \tag{4.28}$$

5 Certain properties of (p, q)-extended elliptic integrals

In this section we obtain certain special values in terms of (p, q)-extended beta function $B_{p,q}(\delta, \sigma)$ and present the connections with G-function of two variables. We also present various derivative and integrals formulas for the (p, q)-extended elliptic-type integrals.

5.1 Special parametric values and connections with G-function

In this subsection, we first find the special values of $K_{p,q}(k), K'_{p,q}(k), E_{p,q}(k)$, and $E'_{p,q}(k)$ in terms of the (p, q)-extended beta function $B_{p,q}(\delta, \sigma)$. It suffices to consider the corresponding defining expressions in Sect. 2 in view of the definition of $B_{p,q}(\delta, \sigma)$ (1.7).

Theorem 5.1 Let $B_{p,q}(\delta, \sigma)$ be the (p, q) -extended beta function defined in (1.7). Then:

$$K_{p,q}(0) = K'_{p,q}(1) = E_{p,q}(0) = E'_{p,q}(1) = \frac{1}{2}B_{p,q}\left(\frac{1}{2}, \frac{1}{2}\right), \tag{5.1}$$

$$E_{p,q}(1) = E'_{p,q}(0) = \frac{1}{2}B_{p,q}\left(\frac{1}{2}, 1\right), \tag{5.2}$$

and

$$K_{p,q}(1) = K'_{p,q}(0) = \frac{1}{2}B_{p,q}\left(\frac{1}{2}, 0\right). \tag{5.3}$$

Next, observe that Meijer’s G -function of two variables [31, p.7, Eq. (1.2.3) and p.88, Eq. (6.4.1)] is expressible in terms of the (p, q) -extended beta function $B_{p,q}(\delta, \sigma)$ for $\Re(p), \Re(q) > 0$ as follows [11, p.350, Eq. (5.6)]:

$$B_{p,q}(\delta, \sigma) = G_{1,0:0,2;0,2}^{0,0:2,0;2,0} \left[\begin{matrix} (\delta + \sigma; 1, 1) : \text{---} ; \text{---} \\ \text{---} : (0, 1), (\delta, 1); (0, 1), (\sigma, 1) \end{matrix} ; p, q \right]. \tag{5.4}$$

Now, by making use of relationship (5.4) to (5.1) through (5.3) in Theorem 5.1, we can establish various representations of (p, q) -extended (complete) elliptic integrals that are given in Corollary 5.1.

Corollary 5.1 The following relations hold true:

$$\begin{aligned} K_{p,q}(0) &= K'_{p,q}(1) = E_{p,q}(0) = E'_{p,q}(1) \\ &= G_{1,0:0,2;0,2}^{0,0:2,0;2,0} \left[\begin{matrix} (1; 1, 1) : \text{---} ; \text{---} \\ \text{---} : (0, 1), (1/2, 1); (0, 1), (1/2, 1) \end{matrix} ; p, q \right], \end{aligned} \tag{5.5}$$

$$E_{p,q}(1) = E'_{p,q}(0) = G_{1,0:0,2;0,2}^{0,0:2,0;2,0} \left[\begin{matrix} (3/2; 1, 1) : \text{---} ; \text{---} \\ \text{---} : (0, 1), (1/2, 1); (0, 1), (1, 1) \end{matrix} ; p, q \right], \tag{5.6}$$

and

$$K_{p,q}(1) = K'_{p,q}(0) = G_{1,0:0,2;0,2}^{0,0:2,0;2,0} \left[\begin{matrix} (1/2; 1, 1) : \text{---} ; \text{---} \\ \text{---} : (0, 1), (1/2, 1); (0, 1), (0, 1) \end{matrix} ; p, q \right]. \tag{5.7}$$

5.2 Differential and integral formulas

In this subsection, we present various differential and integral formulas for (p, q) -extended elliptic-type integrals. The proofs are omitted.

Theorem 5.2 The following derivative formulas hold true for $K_{p,q}(k)$ and $E_{p,q}(k)$:

$$\frac{d}{dk} \{E_{p,q}(\psi, k)\} = \frac{1}{k} [E_{p,q}(\psi, k) - F_{p,q}(\psi, k)], \tag{5.8}$$

$$\frac{d}{dk} \{E_{p,q}(k)\} = \frac{1}{k} [E_{p,q}(k) - K_{p,q}(k)], \tag{5.9}$$

$$\frac{d}{dk} \{E'_{p,q}(k)\} = -\frac{k}{k'^2} [E'_{p,q}(k) - K'_{p,q}(k)], \tag{5.10}$$

$$\frac{d}{dk'} \{E_{p,q}(k)\} = -\frac{k'}{k^2} [E_{p,q}(k) - K_{p,q}(k)], \tag{5.11}$$

$$\frac{d^m}{d(k^2)^m} \{K_{p,q}(k)\} = \frac{\pi [(2m)!]^2}{2(m!)^3 16^m} F_{p,q} \left(\frac{1}{2} + m, \frac{1}{2} + m, 1 + m; k^2 \right), \tag{5.12}$$

$$\frac{d^m}{d(k^2)^m} \{E_{p,q}(k)\} = \frac{\pi (\frac{1}{2})_m (-\frac{1}{2})_m}{2 m!} F_{p,q} \left(-\frac{1}{2} + m, \frac{1}{2} + m, 1 + m; k^2 \right). \tag{5.13}$$

Theorem 5.3 *The following integral formulas hold true for $K_{p,q}(k)$ and $E_{p,q}(k)$:*

$$\int [K_{p,q}(k) - E_{p,q}(k)] \frac{dk}{k} = -E_{p,q}(k), \tag{5.14}$$

$$\int K_{p,q}(k) \frac{dk}{k^2} = -\frac{1}{k} E_{p,q}(k), \tag{5.15}$$

$$\int \frac{k}{k'^2} [K'_{p,q}(k) - E'_{p,q}(k)] dk = E'_{p,q}(k), \tag{5.16}$$

and

$$\int \frac{k'}{k^2} [K_{p,q}(k) - E'_{p,q}(k)] dk' = E_{p,q}(k). \tag{5.17}$$

6 Concluding remark and observations

In our present studies, we have introduced and extensively investigated the family of (*incomplete and complete*) (p, q) -extended elliptic-type integrals and presented connections with (p, q) -extended beta function, (p, q) -extended Gauss' hypergeometric function, and (p, q) -extended Appell's double hypergeometric function F_1 . Moreover, we obtained the connection with Meijer G -function of two variables. Turán-type inequalities were proved by using log-convexity property for these (p, q) -extended complete elliptic-type integrals. Further, we established various Mellin transform formulas and obtained certain infinite series representations containing Laguerre polynomials. We also obtained some relationship between these (p, q) -extended elliptic-type integrals and as special values and deduced numerous differentiation and integral formulas. In conclusion, we introduced (p, q) -extension of the Epstein–Hubbell elliptic-type integral.

It is worth mentioning, as a main conclusion to observe, that Epstein and Hubbell [12] studied and investigated the following extensions of $K(k)$ and $E(k)$, which was encountered in a Legendre polynomial expansion method when applied to certain problems involving computation of the radiation field off-axis from a uniform circular disk radiating according to an arbitrary angular distribution law [14] (see also Weiss [37]):

$$\Omega_j(\kappa) := \int_0^\pi \frac{d\theta}{(1 - \kappa^2 \cos \theta)^{j+\frac{1}{2}}} \tag{6.1}$$

$$(0 \leq \kappa < 1; j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, 3, \dots\}).$$

Note that, by comparing definitions (1.5), (1.6), and (6.1), we can deduce the following connections:

$$\Omega_0(\kappa) = \frac{k\sqrt{2}}{\kappa} K(k) \quad \text{and} \quad \Omega_1(\kappa) = \frac{k\sqrt{2}}{\kappa(1 - \kappa^2)} E(k) \quad \left(k^2 := \frac{2\kappa^2}{1 + \kappa^2} \right).$$

These Epstein–Hubbell elliptic-type integrals can be extended by using definitions, which we have considered in this paper, can be generalized by introducing (p, q) -extended Epstein–Hubbell elliptic-type integral in (6.1):

$$\Omega_{\gamma,p,q}(\kappa) := \int_0^\pi \frac{1}{(1 - \kappa^2 \cos \theta)^{\gamma + \frac{1}{2}}} e^{-\frac{p}{\sin^2(\theta/2)} - \frac{q}{\cos^2(\theta/2)}} d\theta \tag{6.2}$$

$(\min\{\Re(p), \Re(q)\} > 0; 0 \leq \kappa \leq 1; \gamma \in \mathbb{C}_0 \text{ when } p = q = 0).$

It is worth noting that if we substitute $t = \cos(\frac{\theta}{2}) = \sqrt{\frac{(1+\cos \theta)}{2}}$ in (2.4), we obtain the following connection with the (p, q) -extended elliptic-type integrals $\mathcal{H}_{p,q}(k, \gamma)$ defined by (2.4) and $\Omega_{\gamma,p,q}(k)$ defined by (6.2)

$$\mathcal{H}_{p,q}(k, \gamma) = \frac{(2 - k^2)^{\gamma - \frac{1}{2}}}{2^{\gamma + \frac{1}{2}}} \Omega_{-\gamma,p,q}\left(\frac{k}{\sqrt{2 - k^2}}\right).$$

Again by letting $\cos(\theta/2) = t$ and then $t^2 = u$ in (6.2), we can easily obtain relationship (6.3) in terms of (p, q) -extended Gauss’ hypergeometric function (1.8)

$$\Omega_{\gamma,p,q}(\kappa) := \frac{\pi}{(1 + \kappa^2)^{\gamma + \frac{1}{2}}} F_{p,q}\left(\gamma + \frac{1}{2}, \frac{1}{2}; 1; \frac{2\kappa^2}{1 + \kappa^2}\right). \tag{6.3}$$

Note that expression (6.3) for $p = q = 0$ reduces to the known result [37]. Further many of properties such as Mellin transform formulas, infinite series representations containing Laguerre polynomials, log-convexity property, and Turán-type inequalities and differentiation and integral formulas can be easily derived. As a result, the information involved can be left as an exercise for the curious reader. More various concavity, convexity, and monotonicity properties of our findings can be studied and investigated parallel to the recent papers [1, 28, 36, 38, 39].

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The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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