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Extended Jacobi Elliptic Function Expansion Method for Nonlinear Benjamin-Bona-Mahony Equations

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Abstract

The Benjamin-Bona-Mahony equation has improved short-wavelength behaviour, as compared to the Korteweg-de Vries equation, and is another uni-directional wave equation with cnoidal wave solutions. Cnoidal wave solutions can appear in other applications than surface gravity waves as well, for instance to describe ion acoustic waves in plasma physics. Using a computerized symbolic computation technique, we construct the interesting Jacobi doubly periodic wave solutions for these equations by applying the extended Jacobi elliptic function expansion method.

Keywords: Extended Jacobi elliptic function expansion method; nonlinear physical phenomena; Benjamin-Bona-Mahony equation; modified Benjamin-Bona-Mahony equation

1 Introduction

The Benjamin-Bona-Mahony equation (BBME) was first introduced by Benjamin et al. [1] as an improvement of the Korteweg-de Vries equation (KdV) for modeling long waves of small amplitude in 1+1 dimensions. They show the stability and uniqueness of solutions to the BBM equation. This contrasts with the KdV equation, which is unstable in its high wave number components. The BBM equation describes the uni-directional propagation of small-amplitude long waves on the surface of water in a channel.

The BBM equation is not only convenient for shallow water waves but also for hydromagnetic and acoustic waves, and therefore it has some advantages compared with the KdV equation [2]. The BBM equation can be writen in the form:

$$u_t + u_x + auu_x + u_{xxt} = 0, \qquad \text{where } a \text{ is constant.} \tag{1.1}$$

The second equation, we study in this paper, is the modified BBME (MBBME):

$$u_t + u_x + au^2u_x + u_{xxt} = 0, \qquad \text{where } a \text{ is constant.} \tag{1.2}$$

The existence of the solutions of initial value problems for the mBBM equation has been considered in [3]. Many solutions obtained for MBBME using a several methods such as the tanh and the sine-cosine methods [4], and the hyperbolic auxiliary function method [5], the first integral method [6], the homogeneous balance method [7], an algebraic method [8], the variable-coefficient balancing-act method [9] and the Jacobi elliptic function expansion method [10].

Nonlinear problems are of interest to engineers, physicists and mathematicians because most physical systems are inherently nonlinear in nature. Nonlinear partial differential equations (NPDEs) are difficult to solve and give rise to interesting phenomena such as chaos. The exact solutions of these NPDEs plays an important role in the study of nonlinear phenomena. In the past decades, many methods were developed for finding exact solutions of NPDEs as Hirota's bilinear method [11], new similarity transformation method [12], homogeneous balance method [13], extended hyperbolic function method [14], Exp-function method [15, 16], tanh function methods [17, 19], Jacobi and Weierstrass elliptic function method [20, 21]... etc.

In this paper, we extend the extended JEF method with symbolic computation to such special equations for constructing their interesting Jacobi doubly periodic wave solutions. It is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. In addition the algorithm that we use here also a computerized method, in which generating an algebraic system. Two key procedures and laborious to do by hand. But they can be implemented on a computer with the help of mathematica. The outputs of solving the algebraic system from a computer comprise a list of constants. In general if any of the parameters is left unspecified. We only consider the expansion in terms of the Jacobi functions $sn\xi$ and $cn\xi$. Further studies show that different Jacobi function expansions may lead to new periodic wave solutions.

2 Extended Jacobi elliptic function method

In this section, we introduce a simple description of the extended JEF method, for a given partial differential equation

$$G(u, u_x, u_y, u_t, u_{xy}, ...) = 0. \tag{2.1}$$

We like to know whether travelling waves (or stationary waves) are solutions of Eq. (2.1). The first step is to unite the independent variables x, y and t into one particular variable through the new variable

$$\zeta = x + y + \nu t, \quad u(x, y, t) = U(\zeta),$$

where ν is the wave speed, reduce Eq. (2.1) to an ordinary differential equation (ODE)

$$G(U, U^{'}, U^{''}, U^{'''}, ...) = 0. \tag{2.2}$$

Our main goal is to find exact or at least approximate solutions, if possible, for this ODE. For this purpose, using the extended Jacobi elliptic function expansion method, $U(\zeta)$ can be expressed as a finite series of JEF, snζ,

$$u(x, y) = U(\zeta) = \sum_{i=0}^{N} a_i sn(\zeta)^i + \sum_{i=1}^{N} a_{-i} sn(\zeta)^{-i}. \tag{2.3}$$

The parameter N is determined by balancing the linear term(s) of highest order with the nonlinear one(s). And

$$cn^2(\zeta) = 1 - sn^2(\zeta), \qquad dn^2(\zeta) = 1 - m^2 sn^2(\zeta), \tag{2.4}$$

$$\frac{d}{d\zeta} sn\zeta = cn\zeta dn\zeta, \quad \frac{d}{d\zeta} cn\zeta = -sn\zeta dn\zeta, \quad \frac{d}{d\zeta} dn\zeta = -m^2 sn\zeta cn\zeta, \tag{2.5}$$

where $cn\zeta$ and $dn\zeta$ are the Jacobi elliptic cosine function and the JEF of the third kind, respectively, with the modulus m $(0 < m < 1)$. Since the highest degree of $\frac{d^pU}{d\zeta^p}$ is taken as

$$O(\frac{d^pU}{d\zeta^p}) = N + p, \qquad p = 1, 2, 3, \cdots, \tag{2.6}$$

$$O(U^q \frac{d^pU}{d\zeta^p}) = (q + 1)N + p, \quad q = 0, 1, 2, \cdots, p = 1, 2, 3, \cdots. \tag{2.7}$$

Normally N is a positive integer, so that an analytic solution in closed form may be obtained. Substituting Eqs. (2.3)-(2.7) into Eq. (2.2) and comparing the coefficients of each power of $sn\zeta$ in both sides, to get an over-determined system of nonlinear algebraic equations with respect to ν, a_i and a_{-i}, $i = 1, \cdots, N$. Solving the over-determined system of nonlinear algebraic equations by use of Mathematica. We can get other kinds of Jacobi doubly periodic wave solutions.

When $m \longrightarrow 1$, the Jacobi functions degenerate to the hyperbolic functions,

$$sn\zeta \longrightarrow tanh\zeta, \quad cn\zeta \longrightarrow sech\zeta \quad \text{and} \quad dn\zeta \longrightarrow sech\zeta.$$

When $m \longrightarrow 0$, the Jacobi functions degenerate to the triangular functions,

$$sn\zeta \longrightarrow sin\zeta, \quad cn\zeta \longrightarrow cos\zeta \quad \text{and} \quad dn \longrightarrow 1.$$

3 Benjamin-Bona-Mahony equation

We first consider the BBME in the following form:

$$u_t + u_x + auu_x + u_{xxt} = 0. \tag{3.1}$$

If we use the transformations

$$u(x,t) = U(\zeta), \qquad \zeta = x + \nu t. \tag{3.2}$$

It carries Eq. (3.1) to the ODE,

$$\nu U^{'''} + aUU^{'} + (\nu+1)U^{'} = 0. \tag{3.3}$$

where by integrating once we obtain, upon setting the constant of integration to zero,

$$\nu U^{''} + \frac{a}{2}U^2 + (\nu+1)U = 0. \tag{3.4}$$

Balancing the term $U^{''}$ with the term U^2 we obtain $N = 2$ then

$$U(\zeta) = \sum_{i=0}^{2} a_i sn^i(\zeta) + \sum_{i=1}^{2} a_{-i}(sn(\zeta))^{-i}. \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.4) and comparing the coefficients of each power of $sn(\zeta)$ in both sides, getting an over-determined system of nonlinear algebraic equations with respect to ν, a_i; $i = 0,\, 1,\, -1,\, 2,\, -2$. Solving the over-determined system of nonlinear algebraic equations using Mathematica, we obtain three groups of constants:

a)

$$a_{-1} = a_1 = a_2 = 0, \quad a_{-2} = \frac{12}{a(1 \pm 4\sqrt{1-m^2+m^4})}, \quad \nu = \frac{1}{-1 \mp 4\sqrt{1-m^2+m^4}}$$
$$\text{and} \quad a_0 = \frac{12m^2}{a(\pm(5m^2-4m^4-3) + (3+4m^2)\sqrt{1-m^2+m^4})}, \tag{3.6}$$

b)

$$a_{-1} = a_1 = a_{-2} = 0, \quad a_2 = \frac{12m^2}{a(1 \pm 4\sqrt{1-m^2+m^4})}, \quad \nu = \frac{1}{-1 \mp 4\sqrt{1-m^2+m^4}}$$
$$\text{and} \quad a_0 = \frac{12m^2}{a(\pm(5m^2-4m^4-3) + (3+4m^2)\sqrt{1-m^2+m^4})}, \tag{3.7}$$

c)

$$a_{-1} = a_1 = 0, \quad a_2 = \frac{12m^2}{a(1+4\sqrt{1+14m^2+m^4})}, \quad a_{-2} = \frac{12}{a(1+4\sqrt{1+14m^2+m^4})},$$
$$\nu = \frac{1}{-1-4\sqrt{1+14m^2+m^4}} \quad \text{and} \quad a_0 = \frac{48m^2}{a((3+4m^2)\sqrt{1+14m^2+m^4}-(3+55m^2+4m^4))}. \tag{3.8}$$

d)

$$a_{-1} = a_1 = 0, \quad a_2 = \frac{12m^2}{a(1-4\sqrt{1+14m^2+m^4})}, \quad a_{-2} = \frac{12}{a(1-4\sqrt{1+14m^2+m^4})},$$
$$\nu = \frac{1}{-1+4\sqrt{1+14m^2+m^4}} \quad \text{and} \quad a_0 = \frac{4((3+4m^2)\sqrt{1+14m^2+m^4}-(3+55m^2+4m^4)}{a(15+16m^2(14+m^2))}. \tag{3.9}$$

We find the following solutions of Eq. (3.4)

a)

$$\begin{aligned} U_1 =& \frac{12m^2}{a(\pm(5m^2-4m^4-3)+(3+4m^2)\sqrt{1-m^2+m^4})} \\ &+ \frac{12}{a(1\pm 4\sqrt{1-m^2+m^4})}(sn\zeta)^{-2}, \end{aligned} \tag{3.10}$$

b)

$$\begin{aligned} U_2 =& \frac{12m^2}{a(\pm(5m^2-4m^4-3)+(3+4m^2)\sqrt{1-m^2+m^4})} \\ &+ \frac{12m^2}{a(1\pm 4\sqrt{1-m^2+m^4})}(sn\zeta)^{2}, \end{aligned} \tag{3.11}$$

c)

$$\begin{aligned} U_3 =& \frac{48m^2}{a((3+4m^2)\sqrt{1+14m^2+m^4}-(3+55m^2+4m^4))} \\ &+ \frac{12}{a(1+4\sqrt{1+14m^2+m^4})}[m^2(sn\zeta)^2+(sn\zeta)^{-2}]. \end{aligned} \tag{3.12}$$

d)

$$\begin{aligned} U_4 =& \frac{4((3+4m^2)\sqrt{1+14m^2+m^4}-(3+55m^2+4m^4))}{a(15+16m^2(14+m^2))} \\ &+ \frac{12}{a(1-4\sqrt{1+14m^2+m^4})}[m^2(sn\zeta)^2+(sn\zeta)^{-2}]. \end{aligned} \tag{3.13}$$

Then the solutions of the BBME (3.1) are:

$$\begin{aligned} u_1 =& \frac{12m^2}{a(\pm(5m^2-4m^4-3)+(3+4m^2)\sqrt{1-m^2+m^4})} \\ &+ \frac{12}{a(1\pm 4\sqrt{1-m^2+m^4})}(sn(x+\frac{1}{-1\mp 4\sqrt{1-m^2+m^4}}t))^{-2}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} u_2 =& \frac{12m^2}{a(\pm(5m^2-4m^4-3)+(3+4m^2)\sqrt{1-m^2+m^4})} \\ &+ \frac{12m^2}{a(1\pm 4\sqrt{1-m^2+m^4})}(sn(x+\frac{1}{-1\mp 4\sqrt{1-m^2+m^4}}t))^{2}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} u_3 =& \frac{48m^2}{a((3+4m^2)\sqrt{1+14m^2+m^4}-(3+55m^2+4m^4)} + \frac{12}{a(1+4\sqrt{1+14m^2+m^4})} \\ &\times [m^2(sn(x+\frac{1}{-1-4\sqrt{1+14m^2+m^4}}t))^2 + (snx+\frac{1}{-1-4\sqrt{1+14m^2+m^4}}t)^{-2}]. \end{aligned} \tag{3.16}$$

$$\begin{aligned} u_4 =& \frac{4((3+4m^2)\sqrt{1+14m^2+m^4}-(3+55m^2+4m^4)}{a(15+16m^2(14+m^2))} + \frac{12}{a(1-4\sqrt{1+14m^2+m^4})} \\ &\times [m^2(sn(x+\frac{1}{-1+4\sqrt{1+14m^2+m^4}}t))^2 + (sn(x+\frac{1}{-1+4\sqrt{1+14m^2+m^4}}t))^{-2}]. \end{aligned} \tag{3.17}$$

The modulus of solitary wave solution u_1 (Eq. 3.14) and u_2 (Eq. 3.15) are displayed in figures 1 and 2 respectively, with values of parameters listed in their captions.

4 Modified Benjamin-Bona-Mahony equation

We first consider the MBBME in the following form:

$$u_t + u_x + au^2u_x + u_{xxt} = 0. \tag{4.1}$$

If we use the transformations

$$u(x,t) = U(\zeta), \qquad \zeta = x + \nu t. \tag{4.2}$$

It carries Eq. (4.1) to the ODE,

$$\nu U^{'''} + aU^2U^{'} + (\nu+1)U^{'} = 0. \tag{4.3}$$

Where by integrating once we obtain, upon setting the constant of integration to zero,

$$\nu U'' + \frac{a}{3}U^3 + (\nu+1)U = 0. \tag{4.4}$$

Balancing the term U'' with the term U^3 we obtain $N = 1$ then

$$U(\zeta) = \sum_{i=0}^{1} a_i sn^i(\zeta) + \sum_{i=1}^{1} a_{-i}(sn(\zeta))^{-i}. \tag{4.5}$$

Proceeding as in the previous case we obtain

a)

$$a_{-1} = a_0 = 0, \quad a_1 = \pm i\sqrt{\frac{6}{a}} \quad \text{and} \quad \nu = \frac{1}{m^2}, \tag{4.6}$$

b)

$$a_0 = 0, \quad a_1 = \pm m\sqrt{\frac{6}{a(6-m)m}}, \quad a_{-1} = \mp\sqrt{\frac{6}{a(m-6)m}} \quad \text{and} \quad \nu = \frac{1}{a(6-m)m} \tag{4.7}$$

c)

$$a_0 = 0, \quad a_1 = \pm mi\sqrt{\frac{6}{a(6+m)m}}, \quad a_{-1} = \pm i\sqrt{\frac{6}{a(m+6)m}} \quad \text{and} \quad \nu = \frac{1}{a(6+m)m} \tag{4.8}$$

We find the following solutions of Eq. (4.1)

a)

$$U_1 = \pm i\sqrt{\frac{6}{a}} sn\zeta, \tag{4.9}$$

b)

$$U_2 = \mp\sqrt{\frac{6}{a(m-6)m}}[msn\zeta - sn\zeta], \tag{4.10}$$

c)

$$U_3 = \mp i\sqrt{\frac{6}{a(m+6)m}}[msn\zeta + sn\zeta]. \tag{4.11}$$

Then the solutions of the BBME (4.1) are:

$$u_1 = \pm i\sqrt{\frac{6}{a}} sn(x + \frac{1}{m^2}t), \tag{4.12}$$

$$u_2 = \mp\sqrt{\frac{6}{a(m-6)m}}[msn(x + \frac{1}{a(6-m)m}t) - sn(x + \frac{1}{a(6-m)m}t)], \tag{4.13}$$

$$u_3 = \mp i\sqrt{\frac{6}{a(m+6)m}}[msn(x + \frac{1}{a(6+m)m}t) + sn(x + \frac{1}{a(6+m)m}t)]. \tag{4.14}$$

5 Conclusion

We extend the extended JEF method with symbolic computation to two nonlinear equations for constructing their interesting Jacobi doubly periodic wave solutions. It is shown that soliton solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. When $m \longrightarrow 1$, the Jacobi functions degenerate to the hyperbolic functions and given the solutions by the extended the hyperbolic functions methods. When $m \longrightarrow 0$, the Jacobi functions degenerate to the triangular functions and given the solutions by extended triangular functions methods. Moreover we can find a several solutions by By replacing $\mathrm{sn}\,\xi$ in the expansion (2.3) by other kinds of Jacobi functions and repeating the same process as before.

References

[1] T. B. Benjamin, J. L. Bona , J. J. Mahony, Model equations for long waves in nonlinear dispersive systems, Philos Trans R Soc London, Ser A 272 (1972) 47-78.

[2] P.G. Estévez, Ş. Kuru , J. Negro, L.M. Nieto, Travelling wave solutions of the generalized Benjamin-Bona-Mahony equation, Chaos, Solitons and Fractals 40 (2009) 20312040.

[3] T. Tso, Existence of solutions of the modified Benjamin-Bona-Mahony equation, Chin J Math, 24 (1996) 327-36.

[4] E. Yusufoğlu, A. Bekir, The tanh and the sine-cosine methods for exact solutions of the MBBM and the Vakhnenko equations. Chaos, Solitons and Fractals 38 (2008) 1126-1133.

[5] O. P. Layeni, A. P. Akinola, A new hyperbolic auxiliary function method and exact solutions of the mBBM equation, Commun Nonlinear Sci Numer Simulat. 15 (2010) 135-138.

[6] S. Abbasbandy, A. Shirzadi, The first integral method for modified Benjamin-Bona-Mahony equation, Commun Nonlinear Sci Numer Simulat 15 (2010) 1759-1764.

[7] A.S. Abdel Rady, E.S. Osman, M. Khalfallah, The homogeneous balance method and its applications for the Benjamin-Bona-Mahony equation, Appl. Math. Comput. 4 (217) (2010) 1385-1390.

[8] Y. Tang, W. Xu, L. Gao, J. Shen, An algebraic method with computerized symbolic computation for the one-dimensional generalized BBM equation of any order, Chaos Solitons Fractals 32 (2007) 1846-1852.

[9] Y. Chen, B. Li, H. Zhang, Exact solutions for two nonlinear wave equations with nonlinear terms of any order, Commun. Nonlinear Sci. Numer. Simul. 10 (2005) 133-138.

[10] J.Y. An, W.G. Zhang, Exact periodic solutions to generalized BBM equation and relevant conclusions, Acta Math. Appl. Sin. Engl. Ser. 3 (22) (2006) 509-516.

[11] A. M. Wazwaz, The Hirota's bilinear method and the tanh-coth method for multiple-soliton solutions of the Sawada-Kotera-Kadomtsev-Petviashvili equation, Applied Mathematics and Computation, 200 (1) (2008) 160-166.

[12] A.N. Beavers Jr., E.D. Denman, A new similarity transformation method for eigenvalues and eigenvectors, Mathematical Biosciences, 21 (12) (1974) 143-169.

[13] J. F. Zhang, Homogeneous balance method and chaotic and fractal solutions for the Nizhnik-Novikov-Veselov equation, Physics Letters A 313 (56) (2003) 401-407.

[14] Y. Huang, Y. Shang, The extended hyperbolic function method for generalized forms of nonlinear heat conduction and Huxley equations, Journal of Applied Mathematics, (2012) In press.

[15] A.H. Bhrawy, A. Biswas, M. Javidi, Wen-Xiu Ma, Z. Pinar, A. Yildirim, New Solutions for $(1+1)$-Dimensional and $(2+1)$-Dimensional Kaup-Kupershmidt Equations, Results. Math. (2012) DOI 10.1007/s00025-011-0225-7.

[16] H. Naher, F. A. Abdullah, M. A. Akbar, New Traveling Wave Solutions of the Higher Dimensional Nonlinear Partial Differential Equation by the Exp-Function Method, Journal of Applied Mathematics Volume 2012, Article ID 575387, 14 pages doi:10.1155/2012/575387

[17] A. H. Khater, D. K. Callebaut and M. A. Abdelkawy, Two-dimensional force-free magnetic fields described by some nonlinear equations, Phys. Plasmas 17 (2010) 122902 10pp.

[18] A. H. Khater, D. k. Callebaut and M. A. Abdelkawy, Nonlinear periodic solutions for isothermal magnetostatic atmospheres, 5-9 July ICCAM, Leuven, Belgium (2010).

[19] E. G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys Lett A 277 (2000) 212-218.

[20] S. K. Liu, Z. T. Fu, S. D. Liu, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys Lett A. 289 (2001) 69-74.

[21] K.A. Gepreel, A.R. Shehata, Jacobi elliptic solutions for nonlinear differential difference equations in mathematical physics, Journal of Applied Mathematics, (2012) In press.

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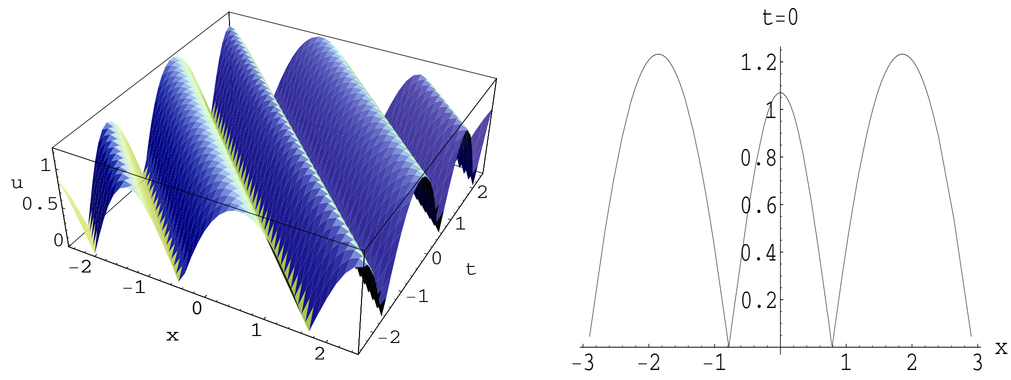

Fig. 1 The modulus of solitary wave solution u_1 (Eq. 3.14) where $m = a = 0.5$

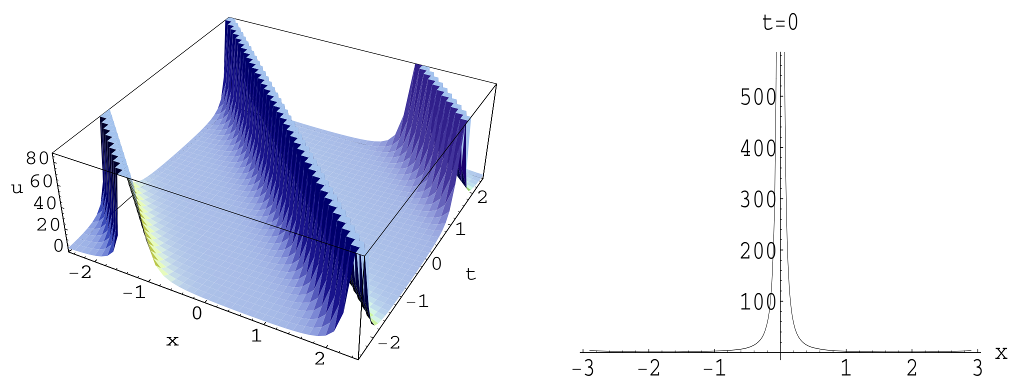

Fig. 2 The modulus of solitary wave solution u_2 (Eq. 3.15) where $m = a = 0.5$