EXTENDED LIMIT DESIGN THEOREMS FOR CONTINUOUS MEDIA[1]

BY

D. C. DRUCKER, W. PRAGER

Brown University

AND

H. J. GREENBERG

Carnegie Institute of Technology

Summary. Earlier results [1,2][2] on safe loads for a Prandtl-Reuss material subject to surface tractions or displacements which increase in ratio are here extended to any perfectly plastic material and any history of loading.

1. Introduction. The general limit design problem is concerned with a body or assemblage of bodies made of *perfectly plastic* (i.e. non-workhardening) material and subject to an arbitrary history of loading. In many cases, only the extreme values of the loads are given, but the order in which these loads are applied to the body is not specified. An important question is whether the body will "collapse", that is deform appreciably under essentially constant loads, or whether its deformation will be contained although substantial portions may go plastic. In engineering design, the actual problem is to insure a reasonable margin of safety against such collapse.

In the present paper a somewhat restricted form of this general problem is discussed: the actual history of loading is assumed to be completely specified rather than only the extreme values of all loads. The given loading history may be very simple; for instance, all loads may increase so as to preserve their ratios (*proportional loading*). On the other hand, the loading program may be very elaborate; additional loads may be superimposed on a state of initial stress, as is the case when traffic loads come on a bridge, or when torsion and bending are applied to a shot peened and hence prestressed axle.

The boundary conditions are assumed to be of the *stress type* for a single body or assemblage of bodies. At each point of the surface of the body or assemblage of bodies each component of the surface traction is specified except when the corresponding component of the displacement is prescribed to be zero.

Several terms must be introduced, and a number of concepts must be discussed before the main theorems can be stated and proved.

2. Perfect plasticity. In the following discussion, the general stress-strain relation for a perfectly plastic material will be used so that the results will apply to a wide variety of materials. By definition, a perfectly plastic material in simple tension has a stress-strain diagram of the form shown in Fig. 1. The essential feature of this diagram is the flat yield which produces a sharp boundary between elastic behavior and unrestricted plastic flow at points such as B in Fig. 1.

To describe this kind of behavior mathematically for more general types of loading, it is convenient to use tensor notation. Latin subscripts take the range of values 1, 2, 3,

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[2]Numbers in square brackets refer to the Bibliography at the end of the paper.

and the summation convention concerning repeated letter subscripts is adopted. The coordinates x_i used in the following are rectangular and Cartesian. Differentiation with respect to a coordinate is indicated by a comma followed by the appropriate subscript ($u_{i,j} = \partial u_i/\partial x_j$).

The mechanical behavior of a perfectly plastic material is completely characterized by its *yield function.* For a homogeneous material, the yield function f depends only on the nine stress components σ_{ij} ; it is positive definite and is symmetric with respect to the conjugate shearing stresses σ_{ij} and σ_{ji} ($i \neq j$) which are formally treated as independent variables. Plastic flow can occur only under states of stress for which $f = 1$.

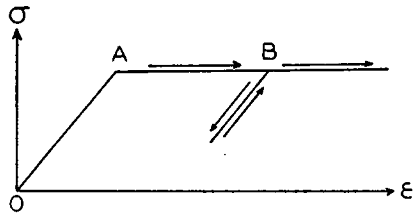

FIG. 1

States of stress for which $f > 1$ are not possible in a perfectly plastic material. For the completely stress-free state $f = 0$; for any other state of stress within the elastic range $0 < f < 1$. In the following, states of stress for which $f < 1$ will be called *safe.*

The most frequently used form of the yield function is $f = s_{ij}s_{ij}/2k^2$, where $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$ is the stress deviation and k the yield stress in simple shear. However, more complicated forms may be used to represent, for instance, various types of anisotropy. For a non-homogeneous material, the yield function may vary from particle to particle.

The stress-strain law of a perfectly plastic material does not contain the strain itself but only the strain rate. Since viscosity effects are disregarded, this law must contain the rates of stress and strain in a homogeneous manner. The strain rate ϵ_{ij} can be decomposed into an elastic component ϵ^e_{ij} and a plastic component ϵ^p_{ij} :

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij} . \tag{1}$$

The elastic strain rate ϵ^e_{ij} is related to the rate of stress σ'_{ij} by the generalized form of Hooke's law. For the following, it is not necessary to write down this relation between ϵ^e_{ij} and σ'_{ij} ; it suffices to note that

$$\epsilon^e_{ij}\sigma'_{ij} > 0 \qquad \text{except when} \qquad \epsilon^e_{ij} = \sigma'_{ij} = 0. \tag{2}$$

The plastic strain rate is related to the state of stress. Since viscosity effects are neglected, the stress components must be homogeneous of the order zero in the components of the plastic strain rate. In other terms, the stress tensor σ_{ij} determines the tensor of the plastic strain rate ϵ^p_{ij} only to within an arbitrary factor.

The following geometric representation of states of stress and plastic strain rate is often useful. In a nine-dimensional Euclidean space, consider a *fixed* system of rectangular Cartesian coordinates. The state of stress σ_{ij} will be represented by the point with coordinates proportional to σ_{ij} . The plastic strain rate ϵ^p_{ij} will be represented by the ray with direction cosines proportional to ϵ^p_{ij} . This manner of representing the plastic strain rate by a ray rather than a point is suggested by the fact that the state of stress determines the plastic strain rate only to within an arbitrary factor.

The yield condition $f = 1$ defines a surface in this nine-dimensional space. We assume this *yield surface* to be convex and to possess a unique normal in each of its points. The mechanical significance of this assumption will become clear from the following discussion.

Consider a state of stress for which $f = 1$. As has been shown in earlier papers [3, 4], any plastic strain rate ϵ^p_{ij} associated with this state of stress is represented by the ray which has the direction of the exterior normal of the yield surface at the point σ_{ij}. From this fact and the assumption regarding the yield surface, there follow two important lemmas.

Lemma 1. The stress rate σ'_{ij} and the plastic strain rate ϵ^p_{ij} satisfy

$$\sigma'_{ij}\epsilon^p_{ij} = 0. \tag{3}$$

Proof. If the plastic strain rate is not to vanish, the stress rate must correspond to the change from one point on the yield surface to a neighboring point. Thus, the stress rate is represented by a vector which is tangential to the yield surface, whereas the plastic strain rate is represented by a ray normal to the yield surface. Equation (3) expresses the orthogonality between this vector and ray. The special case $\sigma'_{ij} \equiv 0$ also satisfies Eq. (3).

Lemma 2. There exists a function F, homogeneous of the first order in the components of the plastic strain rate, such that in flow with the plastic strain rate ϵ^p_{ij}, energy is dissipated at the rate $F(\epsilon^p_{ij})$, whereas, for any safe state of stress σ^s_{ij},

$$\sigma^s_{ij}\epsilon^p_{ij} < F(\epsilon^p_{ij}). \tag{4}$$

Proof. If the plastic strain rate ϵ^p_{ij} is associated with the stress σ_{ij}, the rate of dissipation of energy is $\sigma_{ij}\epsilon^p_{ij}$. Since the stress components are homogeneous of the order zero

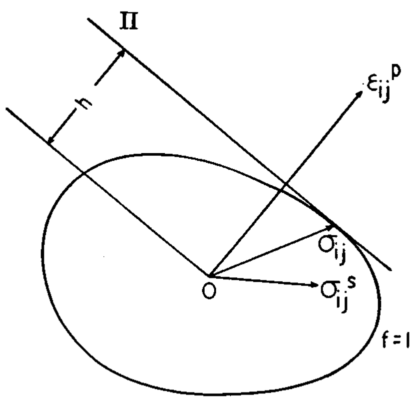

FIG. 2

in the components of the plastic strain rate, the dissipated energy is homogeneous of the first order in these strain rate components. Let Π be the tangent plane of the yield surface at the point σ_{ij} and h its distance from the origin 0 (Fig. 2). The rate of dissipation of energy is then given by $(\epsilon^p_{ij}\epsilon^p_{ij})^{1/2}h$. The inequality (4) expresses the fact that any safe state of stress is represented by a point which lies on the same side of Π as the origin. (Note that the function F is the supporting function of the convex yield surface.)

3. Collapse. From the point of view of the practical engineer, the term collapse implies that appreciable changes in the geometry of a structure will occur under essentially constant loads. For the purpose of the following discussion, however, it is more convenient to use the term collapse to refer to conditions for which plastic flow would occur under constant loads *if the accompanying change in the geometry of the structure or body were disregarded.* In the discussion of this type of collapse, the equilibrium conditions can be set up for the undeformed rather than the deformed body. This obviously represents a great simplification because the deformation occurring during collapse is not known beforehand but constitutes one of the unknowns of the collapse problem.

To illustrate the relation between these two definitions of collapse, consider first a hollow sphere of uniform wall thickness under gradually increasing interior pressure. The material at the interior surface reaches the yield limit first, but its plastic deformation is contained by the surrounding shell of still elastic material. As the pressure continues to increase, the boundary between the elastic and plastic regions moves towards the exterior surface; only when it has reached this surface does large plastic flow become possible. Let us now compare the simplified stress analysis which neglects all changes of geometry during the loading process to the complete analysis which takes account of these changes. Even in the elastic range, and also in the subsequent range of contained plastic deformation, the hollow sphere expands and its wall thickness diminishes somewhat. If those effects are taken into account, the pressure for which the entire sphere becomes plastic is found somewhat smaller than when they are neglected. However, the difference between the two pressure values is small because the deformations occurring up to the end of the range of contained plastic deformation are of the order of magnitude of elastic deformations. Continuing the analysis into the subsequent range of unrestricted plastic flow, we find that the simplified analysis predicts flow under constant interior pressure, whereas the complete analysis predicts flow under gradually decreasing pressure. In either case the sphere loses its usefulness as a pressure vessel. As far as this first example is concerned, the two definitions of collapse are therefore in substantial agreement as to the pressure under which collapse sets in.

Next, consider a blunt, rigid wedge which is pressed against the flat surface of a large block of perfectly plastic material. It is obvious that the term "collapse" can not be applied to this problem in the sense attributed to this term by the practical engineer. Indeed, as the wedge is pressed into the plastic material, the area of contact continues to increase; thus, the load on the wedge must be increased steadily if plastic flow is to be maintained. On the other hand, there exists a well-defined "collapse load" under which plastic flow would continue if all changes in geometry could be disregarded. The physical meaning of this "collapse load" is less clear than that of the collapse pressure obtained in the preceding example. Obviously, the collapse load in the case of the wedge is not the load under which the first indentation occurs, because some identation takes place even in the elastic range. It is likely, however, that the collapse load indicates the load intensity at which the permanent indentation begins to increase considerably faster than the elastic identation. Thus, the collapse load in the case of the wedge has the nature of a *conventional* yield limit, whereas the collapse pressure in the case of the hollow sphere represents a *natural* yield limit.

To arrive at a mathematical characterization of collapse, let the velocities be denoted by v_i, the surface tractions by T_i, and the body forces (per unit volume) by F_i. Furthermore, use the prime to indicate rates of change and the superscript c to refer to collapse. From the definition of collapse introduced above it follows then that during collapse

$$\int T_i'^c v_i^c \, dS = 0 \qquad \text{and} \qquad F_i'^c = 0 \qquad \text{for some} \qquad v_i^c \neq 0, \tag{5}$$

where the integration is extended over the surface S which bounds the considered body or assemblage of bodies. Indeed, since collapse is to occur under constant loads, the rates $F_i'^c$ must vanish throughout V; moreover, for those components of the surface traction which are prescribed by the boundary conditions the rates T_i' must vanish, whereas the surface velocities v_i^c corresponding to the remaining components of the surface traction must vanish according to our definition of stress boundary conditions. Thus, the integral in the first Eq. (5) is seen to vanish.

An expression for the rate at which work is done during collapse is useful. If the velocities v_i^c considered as functions of the coordinates are continuous and have continuous first derivatives, the principle of virtual displacements yields the following equation:

$$\int T_i^c v_i^c \, dS + \int F_i^c v_i^c \, dV = \int \sigma_{ij}^c \epsilon_{ij}^c \, dV. \tag{6}$$

4. Admissible states. Consider first a state of stress for which the components σ_{ij} are continuous functions of the coordinates. Such a state is called *statically admissible* if it satisfies (i) the conditions of equilibrium

$$\sigma_{ij,j} + F_i = 0 \tag{7}$$

throughout V and (ii) the boundary condition

$$\sigma_{ij} n_j = T_i \tag{8}$$

on those portions of the surface where the component T_i of the surface traction is given. In (8), the unit vector along the exterior normal of S is denoted by n_i.

The preceding definition may be generalized to include stress fields with a finite number of surfaces of discontinuity. On either side of such a surface, the stresses must then satisfy (7). Moreover, if n_j^* denotes the unit normal vector of the surface of discontinuity, the expression $\sigma_{ij} n_j^*$ must have the same value whether it is evaluated from the stresses on one or the other side of the surface of discontinuity.

A velocity field v_i is called *kinematically admissible* if the velocity component v_i vanishes on those portions of the surface S where the corresponding component T_i of the surface traction is not prescribed. A kinematically admissible velocity field may represent rigid body motions for certain portions of the body and genuine deformations for the remainder. Special discontinuous velocity fields are also permissible and often useful; they will be discussed in some detail later. For the present, however, we consider only kinematically admissible velocity fields for which the velocity components are continuous functions of the coordinates.

Such a velocity field is said to define a *kinematically admissible state of collapse* if the rate at which the actual surface tractions and body forces do work on the velocities v_i^k equals or exceeds the rate of dissipation of energy computed from the strain rates

$$\epsilon_{ij}^k = \tfrac{1}{2}(v_{i,j}^k + v_{j,i}^k) \tag{9}$$

treated as purely plastic strain rates. Thus, for a kinematically admissible collapse state

$$\int T_i v_i^k \, dS + \int F_i v_i^k \, dV \geq \int F(\epsilon_{ij}^k) \, dV. \tag{10}$$

5. Collapse theorems. The following theorems were previously established for special cases [1, 2]; they can now be shown to hold generally.

Theorem 1. If all changes in geometry occurring during collapse are neglected, all stresses are found to remain constant during collapse.

Proof. Applying the principle of virtual work to the velocity field and the rates of change of the surface tractions, body forces, stresses and strains during collpase, we obtain

$$\int T_i'^c v^c \, dS + \int F_i'^c v_i^c \, dV = \int \sigma_{ij}'^c \epsilon_{ij}^c \, dV. \tag{11}$$

According to (5), the left-hand side of (11) must vanish for the considered collapse state. The strain rate on the right-hand side of (11) can be decomposed into its elastic and plastic components; thus,

$$\int \sigma_{ij}'^c \epsilon_{ij}^{ec} \, dV + \int \sigma_{ij}'^c \epsilon_{ij}^{pc} \, dV = 0. \tag{12}$$

The second integral in (12) vanishes according to (3). The relation (2) shows then that (12) can be satisfied only if the stress rate $\sigma_{ij}'^c$ vanishes throughout V.

Theorem 2. If a safe statically admissible state of stress can be found at each stage of loading, collapse will not occur under the given loading schedule.

Proof. Suppose this theorem to be false. Then, at some stage of loading, a collapse state v_i^c would exist although a safe statically admissible state of stress σ_{ij}^s could be found. Applying the principle of virtual work to the actual surface tractions T_i^c and body forces F_i^c at this collapse stage, the stresses σ_{ij}^s with which these are in equilibrium, the velocities v_i^c , and the corresponding strain rates ϵ_{ij}^c , we obtain

$$\int T_i^c v_i^c \, dS + \int F_i^c v_i^c \, dV = \int \sigma_{ij}^s \epsilon_{ij}^c \, dV. \tag{13}$$

According to Theorem 1, the stresses and hence the elastic strain remain constant during the collapse described by the field v_i^c . Thus, the strain rate ϵ_{ij}^c is purely plastic. From the first part of Lemma 2 it follows then that the right-hand side of (6) can be written as $\int F(\epsilon_{ij}^c) \, dV$. Combining this form of (6) with (13), we obtain

$$\int \sigma_{ij}^s \epsilon_{ij}^c \, dV = \int F(\epsilon_{ij}^c) \, dV \tag{14}$$

which is in contradiction to the second part of Lemma 2, since the strain rate ϵ_{ij}^c is purely plastic.

Theorem 3. As long as collapse does not occur, a safe statically admissible state of stress can be found at each stage of loading.

To prove this converse of Theorem 2, consider a generic stage of loading defined by the surface tractions T_i and the body forces F_i. If the actual stresses at this stage of loading are denoted by σ_{ij}, we have $f(\sigma_{ij}) \leq 1$, because collapse is not supposed to occur.

Consider now the loading specified by NT_i and NF_i, where N is constant throughout the body. For $N = 1$, collapse does not occur according to the condition of the theorem. Since the equations of equilibrium are linear in the stresses, body forces, and surface tractions, the stresses $N\sigma_{ij}$ will be in equilibrium with NT_i and NF_i. Moreover, it follows from the convexity of the yield surface that $f(N\sigma_{ij}) < 1$ if $N < 1$. Thus, collapse can occur under the loads NT_i, NF_i only if $N > 1$. Let σ_{ij}^c denote the actual stresses for such a state of collapse. These stresses are in equilibrium with NT_i, NF_i and satisfy $f(\sigma_{ij}^c) \leq 1$. Therefore, the stresses σ_{ij}^c/N are in equilibrium with T_i, F_i and satisfy $f(\sigma_{ij}^c/N) < 1$; in other terms, these stresses define a safe statically admissible state of stress for the loads T_i, F_i.

Theorem 4. If a kinematically admissible collapse state can be found at any stage of loading, collapse must impend or have taken place previously.

Proof. Suppose this theorem to be false. According to Theorem 3, a safe state of stress σ_{ij}^s could then be found in spite of the existence of a kinematically admissible collapse state v_i^k. Applying the principle of virtual work to the actual surface tractions T_i and body forces F_i at the considered stage of loading, the stresses σ_{ij}^s with which these are in equilibrium, the velocities v_i^k and the corresponding strain rates ϵ_{ij}^k, we obtain

$$\int T_i v_i^k \, dS + \int F_i v_i^k \, dV = \int \sigma_{ij}^s \epsilon_{ij}^k \, dV. \tag{15}$$

On the other hand, we may use (10) since v_i^k is a kinematically admissible collapse state. Thus,

$$\int \sigma_{ij}^s \epsilon_{ij}^k \, dV \geq \int F(\epsilon_{ij}^k) \, dV \tag{16}$$

which is in contradiction to (4) since the strain rates ϵ_{ij}^k of a kinematically admissible collapse state are treated as purely plastic.

6. Discontinuous velocity fields. It is often useful to consider discontinuous velocity fields. However, it should be kept in mind that in plastic flow, as distinct from fracture, actual discontinuities cannot occur across a fixed surface. The type of discontinuity to be considered in the following is simply an idealization of a continuous distribution in which the velocity changes very rapidly across a thin transition layer (Fig. 3). This idealization is permissible provided the stresses on the assumed discontinuity surface are chosen as the limiting values of the stresses on the surfaces bounding the transition layer as the thickness of this layer approaches zero. For plane plastic flow in a Prandtl-Reuss material, for instance, the line of discontinuity must be a shear line for each of the stress fields on the two sides of the line of discontinuity. If the yield function f

depends upon the mean normal stress, it will be found that a discontinuity in tangential velocity implies separation or overlap of the material on the two sides of the discontinuity. In such a case, the actual transition layer must have appreciable thickness, but the idealization to a discontinuity surface may still be useful.*

The theorems of Sec. 5 are obviously valid in the presence of a transition layer. They will, therefore, remain valid in the limit as the thickness of the transition layer approaches

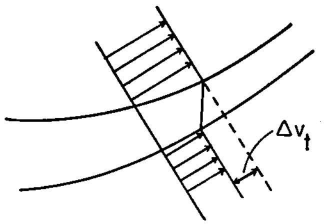

FIG. 3

zero, provided it is kept in mind that the rate of dissipation of energy in the transition layer approaches a finite value in the limit. If the transition layer is replaced by a surface of discontinuity, the expression $\int F(\epsilon_{ij})\, dV$ must, therefore, be replaced by

$$\int F(\epsilon_{ij})\, dV + \int T_i \Delta v_i\, dA \tag{17}$$

where dA is the element of area of the discontinuity surface, T_i the traction and Δv_i the velocity jump across this surface. Thus, in the definition (10) of a kinematically admissible collapse state, the right-hand side must be replaced by (17).

Actual discontinuity of velocity can occur in the case of an assemblage of bodies; for instance slip may occur between a punch and the indented material. If there is no friction between the bodies of an assemblage, Theorems 1 through 4 remain valid in spite of such discontinuities in the velocity because no energy is dissipated on the contact surface in the absence of friction.

7. Additional theorems. The following theorems are intuitively obvious but their statement and indication of proof seems worthwhile.

Theorem 5. Addition to the body of (weightless) material cannot result in a lower collapse load. The proof follows directly from the fact that the collapse stresses σ_{ij}^c for the original body and zero stresses in the added material constitute a statically admissible state for the new body.

Corollary. Removal of material cannot increase the collapse load.

If the yield surface of one material contains that of a second material, the first material will be said to have *higher yield strength* than the second.

Theorem 6. Increasing the yield strength of the material in any region of a perfectly plastic body cannot weaken the body.

The proof follows from the fact that any statically admissible state of stress which is safe for the unimproved body is also safe for the improved body.

*Application to soil mechanics provides an excellent illustration and will be treated in a separate paper.

Theorem 7. Initial stresses or deformations have no effect on collapse providing the geometry is essentially unaltered.

To prove this, we note that an initial or residual state does not affect any of the equations or statements made. This means, for example, that settlement of supports of a continuous structure or initial plastic torsion of a bar subsequently bent or pulled does not affect the limit loads provided the geometry is not changed appreciably.

It is probably best to end on a note of caution. Just as in elasticity, the concept of an essentially unaltered geometry rules out buckling which must, therefore, be studied separately.

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