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Extended Ohtsuka–Vălean Sums

Robert Reynolds * and **Allan Stauffer**

Department of Mathematics and Statistics, York University, Toronto, ON M3J 1P3, Canada; stauffer@yorku.ca
* Correspondence: milver@my.yorku.ca

Abstract: The Ohtsuka–Vălean sum is extended to evaluate an extensive number of trigonometric and hyperbolic sums and products. The sums are taken over finite and infinite domains defined in terms of the Hurwitz–Lerch zeta function, which can be simplified to composite functions in special cases of integer values of the parameters involved. The results obtained include generalizations of finite and infinite products and sums of tangent, cotangent, hyperbolic tangent and hyperbolic cotangent functions, in certain cases raised to a complex number power.

Keywords: Ohtsuka–Vălean sum; trigonometric functions; Cauchy integral; Hurwitz–Lerch zeta function

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1. Introduction

In the work of Ohtsuka [1] and Vălean [2] the sum involving the difference of two divergent series featuring the cosecant and hyperbolic cosecant functions was introduced, called the Ohtsuka–Vălean sum. In their work the integral over an infinite product with factors containing the secant and the hyperbolic secant with powers of two was studied. The study of divergent series is of high interest to the scientific community. Historically, studies on divergent series started in the work of Euler, Poisson, Fourier and Ramanjuan to name a few, and although it was not until Cauchy that the definitions of convergence were formally stated, these mathematicians were knowledgeable enough to know when a series converged and when it did not. Divergent series have been studied in the works of Silverman [3], Hardy [4], Candelpergher [5] and Mitschi et al. [6]. The product involving trigonometric functions have been studied in the works by Sommen [7] and Zotev [8].

In this paper, we apply the contour integral method in [9] to the Ohtsuka–Vălean sum to produce analogous sums involving the Hurwitz–Lerch zeta function. These sums will be used to derive new products and sums involving trigonometric functions. Double infinite sums in terms mathematical constants are also evaluated. The general flavour of this work is that we are able to write down both finite and infinite sums and products involving special functions and their composite functions. We are also able to compare the infinite form of a series and its partial sum, which could have interesting analysis properties. Our preliminaries start with a contour integral method and a few formulae. Let a, k, m and w be general complex numbers and $n \in [1, \infty)$, the contour integral form [9] of the Ohtsuka–Vălean sums are given by

$$\frac{1}{2\pi i}\int_C \sum_{n=1}^{\infty} a^w w^{-k-1}\left(\csc\left(2^{-n}(m+w)\right) - \operatorname{csch}\left(2^{-n}(m+w)\right)\right)dw = \frac{1}{2\pi i}\int_C a^w w^{-k-1}\left(\coth\left(\frac{m+w}{2}\right) - \cot\left(\frac{m+w}{2}\right)\right)dw, \quad (1)$$

$$\frac{1}{2\pi i}\int_C \sum_{j=0}^{n} a^w w^{-k-1}\left(\csc\left(2^{-j}(m+w)\right)-\operatorname{csch}\left(2^{-j}(m+w)\right)\right)dw$$
$$=\frac{1}{2\pi i}\int_C a^w w^{-k-1}\left(\cot\left(2^{-n-1}(m+w)\right)-\coth\left(2^{-n-1}(m+w)\right)\right.$$
$$\left.-\cot(m+w)+\coth(m+w)\right)dw. \tag{2}$$

The derivations follow the method used by us in [9]. This method involves using a form of the generalized Cauchy's integral formula given by

$$\frac{y^k}{\Gamma(k+1)}=\frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw, \tag{3}$$

where $y, w \in \mathbb{C}$ and C is in general an open contour in the complex plane where the bilinear concomitant [9] has the same value at the end points of the contour. This method involves using a form of Equation (3), multiplying both sides by a function and then taking the sum of both sides. This yields a sum in terms of a contour integral. Then, we multiply both sides of Equation (3) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

2. The Hurwitz–Lerch Zeta Function

We use Equation (1.11.3) in [10] where $\Phi(z,s,v)$ is the Hurwitz–Lerch zeta function, which is a generalization of the Hurwitz zeta $\zeta(s,v)$ and polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}(v+n)^{-s}z^n \tag{4}$$

where $|z|<1, v\neq 0,-1,-2,-3,\ldots$, and is continued analytically by its integral representation given by

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{5}$$

where $Re(v)>0$, and either $|z|\leq 1, z\neq 1$ and $Re(s)>0$, or $z=1$ and $Re(s)>1$. Almost all Hurwitz–Lerch zeta functions have an asymmetrical zero distribution [11].

3. Derivation of the Infinite Sums of the Contour Integral Representation

In this section, we use the Cauchy integral formula and the stated contour integral method to derive the infinite sum of the Hurwitz–Lerch zeta functions in terms of its contour integral representation.

3.1. Left-Hand Side First Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y\to\log(a)+i2^{-n}(2y+1)$, then multiply both sides by $-2ie^{im2^{-n}(2y+1)}$ and take the infinite sums over $y\in[0,\infty)$ and $n\in[1,\infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
\begin{aligned}
&-\sum_{n=1}^{\infty}\frac{i2^{k+1}(i2^{-n})^k e^{im2^{-n}}\Phi\left(e^{i2^{1-n}m},-k,\frac{1}{2}(1-i2^n\log(a))\right)}{\Gamma(k+1)}\\
&\qquad=-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{y=0}^{\infty}\int_C 2ia^w w^{-k-1}e^{i2^{-n}(2y+1)(m+w)}dw\\
&\qquad=-\frac{1}{2\pi i}\int_C\sum_{n=1}^{\infty}\sum_{y=0}^{\infty}2ia^w w^{-k-1}e^{i2^{-n}(2y+1)(m+w)}dw\\
&\qquad=\frac{1}{2\pi i}\int_C\sum_{n=1}^{\infty}a^w w^{-k-1}\csc\left(2^{-n}(m+w)\right)dw
\end{aligned}
\tag{6}
$$

from Equation (1.232.3) in [12], where $Re(m+w)>0$ and $Im(m+w)>0$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 177 in [13] as the summands are of bounded measure over the space $\mathbb{C}\times[1,\infty)\times[0,\infty)$.

3.2. Left-Hand Side Second Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y\to\log(a)+2^{-n}(2y+1)$, then multiply both sides by $2e^{m2^{-n}(2y+1)}$ and take the infinite sums over $y\in[0,\infty)$ and $n\in[1,\infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{2^{k+1}(2^{-n})^k e^{m2^{-n}}\Phi\left(e^{2^{1-n}m},-k,\frac{1}{2}(2^n\log(a)+1)\right)}{\Gamma(k+1)}\\
&\qquad=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{y=0}^{\infty}\int_C 2a^w w^{-k-1}e^{2^{-n}(2y+1)(m+w)}dw\\
&\qquad=\frac{1}{2\pi i}\int_C\sum_{n=1}^{\infty}\sum_{y=0}^{\infty}2a^w w^{-k-1}e^{2^{-n}(2y+1)(m+w)}dw\\
&\qquad=-\frac{1}{2\pi i}\int_C\sum_{n=1}^{\infty}a^w w^{-k-1}\operatorname{csch}\left(2^{-n}(m+w)\right)dw
\end{aligned}
\tag{7}
$$

from Equation (1.232.3) in [12] where $Re(m+w)>0$ and $Im(m+w)>0$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 177 in [13], as the summands are of bounded measure over the space $\mathbb{C}\times[1,\infty)\times[0,\infty)$.

3.3. Right-Hand Side First Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y\to\log(a)+i(y+1)$, then multiply both sides by $2ie^{im(y+1)}$ and take the infinite sums over $y\in[0,\infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
\begin{aligned}
&\frac{2ie^{\frac{1}{2}i(\pi k+2m)}\Phi(e^{im},-k,1-i\log(a))}{\Gamma(k+1)}\\
&\qquad=\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2ia^w w^{-k-1}e^{i(y+1)(m+w)}dw\\
&\qquad=\frac{1}{2\pi i}\int_C\sum_{y=0}^{\infty}2ia^w w^{-k-1}e^{i(y+1)(m+w)}dw\\
&\qquad=\frac{1}{2\pi i}\int_C\left(-a^w w^{-k-1}\cot\left(\frac{m+w}{2}\right)-ia^w w^{-k-1}\right)dw
\end{aligned}
\tag{8}
$$

from Equation (1.232.1) in [12], where $Re(m+w) > 0$ and $Im(m+w) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [13], as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

Derivation of the Additional Contour

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a)$, then multiply both sides by $-i$ and simplify to get

$$-\frac{i\log^k(a)}{\Gamma(k+1)} = -\frac{1}{2\pi i}\int_C i a^w w^{-k-1} dw \tag{9}$$

3.4. Right-Hand Side Second Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a) + y + 1$, then multiply both sides by $-2e^{m(y+1)}$ and take the infinite sums over $y \in [0, \infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$\begin{aligned}
-\frac{2e^m\Phi(e^m, -k, \log(a)+1)}{\Gamma(k+1)} \\
&= -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2a^w w^{-k-1} e^{(y+1)(m+w)} dw \\
&= -\frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} 2a^w w^{-k-1} e^{(y+1)(m+w)} dw \\
&= \frac{1}{2\pi i}\int_C \left(a^w w^{-k-1}\coth\left(\frac{m+w}{2}\right) + a^w w^{-k-1}\right) dw
\end{aligned} \tag{10}$$

from Equation (1.232.1) in [12], where $Re(m+w) > 0$ and $Im(m+w) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [13], as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

Derivation of the Additional Contour

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a)$, then simplify to get

$$\frac{\log^k(a)}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C a^w w^{-k-1} dw \tag{11}$$

4. Derivation of the Finite Sums of the Contour Integral Representation

In this section, we use the Cauchy integral formula and the stated contour integral method to derive the finite sum of the Hurwitz–Lerch zeta function in terms of its contour integral representation.

4.1. Left-Hand Side First Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a) + i2^{-j}(2y+1)$, then multiply both sides by $-2ie^{i2^{-j}m(2y+1)}$ and take the infinite and finite sums over $y \in [0, \infty)$ and $j \in [0, n]$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
\begin{aligned}
-\sum_{j=0}^{n} \frac{i2^{k+1}\left(i2^{-j}\right)^{k} e^{i2^{-j}m}\Phi\left(e^{i2^{1-j}m},-k,\frac{1}{2}\left(1-i2^{j}\log(a)\right)\right)}{\Gamma(k+1)} \\
&= -\frac{1}{2\pi i}\sum_{j=0}^{n}\sum_{y=0}^{\infty}\int_{C} 2ia^{w}w^{-k-1}e^{i2^{-j}(2y+1)(m+w)}dw \\
&= -\frac{1}{2\pi i}\int_{C}\sum_{j=0}^{n}\sum_{y=0}^{\infty} 2ia^{w}w^{-k-1}e^{i2^{-j}(2y+1)(m+w)}dw \\
&= \frac{1}{2\pi i}\int_{C}\sum_{j=0}^{n} a^{w}w^{-k-1}\csc\left(2^{-j}(m+w)\right)dw
\end{aligned} \tag{12}
$$

from Equation (1.232.3) in [12], where $Re(m+w)>0$ and $Im(m+w)>0$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 177 in [13], as the summands are of bounded measure over the space $\mathbb{C}\times[0,n]\times[0,\infty)$.

4.2. Left-Hand Side Second Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a) + 2^{-j}(2y+1)$, then multiply both sides by $2e^{2^{-j}m(2y+1)}$ and take the infinite sums over $y \in [0,\infty)$ and $j \in [0,n]$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
\begin{aligned}
\sum_{j=0}^{n} \frac{2^{k+1}\left(2^{-j}\right)^{k} e^{2^{-j}m}\Phi\left(e^{2^{1-j}m},-k,\frac{1}{2}\left(2^{j}\log(a)+1\right)\right)}{\Gamma(k+1)} \\
&= \frac{1}{2\pi i}\sum_{j=0}^{n}\sum_{y=0}^{\infty}\int_{C} 2a^{w}w^{-k-1}e^{2^{-j}(2y+1)(m+w)}dw \\
&= \frac{1}{2\pi i}\int_{C}\sum_{j=0}^{n}\sum_{y=0}^{\infty} 2a^{w}w^{-k-1}e^{2^{-j}(2y+1)(m+w)}dw \\
&= -\frac{1}{2\pi i}\int_{C}\sum_{j=0}^{n} a^{w}w^{-k-1}\operatorname{csch}\left(2^{-j}(m+w)\right)dw
\end{aligned} \tag{13}
$$

from Equation (1.232.3) in [12], where $Re(m+w)>0$ and $Im(m+w)>0$ in order for the sums to converge. We apply Tonelli's theorem for multiple sums, see page 177 in [13], as the summand is of bounded measure over the space $\mathbb{C}\times[0,n]\times[0,\infty)$.

4.3. Right-Hand Side First Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a) + 2i(y+1)$, then multiply both sides by $2ie^{2im(y+1)}$ and take the infinite sum over $y \in [0,\infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$
\begin{aligned}
\frac{(2i)^{k+1}e^{2im}\Phi\left(e^{2im},-k,1-\frac{1}{2}i\log(a)\right)}{\Gamma(k+1)} \\
&= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_{C} 2ia^{w}w^{-k-1}e^{2i(y+1)(m+w)}dw \\
&= \frac{1}{2\pi i}\int_{C}\sum_{y=0}^{\infty} 2ia^{w}w^{-k-1}e^{2i(y+1)(m+w)}dw \\
&= -\frac{1}{2\pi i}\int_{C}\left(a^{w}w^{-k-1}\cot(m+w)+ia^{w}w^{-k-1}\right)dw
\end{aligned} \tag{14}
$$

from Equation (1.232.1) in [12], where $Re(m+w) > 0$ and $Im(m+w) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [13], as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

Derivation of the Additional Contour

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a)$, then multiply both sides by $-i$ and simplify to get

$$-\frac{i\log^k(a)}{\Gamma(k+1)} = -\frac{1}{2\pi i}\int_C ia^w w^{-k-1} dw \tag{15}$$

4.4. Right-Hand Side Second Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a) + i2^{-n}(y+1)$, then multiply both sides by $-2ie^{im2^{-n}(y+1)}$ and take the infinite sum over $y \in [0, \infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$\begin{aligned}
&-\frac{2i(i2^{-n})^k e^{im2^{-n}}\Phi\left(e^{i2^{-n}m}, -k, 1 - i2^n\log(a)\right)}{\Gamma(k+1)} \\
&= -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2ia^w w^{-k-1} e^{i2^{-n}(y+1)(m+w)} dw \\
&= -\frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} 2ia^w w^{-k-1} e^{i2^{-n}(y+1)(m+w)} dw \\
&= \frac{1}{2\pi i}\int_C \left(a^w w^{-k-1}\cot\left(2^{-n-1}(m+w)\right) + ia^w w^{-k-1}\right) dw
\end{aligned} \tag{16}$$

from Equation (1.232.1) in [12], where $Re(m+w) > 0$ and $Im(m+w) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [13], as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

Derivation of the Additional Contour

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a)$, then multiply both sides by i and simplify to get

$$\frac{i\log^k(a)}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C ia^w w^{-k-1} dw \tag{17}$$

4.5. Right-Hand Side Third Contour Integral

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a) + 2(y+1)$, then multiply both sides by $-2e^{2m(y+1)}$ and take the infinite sum over $y \in [0, \infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$\begin{aligned}
&-\frac{2^{k+1}e^{2m}\Phi\left(e^{2m}, -k, \frac{\log(a)}{2} + 1\right)}{\Gamma(k+1)} \\
&= -\frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2a^w w^{-k-1} e^{2(y+1)(m+w)} dw \\
&= -\frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} 2a^w w^{-k-1} e^{2(y+1)(m+w)} dw \\
&= \frac{1}{2\pi i}\int_C \left(a^w w^{-k-1}\coth(m+w) + a^w w^{-k-1}\right) dw
\end{aligned} \tag{18}$$

from Equation (1.232.1) in [12], where $Re(m+w) > 0$ and $Im(m+w) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [13], as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

Derivation of the Additional Contour

Using a generalization of Cauchy's integral Formula (3), we first replace $y \to \log(a)$, then simplify to get

$$\frac{\log^k(a)}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_C a^w w^{-k-1} dw \tag{19}$$

4.6. Right-Hand Side Fourth Contour Integral

Using a generalization of Cauchy's integral Formula (3) we first replace $y \to \log(a) + 2^{-n}(y+1)$, then multiply both sides by $2e^{m2^{-n}(y+1)}$ and take the infinite sums over $y \in [0, \infty)$ and finally, we simplify in terms of the Hurwitz–Lerch zeta function to get

$$\begin{aligned}
&\frac{2(2^{-n})^k e^{m2^{-n}}\Phi\left(e^{2^{-n}m}, -k, 2^n\log(a)+1\right)}{\Gamma(k+1)} \\
&= \frac{1}{2\pi i}\sum_{y=0}^{\infty}\int_C 2a^w w^{-k-1} e^{2^{-n}(y+1)(m+w)} dw \\
&= \frac{1}{2\pi i}\int_C \sum_{y=0}^{\infty} 2a^w w^{-k-1} e^{2^{-n}(y+1)(m+w)} dw \\
&= -\frac{1}{2\pi i}\int_C \left(a^w w^{-k-1}\coth\left(2^{-n-1}(m+w)\right) + a^w w^{-k-1}\right) dw
\end{aligned} \tag{20}$$

from Equation (1.232.1) in [12], where $Re(m+w) > 0$ and $Im(m+w) > 0$ in order for the sum to converge. We apply Fubini's theorem for integrals and sums, see page 178 in [13], as the summand is of bounded measure over the space $\mathbb{C} \times [0, \infty)$.

Derivation of the Additional Contour

Using a generalization of Cauchy's integral Formula (3) we first replace $y \to \log(a)$, then multiply both sides by -1 and simplify to get

$$-\frac{\log^k(a)}{\Gamma(k+1)} = -\frac{1}{2\pi i}\int_C a^w w^{-k-1} dw \tag{21}$$

5. Finite and Infinite Sums of the Hurwitz–Lerch Zeta Functions in Terms of the Hurwitz–Lerch Zeta Functions

In this section, we develop the main theorems used in this work to evaluate special cases and produce double sum formulae and a Table of infinite products of trigonometric functions.

Theorem 1. *For all $k, a, m \in \mathbb{C}$ then,*

$$\begin{aligned}
&\sum_{n=1}^{\infty} 2^{k+1}\left(\left(2^{-n}\right)^k e^{m2^{-n}}\Phi\left(e^{2^{1-n}m}, -k, \frac{1}{2}(2^n\log(a)+1)\right)\right. \\
&\left.\quad -i\left(i2^{-n}\right)^k e^{im2^{-n}}\Phi\left(e^{i2^{1-n}m}, -k, \frac{1}{2}(1-i2^n\log(a))\right)\right) \\
&= 2ie^{\frac{1}{2}i(\pi k+2m)}\Phi\left(e^{im}, -k, 1-i\log(a)\right) - 2e^m\Phi(e^m, -k, \log(a)+1) + (-1+i)\log^k(a)
\end{aligned} \tag{22}$$

Proof. Since the addition of the right-hand sides of Equations (6) and (7) is equivalent to the addition of the right-hand sides of Equations (8)–(11), we may equate the left-hand sides, apply Equation (3.303) in [2] and simplify the Gamma function to yield the stated result. □

Theorem 2. *For all $k, a, m \in \mathbb{C}, n \in \mathbb{Z}_+$ then,*

$$\sum_{j=0}^{n} 2^{k+1}\left(\left(2^{-j}\right)^k e^{2^{-j}m}\Phi\left(e^{2^{1-j}m}, -k, \frac{1}{2}\left(2^j \log(a)+1\right)\right)\right.$$
$$\left.-i\left(i2^{-j}\right)^k e^{i2^{-j}m}\Phi\left(e^{i2^{1-j}m}, -k, \frac{1}{2}\left(1-i2^j \log(a)\right)\right)\right)$$
$$= (2i)^{k+1}e^{2im}\Phi\left(e^{2im}, -k, 1-\frac{1}{2}i\log(a)\right) - 2\left(i\left(i2^{-n}\right)^k e^{im2^{-n}}\Phi\left(e^{i2^{-n}m}, -k, 1-i2^n \log(a)\right)\right.$$
$$\left.-\left(2^{-n}\right)^k e^{m2^{-n}}\Phi\left(e^{2^{-n}m}, -k, 2^n \log(a)+1\right) + 2^k e^{2m}\Phi\left(e^{2m}, -k, \frac{\log(a)}{2}+1\right)\right) \tag{23}$$

Proof. Since the addition of the right-hand sides of Equations (12) and (13) is equivalent to the addition of the right-hand sides of Equations (14)–(21), we may equate the left-hand sides, apply Equation (3.303) in [2] and simplify the Gamma function to yield the stated result. □

6. Special Cases and Table of Infinite Products of Trigonometric Functions

In this section, we will evaluate Equations (22) and (23) for various values of the parameters to derive special cases of the Hurwitz–Lerch zeta function in terms of composite functions.

Example 1. *The degenerate infinite case.*

$$\sum_{n=1}^{\infty}\left(\csc\left(m2^{-n}\right) - csch\left(m2^{-n}\right)\right) = \coth\left(\frac{m}{2}\right) - \cot\left(\frac{m}{2}\right) \tag{24}$$

Proof. Use Equation (22), set $k = 0$ and simplify using entry (2) in Table of Section (64:12) in [14]. □

Example 2. *The degenerate finite case.*

$$\sum_{j=0}^{n}\left(\csc\left(2^{-j}m\right) - csch\left(2^{-j}m\right)\right) = \cot\left(m2^{-n-1}\right) - \coth\left(m2^{-n-1}\right) - \cot(m) + \coth(m) \tag{25}$$

Proof. Use Equation (23), set $k = 0$ and simplify using entry (2) in Table of Section (64:12) in [14]. □

Example 3. *A sum in terms of the exponential function.*

$$\sum_{n=1}^{\infty} 2^{-n-2}\left(-\csc^2\left(2^{-n-2}\right) + \sec^2\left(2^{-n-2}\right) + 2\sinh\left(2^{-n}\right)csch^3\left(2^{-n-1}\right)\right)$$
$$= \frac{1}{2}\csc^2\left(\frac{1}{4}\right) - \frac{2\sqrt{e}}{\left(\sqrt{e}-1\right)^2} \tag{26}$$

Proof. Use Equation (22), set $k = 1, a = 1, m = 1/2$ and simplify using entry (3) in Table of Section (64:12) in [14]. □

Example 4. *A double sum in terms of the product hyperbolic functions.*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} 2^{-2(j+n+1)}\left(-4\csc^3\left(\pi 2^{-j}\right)+2\csc\left(\pi 2^{-j}\right)+\left(\cosh\left(\pi 2^{1-j}\right)+3\right)csch^3\left(\pi 2^{-j}\right)\right)$$
$$\left(\left(\cos\left(\pi 2^{1-n}\right)+3\right)\csc^3\left(\pi 2^{-n}\right)-\left(\cosh\left(\pi 2^{1-n}\right)+3\right)csch^3\left(\pi 2^{-n}\right)\right)$$
$$=-\frac{1}{16}\sinh^2(\pi)csch^8\left(\frac{\pi}{2}\right) \tag{27}$$

Proof. Form two equations using (22) by setting $k=2, a=1, m=\pi$, and $k=2, a=1, m=-\pi$, replace $n\to j$, then multiply both equations and simplify using entry (2) in Table of Section (64:12) in [14]. □

Example 5. *A double sum in terms of the product hyperbolic functions raised to a power.*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} 2^{-j-n-2}\left(-\csc^2\left(\pi 2^{-j-3}\right)+\sec^2\left(\pi 2^{-j-3}\right)+4\coth\left(\pi 2^{-j-2}\right)csch\left(\pi 2^{-j-2}\right)\right)$$
$$\left(\coth\left(\pi 2^{-n-2}\right)csch\left(\pi 2^{-n-2}\right)-\cot\left(\pi 2^{-n-2}\right)\csc\left(\pi 2^{-n-2}\right)\right)$$
$$=\left(2+\sqrt{2}-\frac{1}{2}csch^2\left(\frac{\pi}{8}\right)\right)^2 \tag{28}$$

Proof. Form two equations using (22) by setting $k=2, a=1, m=\frac{\pi}{4}$ and $k=2, a=1, m=-\frac{\pi}{4}$, replace $n\to j$, then multiply both equations and simplify using entry (3) in Table of Section (64:12) in [14]. □

Example 6. *A double sum in terms of the product hyperbolic functions raised to a power.*

$$\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} 2^{-j-n-4}\left(-\csc^2\left(\pi 2^{-j-2}\right)+\sec^2\left(\pi 2^{-j-2}\right)+2\sinh\left(\pi 2^{-j}\right)csch^3\left(\pi 2^{-j-1}\right)\right)$$
$$\left(-\csc^2\left(\pi 2^{-n-2}\right)+\sec^2\left(\pi 2^{-n-2}\right)+2\sinh\left(\pi 2^{-n}\right)csch^3\left(\pi 2^{-n-1}\right)\right)$$
$$=\frac{1}{4}\left(csch^2\left(\frac{\pi}{4}\right)-2\right)^2 \tag{29}$$

Proof. Form two equations using (22) by setting $k=1, a=1, m=\frac{\pi i}{2}$ and $k=1, a=1, m=-\frac{\pi i}{2}$, replace $n\to j$, then multiply both equations and simplify using entry (3) in Table of Section (64:12) in [14]. □

Example 7. *An infinite product involving the exponential of trigonometric functions.*

$$\prod_{n=1}^{\infty}\exp\left(\csc\left(\frac{2^{-n}x}{\beta}\right)-csch\left(\frac{2^{-n}x}{\beta}\right)-\csc\left(2^{-n}x\right)+csch\left(2^{-n}x\right)\right)$$
$$\left(\cot\left(2^{-n-1}x\right)\tanh\left(2^{-n-1}x\right)\tan\left(\frac{2^{-n-1}x}{\beta}\right)\coth\left(\frac{2^{-n-1}x}{\beta}\right)\right)^{2^n\log(a)}$$
$$=a^{\frac{(1+i)(\beta-1)x}{\beta}}e^{-\cot\left(\frac{x}{2\beta}\right)+\coth\left(\frac{x}{2\beta}\right)+\cot\left(\frac{x}{2}\right)-\coth\left(\frac{x}{2}\right)}$$
$$\left(e^{-\frac{\left(\frac{1}{2}+\frac{i}{2}\right)(\beta-1)x}{\beta}}\sin\left(\frac{x}{2}\right)csch\left(\frac{x}{2}\right)\csc\left(\frac{x}{2\beta}\right)\sinh\left(\frac{x}{2\beta}\right)\right)^{2\log(a)} \tag{30}$$

Proof. Use Equation (22), set $k=1, m=x$ and apply the method in Section (8) in [15]. □

Example 8. *Extended Vălean forms. We write down two forms used to evaluate the preceding infinite products. This equation can be considered an extended form of Equation (23) in [16].*

$$\prod_{n=1}^{\infty}\left(\cot\left(m2^{-n-1}\right)\tanh\left(m2^{-n-1}\right)\tan\left(2^{-n-1}r\right)\coth\left(2^{-n-1}r\right)\right)^{2^n} = \frac{\left(-1+e^{im}\right)^2\left(e^r-1\right)^2}{\left(e^m-1\right)^2\left(-1+e^{ir}\right)^2} \tag{31}$$

or

$$\prod_{n=1}^{\infty}\left(\frac{\tan(2^{-n}r)\tanh(2^{-n}m)}{\tan(2^{-n}m)\tanh(2^{-n}r)}\right)^{2^n} = \left(\frac{\sin(m)\sinh(r)}{\sin(r)\sinh(m)}\right)^2 \tag{32}$$

Proof. Use Equation (22) and form a second equation by replacing m by r, then take their difference and set $k=-1, a=1$ and finally, simplify using entry (3) in Table of Section (64:12) in [14]. The second Equation (32) is derived by multiplying Equation (31) by itself after replacing $m\to -m, r\to -r$ and simplifying. □

Example 9. *Product of trigonometric functions raised to a power expressed in terms of a hyperbolic function.*

$$\prod_{n=1}^{\infty}\left(\tan\left(\pi 2^{-n-2}\right)\cot\left(\pi 2^{-n-3}\right)\tanh\left(\pi 2^{-n-3}\right)\coth\left(\pi 2^{-n-2}\right)\right)^{2^{n+1}} = 8\left(3-2\sqrt{2}\right)\cosh^4\left(\frac{\pi}{8}\right) \tag{33}$$

Proof. Use Equation (32) and set $m=\frac{\pi}{4}, r=\frac{\pi}{2}$; simplify to yield the stated result. □

Example 10. *Product of trigonometric functions raised to a power expressed in terms of a hyperbolic functions.*

$$\prod_{n=1}^{\infty}\left(\tan\left(\pi 2^{-n-2}\right)\cot\left(\frac{1}{3}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-1}\right)\coth\left(\pi 2^{-n-2}\right)\right)^{2^{n+1}} = \frac{1}{64}\left(1+2\cosh\left(\frac{\pi}{6}\right)\right)^4 sech^4\left(\frac{\pi}{12}\right) \tag{34}$$

Proof. Use Equation (32), set $m=\frac{\pi}{3}, r=\frac{\pi}{2}$ and simplify to yield the stated result. □

Example 11. *Product of trigonometric functions raised to a power expressed in terms of a hyperbolic functions.*

$$\prod_{n=1}^{\infty}\left(\tan\left(\pi 2^{-n-3}\right)\cot\left(\frac{1}{3}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-1}\right)\coth\left(\pi 2^{-n-3}\right)\right)^{2^{n+1}} = \frac{1}{512}\left(3+2\sqrt{2}\right)sech^4\left(\frac{\pi}{24}\right)\left(2+sech\left(\frac{\pi}{12}\right)\right)^4 \tag{35}$$

Proof. Use Equation (32), set $m=\frac{\pi}{3}, r=\frac{\pi}{4}$ and simplify to yield the stated result. □

Example 12. *Product of trigonometric functions raised to a power expressed in terms of the quotient of hyperbolic functions.*

$$\begin{aligned}&\prod_{n=1}^{\infty}\left(\tan\left(\frac{1}{5}\pi 2^{-n-1}\right)\cot\left(\pi 2^{-n-2}\right)\tanh\left(\pi 2^{-n-2}\right)\coth\left(\frac{1}{5}\pi 2^{-n-1}\right)\right)^{2^{n+1}}\\&\quad=\frac{256\cosh^4\left(\frac{\pi}{20}\right)}{\left(\sqrt{5}-3\right)^2\left(1+2\cosh\left(\frac{\pi}{10}\right)+2\cosh\left(\frac{\pi}{5}\right)\right)^4}\end{aligned} \tag{36}$$

Proof. Use Equation (32), set $m=\frac{\pi}{2}, r=\frac{\pi}{5}$ and simplify to yield the stated result. □

Example 13. *Product of trigonometric functions raised to a power expressed in terms of a hyperbolic function.*

$$\prod_{n=1}^{\infty}\left(\tan\left(\pi 2^{-n-2}\right)\cot\left(\pi 2^{-n-1}\right)\tanh\left(\pi 2^{-n-1}\right)\coth\left(\pi 2^{-n-2}\right)\right)^{2^{n+1}}=\frac{1}{4}sech^4\left(\frac{\pi}{4}\right) \tag{37}$$

Proof. Use Equation (32), set $m=\pi, r=\frac{\pi}{2}$ and simplify to yield the stated result. □

Example 14. *Product of trigonometric functions with a complex angle raised to a power expressed in terms of the quotient of radicals.*

$$\begin{aligned}&\prod_{n=1}^{\infty}\left(\frac{\tan\left(\left(\frac{8}{3}+i\right)2^{-3-n}\pi\right)\tanh\left(\left(\frac{1}{3}+\frac{i}{5}\right)2^{-1-n}\pi\right)}{\tan\left(\left(\frac{1}{3}+\frac{i}{5}\right)2^{-1-n}\pi\right)\tanh\left(\left(\frac{8}{3}+i\right)2^{-3-n}\pi\right)}\right)^{2^{1+n}}\\&\quad=-\frac{64i(-1)^{\frac{1}{15}-\frac{17i}{30}}\left(i+(-1)^{-\frac{1}{6}+\frac{i}{5}}\right)^4\left(\sqrt{2}-(1-i)e^{-\frac{2\pi}{3}}\right)^4}{\left(i+(-1)^{\frac{1}{6}+\frac{i}{4}}\right)^4\left(1+\sqrt{5}+i\sqrt{2\left(5-\sqrt{5}\right)}-4e^{-\frac{\pi}{3}}\right)^4}\end{aligned} \tag{38}$$

Proof. Use Equation (32) and form two equations by setting $m=\left(\frac{1}{3}+\frac{i}{5}\right)\pi, r=\left(\frac{2}{3}+\frac{i}{4}\right)\pi$ and $m=-\left(\frac{1}{3}+\frac{i}{5}\right)\pi, r=-\left(\frac{2}{3}+\frac{i}{4}\right)\pi$, then multiply and simplify to yield the stated result. □

Example 15. *A finite product involving the exponential of trigonometric functions.*

$$\begin{aligned}&\prod_{j=0}^{n}\exp\left(\csc\left(\frac{2^{-j}x}{\beta}\right)-csch\left(\frac{2^{-j}x}{\beta}\right)-\csc\left(2^{-j}x\right)+csch\left(2^{-j}x\right)\right)\\&\qquad\left(\cot\left(2^{-j-1}x\right)\tanh\left(2^{-j-1}x\right)\tan\left(\frac{2^{-j-1}x}{\beta}\right)\coth\left(\frac{2^{-j-1}x}{\beta}\right)\right)^{2^j\log(a)}\\&=\exp\left(\cot\left(\frac{2^{-n-1}x}{\beta}\right)-\coth\left(\frac{2^{-n-1}x}{\beta}\right)-\cot\left(2^{-n-1}x\right)+\coth\left(2^{-n-1}x\right)\right)\\&\qquad\exp\left(-\cot\left(\frac{x}{\beta}\right)+\coth\left(\frac{x}{\beta}\right)+\cot(x)-\coth(x)\right)\\&\qquad a^{\log\left(\left(\csc\left(2^{-n-1}x\right)\sinh\left(2^{-n-1}x\right)\sin\left(\frac{2^{-n-1}x}{\beta}\right)csch\left(\frac{2^{-n-1}x}{\beta}\right)\right)^{2^{n+1}}\right)+\log\left(\sin(x)csch(x)\csc\left(\frac{x}{\beta}\right)\sinh\left(\frac{x}{\beta}\right)\right)}\end{aligned} \tag{39}$$

Proof. Use Equation (23), set $k=1, m=x$ and apply the method in Section (8) in [15]. □

Example 16. *A finite product involving the product of trigonometric functions raised to a power.*

$$\prod_{j=0}^{n}\left(\cot\left(2^{-j-1}m\right)\tanh\left(2^{-j-1}m\right)\tan\left(2^{-j-1}r\right)\coth\left(2^{-j-1}r\right)\right)^{2^j}$$
$$=e^{(-1+i)(m-r)}\sin(m)csch(m)\csc(r)\sinh(r)\left(\frac{\left(e^{m2^{-n}}-1\right)\left(-1+e^{i2^{-n}r}\right)}{\left(-1+e^{im2^{-n}}\right)\left(e^{2^{-n}r}-1\right)}\right)^{2^{n+1}} \tag{40}$$

and

$$\prod_{j=0}^{n}\left(\cot\left(2^{-j-1}m\right)\tanh\left(2^{-j-1}m\right)\tan\left(2^{-j-1}r\right)\coth\left(2^{-j-1}r\right)\right)^{2^{j+1}}$$
$$=\sin^2(m)csch^2(m)\csc^2(r)\sinh^2(r)$$
$$\left(\csc^2\left(m2^{-n-1}\right)\sinh^2\left(m2^{-n-1}\right)\sin^2\left(2^{-n-1}r\right)csch^2\left(2^{-n-1}r\right)\right)^{2^{n+1}} \tag{41}$$

Proof. Use Equation (23), form a second equation by replacing m by r, take their difference, set $k=-1, a=1$ and simplify using entry (3) in Table of Section (64:12) in [14]. The second Equation (41) is derived by multiplying Equation (40) by itself after replacing $m\to -m, r\to -r$ and simplifying. □

Example 17. *A trigonometric identity.*

$$128\sqrt{2}\sin^8\left(\frac{\pi}{32}\right)\csc^8\left(\frac{\pi}{16}\right)\cosh^8\left(\frac{\pi}{32}\right)sech\left(\frac{\pi}{4}\right)=\frac{\sec^8\left(\frac{\pi}{32}\right)\cosh^8\left(\frac{\pi}{32}\right)sech\left(\frac{\pi}{4}\right)}{\sqrt{2}} \tag{42}$$

Proof. Use Equation (40) and set $m=\frac{\pi}{2}, r=\frac{\pi}{4}, n=2$ and simplify. □

Example 18. *Hurwitz–Lerch zeta transformation formula. This is a very simple and direct method for developing the theory of the Hurwitz–Lerch zeta function. It may be noted that Hurwitz's series can be obtained immediately upon specializing one of the parameters,*

$$\Phi(e^{-\alpha},s,v)=e^{-\alpha/2}(-2^{-s})\Phi\left(e^{-2\alpha},s,\frac{v}{2}+\frac{1}{4}\right)-e^{-3\alpha/2}2^{-s}\Phi\left(e^{-2\alpha},s,\frac{v}{2}+\frac{3}{4}\right)$$
$$+2^s\Phi\left(e^{-\alpha/2},s,2v\right)+i2^se^{\left(\frac{1}{2}+\frac{i}{2}\right)\alpha+\frac{i\pi s}{2}}\Phi\left(e^{\frac{i\alpha}{2}},s,2iv+(1-i)\right)$$
$$-ie^{\left(\frac{1}{2}+\frac{i}{2}\right)\alpha+\frac{i\pi s}{2}}\Phi\left(e^{i\alpha},s,iv+\left(\frac{1}{2}-\frac{i}{2}\right)\right)-i2^{-s}e^{\left(\frac{1}{2}+2i\right)\alpha+\frac{i\pi s}{2}}\Phi\left(e^{2i\alpha},s,\frac{iv}{2}+\left(1-\frac{i}{4}\right)\right)$$
$$-i2^{-s}e^{\left(\frac{1}{2}+i\right)\alpha+\frac{i\pi s}{2}}\Phi\left(e^{2i\alpha},s,\frac{1}{4}i(2v-(1+2i))\right) \tag{43}$$

Proof. Use Equation (23), set $n=1$, replace $m\to\alpha, k\to s, a\to v$ and simplify. This is a new form for the Hurwitz–Lerch zeta function where another form is given by Equation (12) in [17]. □

7. Table of Infinite Products of Trigonometric and Hyperbolic Functions

In this section, we evaluate Equation (32) for various imaginary values of the parameters m and r. For example, the first equation in Table 1 is obtained when $m=\pi i, r=\frac{\pi i}{2}$.

Table 1. Table of Infinite Products in terms of Mathematical Constants.

$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-1})\cot(\pi 2^{-n-2})\tanh(\pi 2^{-n-2})\coth(\pi 2^{-n-1})\right)^{2^n}$	$2\cosh^2\left(\frac{\pi}{4}\right)$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-1})\cot\left(\frac{1}{3}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-1}\right)\coth(\pi 2^{-n-1})\right)^{2^n}$	$\frac{1}{4}\left(1+2\cosh\left(\frac{\pi}{3}\right)\right)^2$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-1})\cot(\pi 2^{-n-3})\tanh(\pi 2^{-n-3})\coth(\pi 2^{-n-1})\right)^{2^n}$	$4\sqrt{2\left(3-2\sqrt{2}\right)}\cosh^2\left(\frac{\pi}{8}\right)\cosh^2\left(\frac{\pi}{4}\right)$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-1})\cot\left(\frac{1}{5}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{5}\pi 2^{-n-1}\right)\coth(\pi 2^{-n-1})\right)^{2^n}$	$\frac{1}{4}\sqrt{\frac{1}{2}\left(7-3\sqrt{5}\right)}\left(1+2\cosh\left(\frac{\pi}{5}\right)+2\cosh\left(\frac{2\pi}{5}\right)\right)^2$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-1})\cot\left(\frac{1}{3}\pi 2^{-n-2}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-2}\right)\coth(\pi 2^{-n-1})\right)^{2^n}$	$\sqrt{7-4\sqrt{3}}\left(\cosh\left(\frac{\pi}{12}\right)+\cosh\left(\frac{\pi}{4}\right)+\cosh\left(\frac{5\pi}{12}\right)\right)^2$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-2})\cot\left(\frac{1}{7}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{7}\pi 2^{-n-1}\right)\coth(\pi 2^{-n-2})\right)^{2^n}$	$2\sin^2\left(\frac{\pi}{14}\right)\sinh^2\left(\frac{\pi}{4}\right)\operatorname{csch}^2\left(\frac{\pi}{14}\right)$
$\prod_{n=1}^{\infty}\left(\tan\left(\frac{1}{3}\pi 2^{-n-1}\right)\cot(\pi 2^{-n-3})\tanh(\pi 2^{-n-3})\coth\left(\frac{1}{3}\pi 2^{-n-1}\right)\right)^{2^n}$	$\frac{4\sqrt{2\left(3-2\sqrt{2}\right)}\left(\cosh\left(\frac{\pi}{24}\right)+\cosh\left(\frac{\pi}{8}\right)\right)^2}{\left(1+2\cosh\left(\frac{\pi}{12}\right)\right)^2}$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-3})\cot\left(\frac{1}{5}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{5}\pi 2^{-n-1}\right)\coth(\pi 2^{-n-3})\right)^{2^n}$	$\frac{\sqrt{\left(3+2\sqrt{2}\right)\left(7-3\sqrt{5}\right)}\left(1+2\cosh\left(\frac{\pi}{20}\right)+2\cosh\left(\frac{\pi}{10}\right)\right)^2}{8\left(\cosh\left(\frac{\pi}{40}\right)+\cosh\left(\frac{3\pi}{40}\right)\right)^2}$
$\prod_{n=1}^{\infty}\left(\tan\left(\frac{1}{5}\pi 2^{-n-1}\right)\cot\left(\frac{1}{3}\pi 2^{-n-2}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-2}\right)\coth\left(\frac{1}{5}\pi 2^{-n-1}\right)\right)^{2^n}$	$\frac{8\sqrt{7-4\sqrt{3}}\left(\cosh\left(\frac{\pi}{60}\right)+\cosh\left(\frac{\pi}{20}\right)+\cosh\left(\frac{\pi}{12}\right)\right)^2}{\left(3-\sqrt{5}\right)\left(1+2\cosh\left(\frac{\pi}{30}\right)+2\cosh\left(\frac{\pi}{15}\right)\right)^2}$
$\prod_{n=1}^{\infty}\left(\tan\left(\frac{1}{3}\pi 2^{-n-2}\right)\cot\left(\frac{1}{7}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{7}\pi 2^{-n-1}\right)\coth\left(\frac{1}{3}\pi 2^{-n-2}\right)\right)^{2^n}$	$\frac{\sqrt{7+4\sqrt{3}}\sin^2\left(\frac{\pi}{14}\right)\left(1+2\cosh\left(\frac{\pi}{42}\right)+2\cosh\left(\frac{\pi}{21}\right)+2\cosh\left(\frac{\pi}{14}\right)\right)^2}{\left(\cosh\left(\frac{\pi}{84}\right)+\cosh\left(\frac{\pi}{28}\right)+\cosh\left(\frac{5\pi}{84}\right)\right)^2}$
$\prod_{n=1}^{\infty}\left(\tan(3\pi 2^{-n-3})\cot\left(\frac{1}{3}\pi 2^{-n-2}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-2}\right)\coth(3\pi 2^{-n-3})\right)^{2^n}$	$\sqrt{\frac{1}{2}\left(3-2\sqrt{2}\right)\left(7-4\sqrt{3}\right)}\sinh^2\left(\frac{3\pi}{8}\right)\operatorname{csch}^2\left(\frac{\pi}{12}\right)$
$\prod_{n=1}^{\infty}\left(\tan\left(\frac{1}{5}\pi 2^{1-n}\right)\cot\left(\frac{5}{3}\pi 2^{-n-2}\right)\tanh\left(\frac{5}{3}\pi 2^{-n-2}\right)\coth\left(\frac{1}{5}\pi 2^{1-n}\right)\right)^{2^n}$	$\sqrt{\frac{2\left(7+4\sqrt{3}\right)}{5\left(3+\sqrt{5}\right)}}\sinh^2\left(\frac{2\pi}{5}\right)\operatorname{csch}^2\left(\frac{5\pi}{12}\right)$
$\prod_{n=1}^{\infty}\left(\tan\left(\frac{\pi 2^{-n}}{7}\right)\cot\left(\frac{1}{3}\pi 2^{-n-2}\right)\tanh\left(\frac{1}{3}\pi 2^{-n-2}\right)\coth\left(\frac{\pi 2^{-n}}{7}\right)\right)^{2^n}$	$\frac{\sqrt{7-4\sqrt{3}}\sinh^2\left(\frac{\pi}{7}\right)\operatorname{csch}^2\left(\frac{\pi}{12}\right)}{2\left(\cos\left(\frac{3\pi}{28}\right)-\sin\left(\frac{3\pi}{28}\right)\right)^2}$
$\prod_{n=1}^{\infty}\left(\tan\left(\frac{1}{9}\pi 2^{-n-1}\right)\cot\left(\frac{1}{7}\pi 2^{-n-1}\right)\tanh\left(\frac{1}{7}\pi 2^{-n-1}\right)\coth\left(\frac{1}{9}\pi 2^{-n-1}\right)\right)^{2^n}$	$\frac{\sin^2\left(\frac{\pi}{14}\right)\csc^2\left(\frac{\pi}{18}\right)\left(1+2\cosh\left(\frac{\pi}{63}\right)+2\cosh\left(\frac{2\pi}{63}\right)+2\cosh\left(\frac{\pi}{21}\right)\right)^2}{\left(1+2\cosh\left(\frac{\pi}{63}\right)+2\cosh\left(\frac{2\pi}{63}\right)+2\cosh\left(\frac{\pi}{21}\right)+2\cosh\left(\frac{4\pi}{63}\right)\right)^2}$
$\prod_{n=1}^{\infty}\left(\tan(\pi 2^{-n-1})\cot\left(\frac{3\pi 2^{-n}}{7}\right)\tanh\left(\frac{3\pi 2^{-n}}{7}\right)\coth(\pi 2^{-n-1})\right)^{2^n}$	$\cos^2\left(\frac{\pi}{14}\right)\sinh^2\left(\frac{\pi}{2}\right)\operatorname{csch}^2\left(\frac{3\pi}{7}\right)$

8. Table of Finite Products of Trigonometric and Hyperbolic Functions

In this section, we evaluate Equation (41) for various values of the parameters m and r. For example, the first equation is obtained by multiplying two equations, when $m = \frac{\pi}{2}, r = \frac{\pi}{4}$.

Example 19. *An example in terms of* $\sqrt{2}$.

$$\prod_{j=0}^{n}\left(\tan\left(\pi 2^{-j-3}\right)\cot\left(\pi 2^{-j-2}\right)\tanh\left(\pi 2^{-j-2}\right)\coth\left(\pi 2^{-j-3}\right)\right)^{2^j}$$
$$= \frac{sech\left(\frac{\pi}{4}\right)\sqrt{\left(\sec^2(\pi 2^{-n-3})\cosh^2(\pi 2^{-n-3})\right)^{2^{n+1}}}}{\sqrt{2}} \tag{44}$$

Example 20. *An example in terms of* $\sqrt{3}$.

$$\prod_{j=0}^{n}\left(\tan\left(\frac{1}{3}\pi 2^{-j-1}\right)\cot\left(\pi 2^{-j-2}\right)\tanh\left(\pi 2^{-j-2}\right)\coth\left(\frac{1}{3}\pi 2^{-j-1}\right)\right)^{2^j}$$

$$= \frac{2\sinh\left(\frac{\pi}{3}\right)csch\left(\frac{\pi}{2}\right)\sqrt{\left(\frac{\cos^2\left(\frac{1}{3}\pi 2^{-n-2}\right)\left(2\cosh\left(\frac{1}{3}\pi 2^{-n-1}\right)+1\right)^2 sech^2\left(\frac{1}{3}\pi 2^{-n-2}\right)}{\left(2\cos\left(\frac{1}{3}\pi 2^{-n-1}\right)+1\right)^2}\right)^{2^{n+1}}}}{\sqrt{3}} \tag{45}$$

Example 21. *An example in terms of* $\sqrt{5}$.

$$\prod_{j=0}^{n}\left(\tan\left(\frac{1}{5}\pi 2^{-j-1}\right)\cot\left(\pi 2^{-j-2}\right)\tanh\left(\pi 2^{-j-2}\right)\coth\left(\frac{1}{5}\pi 2^{-j-1}\right)\right)^{2^j}$$

$$= \sqrt{2+\frac{2}{\sqrt{5}}}\sinh\left(\frac{\pi}{5}\right)csch\left(\frac{\pi}{2}\right)\times \sqrt{\left(\sin^2\left(\frac{1}{5}\pi 2^{-n-1}\right)\csc^2(\pi 2^{-n-2})\sinh^2(\pi 2^{-n-2})csch^2\left(\frac{1}{5}\pi 2^{-n-1}\right)\right)^{2^{n+1}}} \tag{46}$$

Example 22. *An example in terms of* $\sqrt{2}$ *and* $\sqrt{5}$.

$$\prod_{j=0}^{n}\left(\tan\left(\pi 2^{-j-2}\right)\cot\left(\frac{1}{5}\pi 2^{1-j}\right)\tanh\left(\frac{1}{5}\pi 2^{1-j}\right)\coth\left(\pi 2^{-j-2}\right)\right)^{2^j}$$

$$= \frac{\sinh\left(\frac{\pi}{2}\right)csch\left(\frac{4\pi}{5}\right)}{2\sqrt{2}}\times \sqrt{-\left(\left(\sqrt{5}-5\right)\left(\sin^2(\pi 2^{-n-2})\csc^2\left(\frac{1}{5}\pi 2^{1-n}\right)\sinh^2\left(\frac{1}{5}\pi 2^{1-n}\right)csch^2(\pi 2^{-n-2})\right)^{2^{n+1}}\right)} \tag{47}$$

Example 23. *An example in terms of the square root of trigonometric functions.*

$$\prod_{j=0}^{n}\left(\tan\left(\frac{1}{3}\pi 2^{-j-2}\right)\cot\left(\frac{1}{7}\pi 2^{1-j}\right)\tanh\left(\frac{1}{7}\pi 2^{1-j}\right)\coth\left(\frac{1}{3}\pi 2^{-j-2}\right)\right)^{2^j}$$

$$= 2\cos\left(\frac{\pi}{14}\right)\sinh\left(\frac{\pi}{6}\right)csch\left(\frac{4\pi}{7}\right)\times \sqrt{\left(\sin^2\left(\frac{1}{3}\pi 2^{-n-2}\right)\csc^2\left(\frac{1}{7}\pi 2^{1-n}\right)\sinh^2\left(\frac{1}{7}\pi 2^{1-n}\right)csch^2\left(\frac{1}{3}\pi 2^{-n-2}\right)\right)^{2^{n+1}}} \tag{48}$$

Example 24. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\sqrt{\left(\tan^2\left(\pi 2^{-j-4}\right)\cot^2\left(\pi 2^{-j-2}\right)\tanh^2\left(\pi 2^{-j-2}\right)\coth^2\left(\pi 2^{-j-4}\right)\right)^{2^j}}$$
$$=\frac{\sqrt{\frac{1}{2}+\frac{1}{\sqrt{2}}}\sqrt{(-1+i)\left(\frac{\sinh^2(\pi 2^{-n-2})csch^2\left(\pi 2^{-n-4}\right)}{\left(1+e^{i\pi 2^{-n-3}}\right)^2\left(1+e^{i\pi 2^{-n-2}}\right)^2}\right)^{2^{n+1}}}}{\cosh\left(\frac{\pi}{8}\right)+\cosh\left(\frac{3\pi}{8}\right)} \tag{49}$$

Example 25. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\sqrt{\left(\cos^2\left(\pi 2^{-j-2}\right)\sec^4\left(\pi 2^{-j-3}\right)\cosh^4\left(\pi 2^{-j-3}\right)sech^2\left(\pi 2^{-j-2}\right)\right)^{2^j}}$$
$$=sech\left(\frac{\pi}{4}\right)\sqrt{i2^{2^{n+2}-1}\left(\frac{\cosh^2(\pi 2^{-n-3})}{\left(1+e^{i\pi 2^{-n-2}}\right)^2}\right)^{2^{n+1}}} \tag{50}$$

Example 26. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\sqrt{\left(\tan^2\left(3\pi 2^{-j-3}\right)\cot^2\left(\pi 2^{-j-3}\right)\tanh^2\left(\pi 2^{-j-3}\right)\coth^2\left(3\pi 2^{-j-3}\right)\right)^{2^j}}$$
$$=\left(1+2\cosh\left(\frac{\pi}{2}\right)\right)\sqrt{-\left(\frac{e^{i\pi 2^{-n-1}}\left(2\cos(\pi 2^{-n-2})+1\right)^2}{\left(2\cosh(\pi 2^{-n-2})+1\right)^2}\right)^{2^{n+1}}} \tag{51}$$

Example 27. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\sqrt{\left(\cos^4\left(\frac{1}{3}\pi 2^{-j-1}\right)\sec^2\left(\frac{\pi 2^{-j}}{3}\right)\cosh^2\left(\frac{\pi 2^{-j}}{3}\right)sech^4\left(\frac{1}{3}\pi 2^{-j-1}\right)\right)^{2^j}}$$
$$=\cosh\left(\frac{\pi}{3}\right)\sqrt{\left(-2-2i\sqrt{3}\right)2^{-2^{n+2}}\left(\left(1+e^{\frac{1}{3}i\pi 2^{-n}}\right)^2 sech^2\left(\frac{1}{3}\pi 2^{-n-1}\right)\right)^{2^{n+1}}} \tag{52}$$

Example 28. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\sqrt{\left(\tan^2\left(\frac{\pi 2^{-j}}{3}\right)\cot^2\left(\frac{3\pi 2^{-j}}{7}\right)\tanh^2\left(\frac{3\pi 2^{-j}}{7}\right)\coth^2\left(\frac{\pi 2^{-j}}{3}\right)\right)^{2^j}}$$
$$=\frac{2\sin\left(\frac{\pi}{7}\right)\sinh\left(\frac{2\pi}{3}\right)csch\left(\frac{6\pi}{7}\right)\sqrt{(-1)^{\frac{8i}{21}}\left(\frac{\left(e^{\frac{3}{7}\pi 2^{1-n}}-1\right)^2\sin^2\left(\frac{\pi 2^{-n}}{3}\right)\csc^2\left(\frac{3\pi 2^{-n}}{7}\right)}{\left(e^{\frac{1}{3}\pi 2^{1-n}}-1\right)^2}\right)^{2^{n+1}}}}{\sqrt{3}} \tag{53}$$

Example 29. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\left(\frac{\cot\left(2^{-2-j}\pi\right)\cot\left(\frac{1}{3}2^{-1-j}\pi\right)\tanh\left(2^{-2-j}\pi\right)\tanh\left(\frac{1}{3}2^{-1-j}\pi\right)}{\cot\left(2^{-3-j}\pi\right)\cot\left(\frac{1}{5}2^{-1-j}\pi\right)\tanh\left(2^{-3-j}\pi\right)\tanh\left(\frac{1}{5}2^{-1-j}\pi\right)}\right)^{2^j}$$
$$= (-1)^{\frac{23}{60}+\frac{7i}{60}}\sqrt{3+\frac{3}{\sqrt{5}}}\left(\frac{\left(-1+e^{2^{-1-n}\pi}\right)\left(-1+e^{-\frac{1}{3}2^{-n}\pi}\right)}{\frac{\left(\left(-1+e^{i2^{-1-n}\pi}\right)\left(-1+e^{\frac{1}{3}i2^{-n}\pi}\right)\right)\left(\left(-1+e^{2^{-2-n}\pi}\right)\left(-1+e^{-\frac{1}{5}2^{-n}\pi}\right)\right)}{\left(-1+e^{i2^{-2-n}\pi}\right)\left(-1+e^{\frac{1}{5}i2^{-n}\pi}\right)}}\right)^{2^{1+n}}$$
$$csch\left(\frac{\pi}{3}\right)csch\left(\frac{\pi}{2}\right)\sinh\left(\frac{\pi}{5}\right)\sinh\left(\frac{\pi}{4}\right) \tag{54}$$

Example 30. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\sqrt{\left(\tanh^2\left(2^{-j-2}\log(2)\right)\cot^2\left(2^{-j-2}\log(2)\right)\right)^{2^{j+1}}}$$
$$= \left(-1+2^{i}\right)^2\sqrt{2^{1-i}\left(\frac{2^{(-1+i)2^{-n-1}}\left(2^{2^{-n-1}}-1\right)^4}{\left(-1+2^{i2^{-n-1}}\right)^4}\right)^{2^{n+1}}} \tag{55}$$

Example 31. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\left(\frac{\tan\left(\frac{1}{3}2^{-2-j}\pi\right)\tan\left(\frac{5}{3}2^{-2-j}\pi\right)\tanh\left(2^{-2-j}\pi\right)\tanh\left(3\,2^{-2-j}\pi\right)}{\tan\left(2^{-2-j}\pi\right)\tan\left(3\,2^{-2-j}\pi\right)\tanh\left(\frac{1}{3}2^{-2-j}\pi\right)\tanh\left(\frac{5}{3}2^{-2-j}\pi\right)}\right)^{2^j}$$
$$= -4\frac{\left(\sinh\left(\frac{\pi}{6}\right)\sinh\left(\frac{5\pi}{6}\right)\right)}{\sinh\left(\frac{\pi}{2}\right)\sinh\left(\frac{3\pi}{2}\right)}$$
$$\left(\frac{\left(1+2\cos\left(\frac{1}{3}2^{-1-n}\pi\right)+2\cos\left(\frac{2^{-n}\pi}{3}\right)\right)\left(1+2\cosh\left(\frac{1}{3}2^{-1-n}\pi\right)\right)^2\left(1+2\cosh\left(2^{-1-n}\pi\right)\right)}{\left(1+2\cos\left(\frac{1}{3}2^{-1-n}\pi\right)\right)^2\left(1+2\cos\left(2^{-1-n}\pi\right)\right)\left(1+2\cosh\left(\frac{1}{3}2^{-1-n}\pi\right)+2\cosh\left(\frac{2^{-n}\pi}{3}\right)\right)}\right)^{2^{1+n}} \tag{56}$$

Example 32. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\left(\frac{\cot\left(\frac{5}{9}2^{-1-j}\pi\right)\coth\left(\frac{1}{9}2^{1-j}\pi\right)\tan\left(\frac{1}{9}2^{1-j}\pi\right)\tanh\left(\frac{5}{9}2^{-1-j}\pi\right)}{\cot\left(\frac{1}{9}2^{1-j}\pi\right)\coth\left(\frac{5}{9}2^{-1-j}\pi\right)\tan\left(\frac{5}{9}2^{-1-j}\pi\right)\tanh\left(\frac{1}{9}2^{1-j}\pi\right)}\right)^{2^j}$$
$$= (-1)^{\frac{2i}{9}}\left(e^{\frac{2^{-n}\pi}{9}}\csc^2\left(\frac{5}{9}2^{-1-n}\pi\right)csch^2\left(\frac{1}{9}2^{1-n}\pi\right)\sin^2\left(\frac{1}{9}2^{1-n}\pi\right)\sinh^2\left(\frac{5}{9}2^{-1-n}\pi\right)\right)^{2^{1+n}}$$
$$csch^2\left(\frac{5\pi}{9}\right)\sinh^2\left(\frac{4\pi}{9}\right) \tag{57}$$

Example 33. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\left(\frac{\cot\left(\frac{7}{5}2^{-2-j}\pi\right)\coth\left(\frac{3}{5}2^{-2-j}\pi\right)\tan\left(\frac{3}{5}2^{-2-j}\pi\right)\tanh\left(\frac{7}{5}2^{-2-j}\pi\right)}{\cot\left(\frac{3}{5}2^{-2-j}\pi\right)\coth\left(\frac{7}{5}2^{-2-j}\pi\right)\tan\left(\frac{7}{5}2^{-2-j}\pi\right)\tanh\left(\frac{3}{5}2^{-2-j}\pi\right)}\right)^{2^j}$$
$$= (-1)^{4/5}\left(\frac{e^{-\frac{1}{5}i2^{1-n}\pi}\left(\sin^2\left(\frac{3}{5}2^{-2-n}\pi\right)\sinh^2\left(\frac{7}{5}2^{-2-n}\pi\right)\right)}{\sin^2\left(\frac{7}{5}2^{-2-n}\pi\right)\sinh^2\left(\frac{3}{5}2^{-2-n}\pi\right)}\right)^{2^{1+n}} csch^2\left(\frac{7\pi}{10}\right)\sinh^2\left(\frac{3\pi}{10}\right) \quad (58)$$

Example 34. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\left(\frac{\cot\left(3\,2^{-3-j}\pi\right)\coth\left(2^{-2-j}\pi\right)\tan\left(2^{-2-j}\pi\right)\tanh\left(3\,2^{-3-j}\pi\right)}{\cot\left(7\,2^{-4-j}\pi\right)\coth\left(\frac{5}{3}2^{-2-j}\pi\right)\tan\left(\frac{5}{3}2^{-2-j}\pi\right)\tanh\left(7\,2^{-4-j}\pi\right)}\right)^{2^j}$$
$$= \frac{\left((-1)^{\frac{5}{24}+\frac{5i}{24}}\csc\left(\frac{\pi}{8}\right)csch\left(\frac{3\pi}{4}\right)csch\left(\frac{5\pi}{6}\right)\sinh\left(\frac{\pi}{2}\right)\sinh\left(\frac{7\pi}{8}\right)\right)}{2\sqrt{2}}\times$$
$$\left(\frac{\left(-1+e^{3\,2^{-2-n}\pi}\right)\left(-1+e^{i2^{-1-n}\pi}\right)}{\frac{\left(\left(-1+e^{3i2^{-2-n}\pi}\right)\left(-1+e^{2^{-1-n}\pi}\right)\right)\left(\left(-1+e^{7\,2^{-3-n}\pi}\right)\left(-1+e^{\frac{5}{3}i2^{-1-n}\pi}\right)\right)}{\left(-1+e^{7i2^{-3-n}\pi}\right)\left(-1+e^{\frac{5}{3}2^{-1-n}\pi}\right)}}\right)^{2^{1+n}} \quad (59)$$

Example 35. *An example in terms of the square root of a complex number.*

$$\prod_{j=0}^{n}\left(\frac{\tan\left(\left(1+\frac{2i}{3}\right)2^{-2-j}\pi\right)\tanh\left(\left(3+\frac{4i}{5}\right)2^{-3-j}\pi\right)}{\tan\left(\left(3+\frac{4i}{5}\right)2^{-3-j}\pi\right)\tanh\left(\left(1+\frac{2i}{3}\right)2^{-2-j}\pi\right)}\right)^{2^j}$$

$$= (-1)^{\frac{23}{60}+\frac{7i}{60}}\left(\frac{\left(-1+e^{\left(3+\frac{4i}{5}\right)2^{-2-n}\pi}\right)\left(-1+e^{\left(-\frac{2}{3}+i\right)2^{-1-n}\pi}\right)}{\left(-1+e^{\left(-\frac{4}{5}+3i\right)2^{-2-n}\pi}\right)\left(-1+e^{\left(1+\frac{2i}{3}\right)2^{-1-n}\pi}\right)}\right)^{2^{1+n}}$$
$$\cos\left(\left(\frac{1}{6}+\frac{i}{2}\right)\pi\right)csch\left(\left(\frac{3}{4}+\frac{i}{5}\right)\pi\right)sech\left(\frac{\pi}{3}\right)\sinh\left(\left(\frac{1}{5}+\frac{i}{4}\right)\pi\right) \quad (60)$$

9. Discussion

In this work we evaluated a few equations. The two equations we focused on were the infinite product and finite product of the ratio of the tangent function of angles with power of two given by;

$$\prod_{n=1}^{\infty}\left(\frac{\tan\left(2^{-1-n}r\right)\tanh\left(2^{-1-n}m\right)}{\tan\left(2^{-1-n}m\right)\tanh\left(2^{-1-n}r\right)}\right)^{2^n} = \csc^2\left(\frac{r}{2}\right)\operatorname{csch}^2\left(\frac{m}{2}\right)\sin^2\left(\frac{m}{2}\right)\sinh^2\left(\frac{r}{2}\right) \quad (61)$$

and

$$\prod_{j=0}^{n}\left(\frac{\tan\left(2^{-1-j}r\right)\tanh\left(2^{-1-j}m\right)}{\tan\left(2^{-1-j}m\right)\tanh\left(2^{-1-j}r\right)}\right)^{2^j}$$
$$= \csc(r)\operatorname{csch}(m)\sin(m)\sinh(r)$$
$$\left(\csc^2\left(2^{-1-n}m\right)\operatorname{csch}^2\left(2^{-1-n}r\right)\sin^2\left(2^{-1-n}r\right)\sinh^2\left(2^{-1-n}m\right)\right)^{2^n} \quad (62)$$

In future work, we would like to work on analyzing this infinite and partial product and see if any particular mathematical trends can be surfaced. We think this type of analysis could be of interest to the mathematical community since this work involves the product of trigonometric functions.

10. Conclusions

In this paper, we have presented a novel method for deriving new finite and infinite sums and products involving trigonometric functions using contour integration. The method applied in the derivation of the main theorem may be used to derive other sums and products in future work, which could include deriving and extending Melnikov forms such as Equation (23) in [16]. We were able to derive infinite forms involving trigonometric functions and their partial sum counterpart. These types of evaluations allowed us to analyze the infinite sum of functions and the partial sums which they are built upon. The results presented were numerically verified for both real and imaginary and complex values of the parameters in the integrals using Mathematica by Wolfram.

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References

1. Edgar, G.A.; Ullman, D.H.; West, D.B. Problems and Solutions. *Am. Math. Mon.* **2017**, *124*, 465–474. [CrossRef]
2. Vălean, C.I. *(Almost) Impossible Integrals, Sums, and Series*; Springer: Berlin/Heidelberg, Germany, 2019.
3. Silverman, L.L. *On the Definition of the Sum of a Divergent Series*; University of Missouri: Columbia, MO, USA, 1913; Volume 1.
4. Hardy, G.H. *Divergent Series*; Oxford University Press: Oxford, UK, 1949.
5. Candelpergher, B. *Ramanujan Summation of Divergent Series*; Springer: Cham, Switzerland, 2017. [CrossRef]
6. Mitschi, C.; Sauzin, D. *Divergent Series, Summability and Resurgence I*; Springer: Cham, Switzerland, 2016. [CrossRef]
7. Sommen, F. A product and an exponential function in hypercomplex function theory. *Appl. Anal.* **1981**, *12*, 13–26. [CrossRef]
8. Zotev, V.S.; Rebane, T.K. Exponential-trigonometric basis functions in the coulomb four-body problem. *Phys. Atom. Nucl.* **2000**, *63*, 40–42. [CrossRef]
9. Reynolds, R.; Stauffer, A. A Method for Evaluating Definite Integrals in Terms of Special Functions with Examples. *Int. Math. Forum* **2020**, *15*, 235–244. [CrossRef]
10. Erdéyli, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F.G. *Higher Transcendental Functions*; McGraw-Hill Book Company, Inc.: New York, NY, USA; Toronto, ON, Canada; London, UK, 1953; Volume I.
11. Laurincikas, A.; Garunkstis, R. *The Lerch Zeta-Function*; Springer Science & Business Media: New York, NY, USA, 2013; 189p.
12. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 6th ed.; Academic Press: Cambridge, MA, USA, 2000.
13. Gelca, R.; Andreescu, T. *Putnam and Beyond*, 1st ed.; Springer: New York, NY, USA, 2007
14. Oldham, K.B.; Myland, J.C.; Spanier, J. *An Atlas of Functions: With Equator, the Atlas Function Calculator*, 2nd ed.; Springer: New York, NY, USA, 2009.
15. Reynolds, R.; Stauffer, A. A Note on the Infinite Sum of the Lerch function. *Eur. J. Pure Appl. Math.* **2022**, *15*, 158–168. [CrossRef]

16. Melnikov, Y.A. A new approach to the representation of trigonometric and hyperbolic functions by infinite products. *J. Math. Anal. Appl.* **2008**, *344*, 521–534. [CrossRef]
17. Oberhettinger, F. Note on the Lerch zeta function. *Pac. J. Math. Pac. J. Math.* **1956**, *6*, 117–120. [CrossRef]