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Extended Permutation Filters and Their Application to Edge Enhancement

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Abstract

Extended permutation (EP) filters are defined and analyzed in this paper. In particular, we focus on *extended permutation rank selection* (EPRS) filters. These filters are constrained to output an order statistic from an extended observation vector. This extended vector includes N observation samples and K statistics that are functions of the observation samples. The rank permutations from selected samples in this extended observation vector are used as the basis for selecting an order statistic output. We show that by including the sample mean in the extended observation vector, the filters exhibit excellent edge enhancement properties. We also show that several previously defined classes of rank order based edge enhancers (CS, LUM, and WMMR sharpeners) can be formulated as subclasses of EPRS filters. These sharpening subclasses are in addition to the smoothing subclasses, which include *rank conditioned rank selection*, *permutation*, *stack*, and *weighted order statistic* filters. Thus, this novel class of filters provides a broad framework within which many rank order based smoothers and edge enhancers can be unified. Edge enhancement properties are developed and an L_η norm EPRS filter optimization procedure is presented. Finally, extensive computer simulation results are presented comparing the performance of EPRS and other sharpening filters in edge enhancement applications.

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1 Introduction

Nonlinear filters have proven to be exceptionally useful in many signal and image restoration applications. In particular, rank order based filters are well known for their ability to successfully treat heavy tailed noise and non-stationary signals. The common occurrence of such signals, and the poor performance of linear filters operating on them, have motivated the development of rank order filters. The first, and most well known, of these rank order based filters is the median filter [1]. Since its introduction, the median filter has been extensively studied [2, 3, 4, 1]. Building on the success of the median filter, many more sophisticated rank order filters have been proposed. These include *multistage median* filters [5, 6, 7], *center weighted median* (CWM) filters [8, 9, 10], general *weighted median* (WM) and *weighted order statistic* (WOS) filters [4, 11], *stack* filters [12, 13, 14, 15], *permutation* filters [16, 17] and *rank conditioned rank selection* (RCRS) filters [18]. These filters have primarily been utilized as smoothing filters in restoration applications where a signal is corrupted by noise.

All of the above filters can be formulated as *rank selection* (RS) filters [17, 18], since their output is constrained to be one of the order statistics from observation set. However, they differ in the information that they use to perform the selection operation. The permutation filter and the RCRS filter use the ranks of the input samples as the basis for the output rank selection [17, 18]. These filters are highly effective as smoothers [17, 18]. However, they are not suited to perform edge enhancement. This results from the fact that in an edge region, comprised of non-decreasing or non-increasing samples, the ranks of the input samples remain the same for all observation window locations. Thus, they do not help to identify which “side” of an edge the observation window is lies on. Consequently, different rank selections cannot be made on each side of an edge to yield gradient enhancement.

For these reasons, the application of rank order based filters to edge enhancement has received limited attention. However, some edge enhancing RS filters have been proposed. These include the *comparison and selection* (CS) filter [19], the *lower-upper-middle* (LUM) filter [8] and the *weighted majority of samples with minimum range* (WMMR) filter [20, 21]. The CS and LUM filters utilize a mean estimate to aid in rank selection. In particular, the observation sample mean is compared to a specified sample within the observation window to determine which rank ordered sample to output. This comparison helps to identify which “side” of an edge the filters window lies on. In a somewhat similar fashion, WMMR filters use rank ranges to delineate different regions of an edge. Having partitioned the edge into different regions, an appropriate output sample is chosen in each region so as to increase the edge gradient.

In this paper, we develop a filter class which provides a broad framework for rank order based edge

enhancing filters. These filters will be referred to as *extended permutation* (EP) filters and can be viewed as an extension of RCRS and permutation filters. The EP filters are based upon a partitioning of the observation space using rank permutations of samples from an extended observation vector. This extended vector contains N observation samples and K statistics which are functions of the observation samples. A common filtering operation is defined for each partition, or ordering of the extended observation vector. While numerous filtering operations can be performed for each partition, we focus here on rank selection operations, and refer to the resulting filters as *extended permutation rank selection* (EPRS) filters.

The EPRS filters possess excellent noise smoothing capabilities as a result of their use of rank order information and their inclusion of RCRS and permutation filters as subsets. With well chosen statistics in the extended observation vector, the capabilities of EPRS filters can be made to include edge enhancement. We show that the sample mean is such a statistic and that the inclusion of the sample mean in the extended observation vector gives EPRS filters excellent edge enhancement properties. Additionally, the filter formulation combines the advantages of RCRS and permutation filters with those of the rank order based edge enhancing filters. In fact, we show that the CS, LUM, and WMMR filters can all be formulated as subclasses of EPRS filters. Thus, EPRS filters offer a framework under which numerous rank order based smoothing and edge enhancing filters can be unified. This not only helps relate and explain the operations of these previously unrelated filters, but also provides increased performance. This increased performance in edge enhancement is illustrated here using both Markov sequences and images that have been smoothed. Also, we show that EPRS filters avoid many of the shortcomings of linear sharpening filters. Namely, they are relatively insensitive to heavy tailed noise and they do not cause ringing (overshoot and undershoot). With the ability to perform edge gradient enhancement in the presence of noise, the filters may be useful in deblurring or deconvolution applications.

The remainder of this paper is organized as follows. In Section 2, the application of rank order based filters to the edge enhancement problem is described. The EPRS filters are defined in Section 3. In addition, the relationship between other filters and the EPRS filters is explored. Some filter properties are developed in Section 4 along with an optimization procedure. Experimental results are presented in Section 5. These results illustrate the performance of the new filters in 1- and 2-dimensional deblurring applications in comparison to other rank order based sharpeners. Finally, some conclusions are presented in Section 6.

2 Rank Order Filters and the Edge Enhancement Problem

This section addresses the application of rank order based filters to the edge enhancement problem. First, RCRS and permutation filters are defined since EP filters are extensions of these classes and their development builds upon the RCRS and permutation filter definitions. Next, edge gradient enhancement using RS filters is considered. In light of this examination, the EP filters are developed in Section 3. There it will be shown that the EP filter class unifies, under a single definition, each of the filters discussed in this section.

2.1 RCRS and Permutation Filters

Consider the d -dimensional discrete sequences $\{d(\mathbf{n})\}$ and $\{x(\mathbf{n})\}$, where the discrete index $\mathbf{n} = [n_1, n_2, \dots, n_d]$. Let these sequences represent the desired and corrupted versions of a signal respectively. Also, consider a d -dimensional window function that spans N samples and passes over the corrupted sequence in some predetermined fashion. At each location \mathbf{n} , the N observation samples spanned by the window can be indexed and written as a vector, yielding

$$\mathbf{x}(\mathbf{n}) = [x_1(\mathbf{n}), x_2(\mathbf{n}), \dots, x_N(\mathbf{n})]. \quad (1)$$

The windowing and indexing of the observation sequence defines an ordering of the observed samples. Typically, this ordering is temporal for one-dimensional time sequences and spatial, e.g. raster-scan, for two-dimensional signals such as images. Other orderings are possible, as are windows of higher dimension. An ordering that can be universally applied to the observed samples, regardless of signal dimension or window configuration, is rank ordering. The N observation samples ordered according to rank will be written as

$$x_{(1)}(\mathbf{n}) \leq x_{(2)}(\mathbf{n}) \leq \dots \leq x_{(N)}(\mathbf{n}), \quad (2)$$

where $x_{(1)}(\mathbf{n}), x_{(2)}(\mathbf{n}), \dots, x_{(N)}(\mathbf{n})$ are referred to as the order statistics of the observation.

The use of more than one ordering of the observed samples has proved advantages in many filtering problems [17, 18]. For instance, temporal correlations can be exploited if the temporal order of samples is known. In contrast, rank ordering allows for the effective rejection of outliers, since these samples are most often located in the extremes of the ranked set. By utilizing both orderings, results superior to the two marginal cases can be obtained. Thus, to relate the rank of a sample to its (temporal, spatial, etc.) location (index) within the window, we define $r_i(\mathbf{n})$ to be the rank of the sample in window location i . This establishes the equivalence $x_i(\mathbf{n}) \equiv x_{(r_i(\mathbf{n}))}(\mathbf{n})$.

The filtering, or estimation, problem can now be posed as follows. From the set of observation samples, we wish to form an estimate of the desired sample at location δ within the window. This

estimate is denoted as $\hat{d}_\delta(\mathbf{n})$, where $1 \leq \delta \leq N$. In the remainder of the paper, the index \mathbf{n} is assumed and is used explicitly only when necessary for clarity.

By definition, the output \hat{d}_δ of an RS filter is constrained to be an order statistic from the observation vector. Numerous non-linear filters can be cast as RS filters, including WOS [11], stack [13], RCRS [18], and permutation filters [17]. The most general of these formulations are the RCRS and permutation filters, which we define next.

Consider the vector $\mathbf{r} = [r_{\gamma_1}, r_{\gamma_2}, \dots, r_{\gamma_M}]$, which contains the ranks of M selected observation samples $x_{\gamma_1}, x_{\gamma_2}, \dots, x_{\gamma_M}$, where $0 \leq M \leq N$. Note that since $r_{\gamma_i} \in \{1, 2, \dots, N\}$ for $i = 1, 2, \dots, M$, then $\mathbf{r} \in \Omega_{\mathbf{z}} = \{[i_1, i_2, \dots, i_M] : i_j \in \{1, 2, \dots, N\} \text{ and } i_j \neq i_k \forall j \neq k\}$. That is, $\Omega_{\mathbf{z}}$ is the set that contains all permutations of the N indices $1, 2, \dots, N$ taken M at a time and we refer to this as the rank permutation set. The rank permutation set has cardinality $|\Omega_{\mathbf{z}}| = N!/(N-M)!$ and the index $\mathbf{z} = [M, N]$, is used to indicate the dependance of this set on the parameters M and N .

The output of an M^{th} -order RCRS filter with window size N is given by

$$F_{RCRS}(\mathbf{x}) = x_{(\mathcal{S}(\mathbf{r}))}, \quad (3)$$

where $\mathcal{S}(\cdot)$ is said to be the selection rule and $\mathcal{S} : \Omega_{\mathbf{z}} \mapsto \{1, 2, \dots, N\}$ [18]. Thus, RCRS filter estimates are based on the temporal and rank order of M selected samples. If $M = N$, then \mathbf{r} relates the temporal and rank order of each input sample and $\Omega_{\mathbf{z}}$ is the group of permutations. In this case, the full permutation information is used and (3) defines the class of permutation filters [16, 17]. Using \mathbf{r} as the basis for rank selection has been shown to be effective for smoothing and frequency selection/rejection applications [17, 18]. However, as shown in Section 2.2, using sample ranks alone is not effective for edge enhancement.

2.2 Edge Enhancement

The problem of enhancing edges, or transition spans, using RS filters is now addressed. We begin by discussing the “edge dilemma” of strictly rank based rank selection filters (RCRS and Permutation filters). Specifically, consider the case in which the full set of observation ranks \mathbf{r} is used as the basis for a filter output. For locally monotone sequences, the resulting rank vector is given by

$$\mathbf{r} = [1, 2, \dots, N] \quad \text{or} \quad \mathbf{r} = [N, N-1, \dots, 1]. \quad (4)$$

Moreover, as an observation window slides over a monotone signal the rank vector does not change since the ranks of the observation samples remain constant. As an illustration, consider the one-dimensional sequence containing an edge in Fig. 1. An observation window located anywhere along the monotone increasing portion of the signal results in the rank vector $\mathbf{r} = [1, 2, \dots, N]$. Thus, for an RS filter with any rank based selection rule $\mathcal{S}(\cdot)$, the output along a monotone sequence will be an

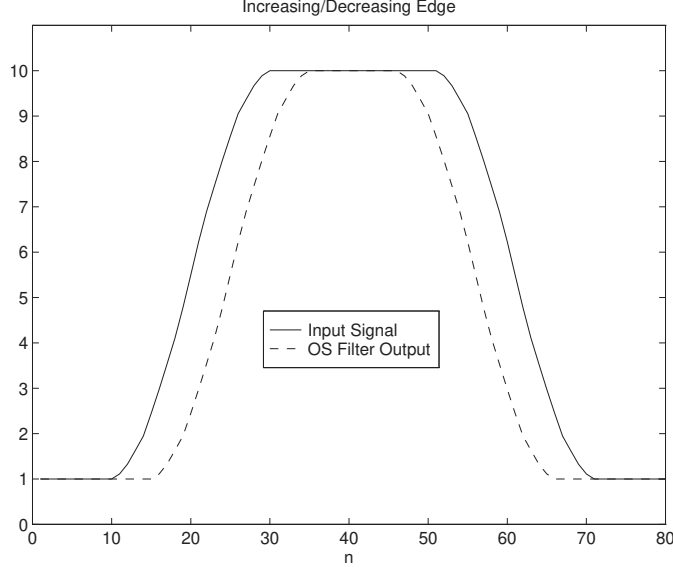


Figure 1: Example of a sequence containing a convex/concave increasing/decreasing edge and the output of an order statistic (OS) filter ($N = 15$, $k = 3$) operating on the sequence. The increasing edge has been retarded while the decreasing edge has been advanced. Neither transition region duration has been reduced.

order statistic $x_{(k)}$, where $k \in \{1, 2, \dots, N\}$. Moreover, since $\mathbf{x} = \mathbf{x}^r$ for this monotone increasing sequence, it is easy to see that the output of a filter $F(\cdot)$ based on $\mathcal{S}(\cdot)$ is simply $F(\{x(n)\}) = \{x(n - (\frac{N+1}{2} - k))\}$. Thus, the sequence is simply shifted in time. If $k < \frac{N+1}{2}$ ($k > \frac{N+1}{2}$) the sequence is retarded (advanced) and if $k = \frac{N+1}{2}$, then the median filter is realized and the sequence is left unaltered. Similar results hold for monotone decreasing sequences. In this case, the sequence is advanced (retarded) for $k < \frac{N+1}{2}$ ($k > \frac{N+1}{2}$) and left unaltered for $k = \frac{N+1}{2}$. This is illustrated in Fig. 1.

The CS, LUM and WMMR filters overcome this problem by utilizing information other than sample ranks in the output rank selection process. In particular, the CS filter compares the observation vector sample mean and median to determine the output rank [19]. In a similar fashion, the LUM filter compares the value of the middle sample in the observation window to the midpoint between an upper and lower order statistic

$$(x_{(l)} + x_{(N-l+1)})/2, \quad (5)$$

where $1 \leq l \leq (N + 1)/2$ [8]. The results of these comparisons helps to indicate which “side” of an edge the observation window lies. Consequently, different rank selections can be made on each side of an edge to produce edge gradient enhancement. We show that by using the rank of the mean, rather than simply comparing it to the median or middle sample, the location of the observation window with respect to an edge can be more accurately determined. This leads to superior performance in

edge enhancement applications.

3 Extended Permutation Filters

In this section, the EP filters are defined and discussed. These filters can be considered an extension of RCRS and Permutation filters. They incorporate the advantages of RCRS and permutation filters with those of the rank order based edge enhancers.

3.1 Filter Definition

The EP filters are based on a partitioning of the observation space using the rank permutations of samples from an extended observation vector. That is, in addition to the ranks of selected observation samples, the rank of additional statistics are utilized. These statistics are computed as functions of the observation vector. Thus, define an extended observation vector as

$$\tilde{\mathbf{x}} = [\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{N+K}] = [x_1, x_2, \dots, x_N, F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_K(\mathbf{x})]. \quad (6)$$

This extended observation vector can be sorted as before, yielding

$$\tilde{x}_{(1)} \leq \tilde{x}_{(2)} \leq \dots \leq \tilde{x}_{(N+K)}. \quad (7)$$

Also, let an extended rank vector be defined as

$$\tilde{\mathbf{r}} = [\tilde{r}_{\gamma_1}, \tilde{r}_{\gamma_2}, \dots, \tilde{r}_{\gamma_M}, \tilde{r}_{\beta_1}, \tilde{r}_{\beta_2}, \dots, \tilde{r}_{\beta_L}] \in \Omega_{\mathbf{z}}, \quad (8)$$

where $1 \leq \gamma_i \leq N$, $N+1 \leq \beta_i \leq N+K$, and the limits on M and L are given by $0 \leq M \leq N$ and $0 \leq L \leq K$. The element \tilde{r}_{γ_i} is the rank of $\tilde{x}_{\gamma_i} = x_{\gamma_i}$ in $\tilde{\mathbf{x}}$, and \tilde{r}_{β_i} is the rank of $\tilde{x}_{\beta_i} = F_{\beta_i-N}(\mathbf{x})$. Thus, the extended rank vector lies in the extended rank permutation space which is denoted as $\Omega_{\mathbf{z}}$, where $\mathbf{z} = [M, N, K, L]$.

Each unique extended rank vector $\tilde{\mathbf{r}} \in \Omega_{\mathbf{z}}$ defines a distinct partition in the R^N observation space. EP filters are defined such that a common filtering operation is applied to each observation vector lying in a given partition. In the general case, the filtering operation performed is a function of the extended observation and can be either linear or nonlinear. For the EPRS filters considered here, the filtering method is restricted to an order statistic operation. That is, in each partition a specific order statistic from $\tilde{\mathbf{x}}$ is selected as the filter output. These filters are formally defined below.

Definition 3.1 *The output of an EPRS filter is given by*

$$F_{EPRS}(\mathbf{x}) = \tilde{x}_{(\mathcal{S}(\tilde{\mathbf{r}}))}, \quad (9)$$

where $\mathcal{S} : \Omega_{\mathbf{z}} \mapsto \{1, 2, \dots, N+K\}$.

The filter operation can be achieved by using straightforward lookup table based on the selection function $\mathcal{S}(\cdot)$. Such a lookup table would be similar to the ordering-output table shown in [22]. A specific order statistic is selected based on the observed extended rank vector $\tilde{\mathbf{r}}$.

The cardinality of the extended rank permutation space depends, in general, on the K functions $F_1(\cdot), F_2(\cdot), \dots, F_K(\cdot)$. For arbitrary observation values, the *ranks* of the function outputs must lie between 1 and $N + K$. Thus, the cardinality is bound above such that $|\Omega_{\mathbf{z}}| \leq (N + K)! / (N - M + K - L)!$, where the inequality is strict if one or more of the extended vector samples are bound by other samples or the corresponding order statistics. The inequality is strict, for instance, if $F_1(\cdot)$ is the sample mean since $x_{(1)} \leq F_1(\mathbf{x}) \leq x_{(N)}$. Similarly, for each observed rank permutation, the number of possible unique EPRS filter outputs is less than or equal to $N + K$. Thus, denoting the class of EPRS filters as $\Psi_{\mathbf{z}}$, the cardinality of the filter class is bound above by $|\Psi_{\mathbf{z}}| \leq (N + K)^{|\Omega_{\mathbf{z}}|}$. The exact number will depend on the specific functions $F_1(\cdot), F_2(\cdot), \dots, F_K(\cdot)$ and the domain of the observation \mathbf{x} . Next, we restrict the extension of the observation set to a single statistic, the α -trimmed mean. Bounds on the general and restricted filter classes are then given as a theorem.

Consider the case where $K = L = 1$ ($\beta_1 = N + 1$) and $\tilde{x}_{N+1} = F_1(\mathbf{x})$ is an α -trimmed sample mean estimate given by

$$F_1(\mathbf{x}) = \frac{1}{N - 2\alpha + 2} \sum_{i=\alpha}^{N-\alpha+1} x_{(i)}, \quad (10)$$

where $1 \leq \alpha \leq (N + 1)/2$ and is selected to provide a robust mean estimate when outliers are present. We show that this is an effective choice for edge enhancement applications. This follows because the rank of the mean, \tilde{r}_{β_1} , provides information about where the observation window lies with respect to an edge midpoint. Furthermore, providing the opportunity to select $F_1(\mathbf{x})$ to be the output can be valuable. The rest of this paper will focus on the case where $K = L = 1$ and $F_1(\mathbf{x})$ is defined in (10). As the size of the extended vector $\tilde{\mathbf{x}}$ is now determined by M , we will refer to M as the order of the EPRS filter.

Theorem 3.1: *The cardinality of the the window size N EPRS filter class with extended observation vector $\tilde{\mathbf{x}} = [x_1, x_2, \dots, x_N, F_1(\mathbf{x}), F_2(\mathbf{x}), \dots, F_K(\mathbf{x})] \in R^{N+K}$ and extended rank vector $\tilde{\mathbf{r}} = [\tilde{r}_{\gamma_1}, \tilde{r}_{\gamma_2}, \dots, \tilde{r}_{\gamma_M}, \tilde{r}_{\beta_1}, \tilde{r}_{\beta_2}, \dots, \tilde{r}_{\beta_L}]$, is denoted as $|\Psi_{\mathbf{z}}|$ and is bound by*

$$|\Psi_{\mathbf{z}}| \leq (N + K)^{\frac{(N+K)!}{(N+K-M-L)!}}, \quad (11)$$

where $0 \leq M \leq N$ and $0 \leq L \leq K$. When $K = L = 1$ and $F_1(\mathbf{x})$ is the α -trimmed sample mean, the bound becomes strict with

$$|\Psi_{\mathbf{z}}| < (N + 1)^{\frac{(N+1)!}{(N-M)!}}. \quad (12)$$

If the input is restricted such there are no constant subsequences in the extended observation vector,

then

$$|\Psi_{\mathbf{z}}| = (N + 1) \frac{(N+1-2\alpha)N!}{(N-M)!}. \quad (13)$$

Proof: The bound in (11) follows from the discussion above where it was shown that $|\Psi_{\mathbf{z}}| \leq (N + K)^{|\Omega_{\mathbf{z}}|}$ and $|\Omega_{\mathbf{z}}| \leq (N+K)!/(N-M+K-L)!$. Setting $K = L = 1$ directly reduces the bound to that in (12) where the relation operator is the inequality \leq . The inequality is strict for $F_1(\mathbf{x})$ taken to be the α -trimmed sample mean since, by definition, $F_1(\mathbf{x})$ averages only over $x_{(\alpha)}, x_{(\alpha+1)}, \dots, x_{(N+1-\alpha)}$. Consequently, $x_{(\alpha)} \leq F_1(\mathbf{x})$ and, due to stable sorting, $\alpha < \tilde{r}_{\beta_1}$. Stable sorting also allows \tilde{r}_{β_1} to take on values up to and including $N + 1$ (for instance when all observation samples have equal value). Thus, $\alpha + 1 \leq \tilde{r}_{\beta_1} \leq N + 1$. The lower bound on \tilde{r}_{β_1} limits the number of permutations $\tilde{\mathbf{r}}$ can take on to fewer than in the general case, causing the bound in (12) to be strict. For the case where all elements of $\tilde{\mathbf{x}}$ are unique, \tilde{r}_{β_1} is bound to the $N + 1 - 2\alpha$ values $\alpha + 1 \leq \tilde{r}_{\beta_1} \leq N + 1 - \alpha$. For this case, the M selected input samples can take on $N(N - 1) \cdots (N - M + 1) = N!/(N - M)!$ ordering combinations. This, with the $N + 1 - 2\alpha$ possible values of \tilde{r}_{β_1} , gives a total of $(N + 1 - 2\alpha)N!/(N - M)!$ possible rank permutations. For each of the permutations, the range of $\mathcal{S}(\tilde{\mathbf{r}})$ is $\{1, 2, \dots, N + 1\}$, corresponding to $N + 1$ possible outputs, each of which is unique. Thus for this case, the number of possible filters is $(N + 1) \frac{(N+1-2\alpha)N!}{(N-M)!}$. \square

An interesting consequence of stable sorting revealed in the proof is that, unlike the straight rank permutation filter, the EPRS can differentiate between a constant input signal and a non-constant non-decreasing signal. That is, $\tilde{r}_{\beta_1} = N + 1$ if and only if $x_1 = x_2 = \dots = x_N$. For non-constant signals, $\tilde{r}_{\beta_1} \leq N$. Next, we relate the EPRS filters to other previously defined filters.

3.2 Relationship to Other Filters

The EPRS filters are a broad class of filters that contain several important, and as yet unrelated, filters as subclasses. Thus, EPRS filters provide a unifying framework that aids in the understanding and analysis of the various filter subclasses. For example, by virtue of the fact that EPRS filters utilize the ranks of selected samples, previous work shows that weighted order statistic, stack, RCRS, and permutation smoothing filters are also subclasses of EPRS filters [17, 18]. The following theorems show that the CS, LUM, and WMMR sharpening filters are also subclasses of EPRS filters.

Theorem 3.2: *The window size N CS filter with parameter j is a subclass of order zero EPRS filters where $K = L = 1$ and $\alpha = 1$ ($F_1(\mathbf{x})$ is the sample mean). The EPRS filter selection function that gives the CS filter class is*

$$\mathcal{S}(\tilde{\mathbf{r}}) = \mathcal{S}(\tilde{r}_{\beta_1}) = \begin{cases} N - j + 2 & \text{if } \tilde{r}_{\beta_1} \leq (N + 1)/2 \\ j & \text{otherwise} \end{cases}, \quad (14)$$

where $1 \leq j \leq (N + 1)/2$.

Proof: If $\tilde{r}_{\beta_1} \leq (N + 1)/2$, then for stable sorting, $F_1(\mathbf{x}) < x_{((N+1)/2)}$ and $\tilde{x}_{(N-j+2)} = x_{(N-j+1)}$. To yield the CS output, equivalent to that defined in [19], the EPRS filter must output $\tilde{x}_{(N-j+2)}$ in this case. If $\tilde{r}_{\beta_1} > (N + 1)/2$ using stable sorting, then $F_1(\mathbf{x}) \geq x_{((N+1)/2)}$ and $\tilde{x}_{(j)} = x_{(j)}$. In this case, the EPRS filter should output $\tilde{x}_{(j)}$. Thus, all possible inputs are accounted for and the proof is complete. \square

The LUM filter can also be formulated as an EPRS filter provided that a slightly modified mean estimate is used. This is shown in the following theorem.

Theorem 3.3: *The window size N LUM filter with parameters k and l is a subclass of order one EPRS filters with $F_1(\mathbf{x}) = (x_{(N-l+1)} + x_{(l)})/2$, and $\tilde{\mathbf{r}} = [\tilde{r}_{\gamma_1}, \tilde{r}_{\beta_1}]$, where $\gamma_1 = \delta$ (usually the index of the center sample in the observation vector). The EPRS filter selection function that gives the LUM filter class is*

$$\mathcal{S}(\tilde{\mathbf{r}}) = \mathcal{S}([\tilde{r}_{\gamma_1}, \tilde{r}_{\beta_1}]) = \begin{cases} N - k + 2 & \text{if } \tilde{r}_{\gamma_1} > N - k + 2 \\ k & \text{if } \tilde{r}_{\gamma_1} < k \\ N - l + 2 & \text{if } \tilde{r}_{\beta_1} < \tilde{r}_{\gamma_1} < N - l + 2 \\ l & \text{if } l < \tilde{r}_{\gamma_1} < \tilde{r}_{\beta_1} \\ \tilde{r}_{\gamma_1} & \text{otherwise} \end{cases}, \quad (15)$$

where $1 \leq k \leq l \leq (N + 1)/2$.

Proof: First note that since $\tilde{x}_{N+1} = F_1(\mathbf{x}) = (x_{(N-l+1)} + x_{(l)})/2$ and $1 \leq k \leq l \leq (N + 1)/2$, then

$$x_{(k)} \leq x_{(l)} \leq \tilde{x}_{N+1} \leq x_{(N-l+1)} \leq x_{(N-k+1)}. \quad (16)$$

Thus, $\tilde{x}_{(N-k+2)} = x_{(N-k+1)}$, $\tilde{x}_{(N-l+2)} = x_{(N-l+1)}$, $\tilde{x}_{(l)} = x_{(l)}$ and $\tilde{x}_{(k)} = x_{(k)}$. If $\tilde{r}_{\gamma_1} > N - k + 2$, then $\tilde{x}_{\gamma_1} > \tilde{x}_{(N-k+2)}$ or equivalently $x_{\gamma_1} > x_{(N-k+1)}$. To yield the LUM filter output, defined in [8], the EPRS filter should output $x_{(N-k+1)} = \tilde{x}_{(N-k+2)}$ in this case. Similarly, if $\tilde{r}_{\gamma_1} < k$, then $x_{\gamma_1} < x_{(k)}$. In this case, the EPRS filter output should be $x_{(k)} = \tilde{x}_{(k)}$. Also, if $\tilde{r}_{\beta_1} < \tilde{r}_{\gamma_1} < N - l + 2$, then $(x_{(N-l+1)} + x_{(l)})/2 < x_{\gamma_1} \leq x_{(N-l+1)}$. Thus, the LUM output in this case is equal to $x_{(N-l+1)} = \tilde{x}_{(N-l+2)}$. If $l < \tilde{r}_{\gamma_1} < \tilde{r}_{\beta_1}$, then $x_{(l)} \leq x_{\gamma_1} \leq (x_{(N-l+1)} + x_{(l)})/2$. Thus, the LUM output in this case is equal to $x_{(l)} = \tilde{x}_{(l)}$. Finally, in any other case, the output of the LUM filter is $x_{\gamma_1} = \tilde{x}_{\gamma_1}$ which has rank \tilde{r}_{γ_1} . This completes the proof. \square

An example of a LUM filter function is shown in Fig. 2 for $N = 25$, $k = 5$ and $l = 8$. Since the case where $\tilde{r}_{\gamma_1} = \tilde{r}_{\beta_1}$ cannot occur, these values are omitted from the plot. Note that the midpoint rank, \tilde{r}_{β_1} , can only range from $l + 1$ to $N + 1$ using stable sorting. Also, note that if $\tilde{r}_{\beta_1} = i$ for $i = N - l + 2, N - l + 3, \dots, N + 1$, then $x_{(l)} = x_{(i-1)}$. This means that any rank selection from l to $i - 1$ produces the same output. The LUM selection function determines the output based

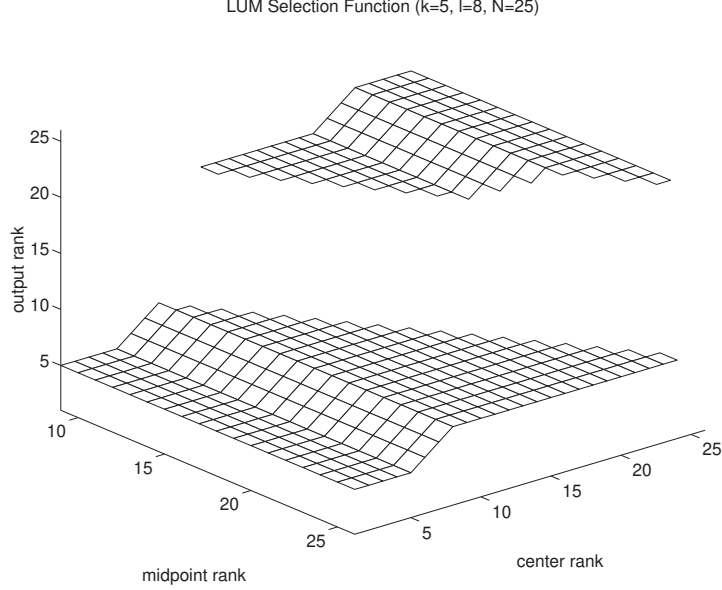


Figure 2: LUM filter selection function $\mathcal{S}([\tilde{r}_{\gamma_1}, \tilde{r}_{\beta_1}])$, where \tilde{r}_{γ_1} is the center sample rank and \tilde{r}_{β_1} is the rank of the LUM midpoint t_l . This function is shown for $N = 25$, $k = 5$ and $l = 8$.

on rank *ranges* of the center sample and mean. Since the EPRS filters utilize the complete rank information of these statistics, the number of functions that can be realized, and their complexity, is increased. This increase in number and complexity allows EPRS filters to be more finely tuned to signal statistics, resulting in superior performance in general.

In addition to the trimmed mean given in (10), other statistics may be useful for edge enhancement either alone or in conjunction with the trimmed mean. For example, the statistic used by the WMMR filter in [20, 21] can be useful. This provides an estimate of the nearest edge plateau. Thus, by letting $F_1(\mathbf{x})$ be the WMMR filter output, $\tilde{\mathbf{r}} = [\tilde{r}_{\gamma_1}, \tilde{r}_{\beta_1}]$, and $\gamma_1 = \delta$, the benefits of order one RCRS and WMMR filters are combined. One clear advantage of this filter is the ability to perform the identity operation if $\mathcal{S}(\tilde{\mathbf{r}}) = \tilde{r}_{\gamma_1}$. This will give the filter significantly better detail preserving characteristics than the WMMR filter possess. Note that the WMMR operation is performed by this EPRS filter for the fixed selection rule $\mathcal{S}(\tilde{\mathbf{r}}) = \tilde{r}_{\beta_1}$. Such a fixed rule can not perform different output selections as may be warranted by local signal statistics. The ability of EPRS filters to select order statistic outputs as a function of local statistics, as measured by the ordering of $\tilde{\mathbf{x}}$, is of great advantage and results in superior performance over such fixed rule filters.

In addition to the sharpening filters discussed above, a number of other relatively simple filters can also be realized as EPRS filters. For example, a standard k rank filter is obtained by the following selection function

$$\mathcal{S}(\tilde{\mathbf{r}}) = \mathcal{S}(\tilde{r}_{\beta_1}) = \begin{cases} k + 1 & \text{if } \tilde{r}_{\beta_1} < k \\ k & \text{otherwise} \end{cases} . \quad (17)$$

Similarly, an α -trimmed mean filter is obtained by using the selection function $\mathcal{S}(\tilde{\mathbf{r}}) = \mathcal{S}(\tilde{r}_{\beta_1}) = \tilde{r}_{\beta_1}$, when $F_1(\mathbf{x})$ is given by (10). As with the other sharpeners, these filters are fixed rule filters and do not take into account local variations in $\{x\}$.

It should be noted that other filtering methods have been extended to incorporate both linear operations and ranking. Most notably, FIR-WOS filters [23] utilize linear combinations of samples and rank selection. However, the rank selection method is that of WOS filters in which samples are weighted (repeated) and an order statistic from the weighted (expanded) set is selected. This weighting allows certain samples to be emphasised and others deemphasized. Such a rank selection method is suited for smoothing applications and has not been successfully applied to sharpening. Due to the different selection methodology, FIR-WOS filters are not a subset of EPRS filters except in the limiting case where $M = N$ and $F_1(\mathbf{x})$ is an FIR operator.

As this section has shown, the EPRS filter class contains a wide array of possible filters. In order to aid the design and analysis of EPRS filters, the next section develops a number of filter properties and an optimization procedure. This optimization procedure returns the optimal selection rule for a given set of training signals.

4 Properties and Filter Optimization

In this section, some deterministic properties of the filters are derived. The first property discussed, which relates to the generalizability of a filter class, is scale and bias invariance. Next, several properties relating the rank of the mean and edges are given. These properties are then related to the edge sharpening capabilities of EPRS filters. While these properties will aid in design and analysis of EPRS filters, it may not be practical to design EPRS filters based solely on them. Thus, an adaptive procedure for optimizing over the filter class is also presented in this section.

4.1 Deterministic Properties

The EPRS filtering operation is clearly nonlinear. Consequently, the superposition property does not hold in its general form. The superposition property does, however, hold for the special case of a change in scale and bias.

Property 4.1 (Scale and bias invariance) *For $K = L = 1$ and $F_1(\mathbf{x})$ defined by (10), the EPRS filters have the property of scale and bias invariance. Specifically, if $\mathbf{y} = a\mathbf{x} + b\mathbf{1}_N$, where $\mathbf{1}_N$ is an N -vector of ones, then*

$$F_{EPRS}(\mathbf{y}) = aF_{EPRS}(\mathbf{x}) + b \quad (18)$$

for $a \geq 0$ and $-\infty < b < \infty$. If the function $\mathcal{S}(\tilde{\mathbf{r}})$ has the symmetry $\mathcal{S}(\tilde{\mathbf{r}}) = N+2 - \mathcal{S}((N+2)\mathbf{1}_{N+1} - \tilde{\mathbf{r}})$, then (18) is valid for $-\infty < a, b < \infty$. \square

The proof can be readily extended from that presented in [18]. Thus, the EPRS filters are not be sensitive to changes in scale and bias. This is important since these parameters often vary from image-to-image.

As stated earlier, the rank of the α -trimmed mean, as defined by (10), provides important information regarding the location of the filter window with respect to an edge. The following property illustrates this for a one-dimensional step edge.

Property 4.2 (α -trimmed mean rank for step edges) *For a step edge defined as*

$$x(n) = \begin{cases} a & \text{for } n < I \\ b & \text{for } n \geq I \end{cases}, \quad (19)$$

the rank of $F_1(\mathbf{x}(n))$ for $a < b$ is given by

$$\tilde{r}_{\beta_1}(n) = \begin{cases} (N + 2(I - n) + 1)/2 & \text{for } I - \frac{N+1}{2} < n \leq I + \frac{N-1}{2} - \alpha \\ N + 1 & \text{otherwise} \end{cases}. \quad (20)$$

For $a > b$, the rank of $F_1(\mathbf{x}(n))$ is given by

$$\tilde{r}_{\beta_1}(n) = \begin{cases} (N + 2(n - I) + 1)/2 & \text{for } I - \frac{N+1}{2} + \alpha \leq n < I + \frac{N-1}{2} \\ N + 1 & \text{otherwise} \end{cases}. \quad (21)$$

Proof: For the case when $n \leq I - \frac{N+1}{2}$, the window spans N samples with value a . Using stable sorting, $\tilde{r}_{\beta_1}(n) = N + 1$. Similarly, when $n \geq I + \frac{N-1}{2}$, the window spans N samples with value b and stable sorting yields $\tilde{r}_{\beta_1}(n) = N + 1$. When $I - \frac{N+1}{2} < n < I + \frac{N-1}{2}$, the filter window spans $z = n - (I - \frac{N+1}{2})$ samples with value b and $N - z$ with value a . In the region $I - \frac{N+1}{2} < n < I - \frac{N+1}{2} + \alpha$, $F_1(\mathbf{x}(n)) = a$. Thus, for $a < b$, $\tilde{r}_{\beta_1}(n) = N - z + 1$ and for $a > b$, $\tilde{r}_{\beta_1}(n) = N + 1$ using stable sorting. In the region $I - \frac{N+1}{2} + \alpha \leq n \leq I + \frac{N-1}{2} - \alpha$, $a < F_1(\mathbf{x}(n)) < b$. Thus, for $a < b$, $\tilde{r}_{\beta_1}(n) = N - z + 1$ and for $a > b$, $\tilde{r}_{\beta_1}(n) = z + 1$ in this region using stable sorting. Finally, in the region $I + \frac{N+1}{2} - \alpha < n < I + \frac{N-1}{2}$, $F_1(\mathbf{x}(n)) = b$. Thus, for $a < b$, $\tilde{r}_{\beta_1}(n) = N + 1$ and for $a > b$, $\tilde{r}_{\beta_1}(n) = z + 1$ using stable sorting. Substituting in the value of z gives rise to expressions (20) and (21). \square

Thus, the filter location relative to a step edge can be determined solely on the basis of the rank of $F_1(\mathbf{x}(n))$ (within a finite region around the edge). An example illustrating this is shown in Fig. 3 for a window size $N = 9$ filter with $\alpha = 1$. The rank of the mean provides detailed information about the location of the filter with respect to non-step edges as well. This is illustrated in Fig. 4, which shows a ramp edge and the resulting window mean and mean rank for all window locations. For simple sequences, the rank of the trimmed window mean can be determined in a straight forward manner. Next, we investigate the relationship between the mean and median for the more complex convex and concave sequences and edges.

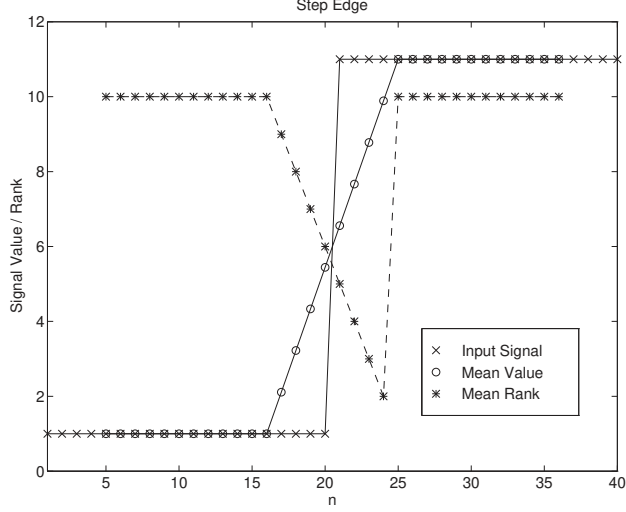


Figure 3: Example of a step edge showing the rank of the mean \tilde{r}_{β_1} for size $N = 9$ filter window ($\alpha = 1$). Notice that for each filter position which spans the edge, the mean takes on a unique rank.

To define convex and concave sequences, the first difference of samples is used. Let $\Delta(n)$ denote the first difference, $\Delta(n) = x(n) - x(n-1)$. Then, $\{x\}$ is convex (concave) if $\Delta(n) \geq \Delta(n-1)$ ($\Delta(n) < \Delta(n-1)$) for all n . Convex and concave sequences can be concatenated to form edges. We consider such edges after relating the mean and median of a window passing over each type of sequence.

Property 4.3 (α -trimmed mean rank bounds for convex and concave sequences) *For a size N window passing over a convex (concave), strictly increasing (decreasing), sequence $\{x\}$, the rank of the α -trimmed mean, $F_1(\mathbf{x})$, is bound below by $\tilde{r}_{\beta_1} \geq (N+1)/2$. Similarly, for a concave (convex), strictly increasing (decreasing), sequence, the rank of the α -trimmed mean is bound above by $\tilde{r}_{\beta_1} < (N+1)/2$.*

Proof: Note that $(N+1)/2$ is the rank of the median for a window size N . Also, since all sequences considered are strictly increasing or decreasing, the median sample is always the center sample in the window. Thus for the symmetric window considered, we can, without consequence, consider the median taken over an arbitrary window size. Also, for a strictly increasing sequence time and rank order are identical. Thus, $\mathbf{x} = \mathbf{x}^r$ and $F_1(\mathbf{x}) = \frac{1}{N'} \sum_{i=\alpha}^{N+1-\alpha} x(i) = \frac{1}{N'} \sum_{i=\alpha}^{N+1-\alpha} x_i = \mu_{N'}$, where $N' = N - 2(\alpha - 1)$ and $\mu_{N'}$ is the mean of an N' sample observation \mathbf{x}' (centered at the same location as \mathbf{x}). The same result holds for strictly decreasing sequences. Since the window size for the median is arbitrary, we choose N' and denote the median of this size window as $\gamma_{N'}$. Consequently, to prove the first assertion of the property it is sufficient to show that $\mu_{N'} \geq \gamma_{N'}$ for strictly increasing convex sequences. To simplify the notation, denote the the first difference of observed samples as

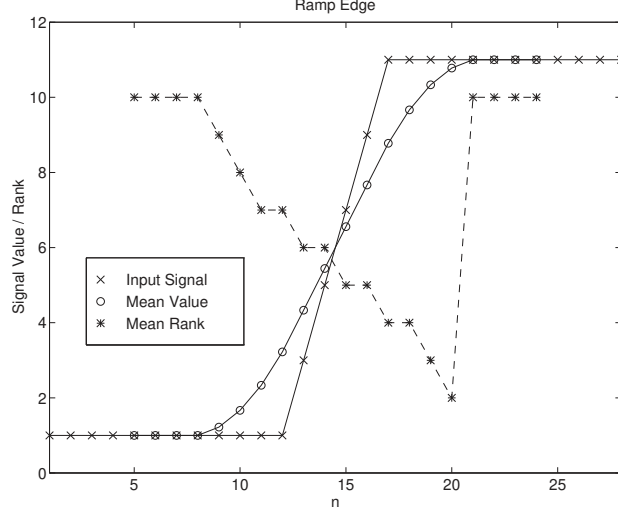


Figure 4: Example of a ramp edge showing the rank of the mean \tilde{r}_{β_1} for size $N = 9$ filter window where $\alpha = 1$. Notice that mean rank provides detailed edge location information.

$\Delta_i = x'_i - x'_{i-1}$ and the center sample in the observation window as x'_δ . Now consider $N'(\gamma_{N'} - \mu_{N'})$,

$$N'(\gamma_{N'} - \mu_{N'}) = N'x'_\delta - \sum_{i=1}^{N'} x'_i = \sum_{i=1}^{N'} x'_\delta - x'_i \quad (22)$$

$$= \sum_{i=1}^{\delta-1} x'_\delta - x'_i - (x'_{N'+1-i} - x'_\delta) = \sum_{i=1}^{\delta-1} \left(\sum_{j=i+1}^{\delta} \Delta_j - \sum_{j=\delta+1}^{N'+1-i} \Delta_j \right) \leq 0 \quad (23)$$

since $\sum_{j=i+1}^{\delta} \Delta_j - \sum_{j=\delta+1}^{N'+1-i} \Delta_j \leq 0$ for $1 \leq i \leq \delta - 1$ and strictly increasing convex sequences. The other assertions are proved similarly. \square

The rank of the α -trimmed mean can thus, for instance, distinguish an increasing convex sequence ($\tilde{r}_{\beta_1} \geq (N+1)/2$) from an increasing concave sequence ($\tilde{r}_{\beta_1} < (N+1)/2$). This discrimination property allows EPRS filters to enhance the gradient of the class of edges formed by concatenating convex and concave sequences. Convex and concave sequences can be concatenated to form the sets of increasing convex/concave, and decreasing concave/convex, edges. Due to their symmetric nature, we need only consider the set of increasing convex/concave edges. Similar results hold for the set of decreasing concave/convex edges.

An increasing sequence $\{x\}$, with first difference $\{\Delta\}$, contains a convex/concave edge with inflection point I if $\Delta(n) \geq \Delta(n-1)$ for $n \leq I$ and $\Delta(n) < \Delta(n-1)$ for $n > I$. Thus, $x(n)$ is convex for $n \leq I$ and concave for $n > I$. The previous theorem can now be used to determine which region of the convex/concave edge an observation window lies in. The theorem states that if the window spans only convex (concave) samples, then $\tilde{r}_{\beta_1} \geq (N+1)/2$ ($\tilde{r}_{\beta_1} < (N+1)/2$). Thus, if the window is centered sufficiently to the left (right) of the edge inflection point, $\tilde{r}_{\beta_1} \geq (N+1)/2$ ($\tilde{r}_{\beta_1} < (N+1)/2$). Moreover, the rank of the α -trimmed mean transitions from $\geq (N+1)/2$ to

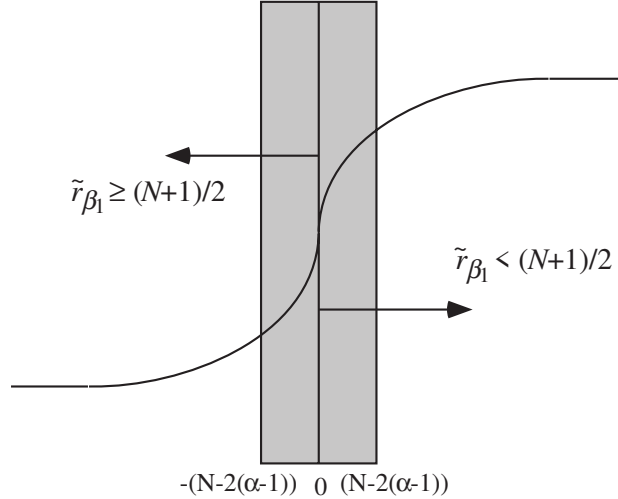


Figure 5: A convex/concave edge with inflection point at time index 0. On the left (right) side of the edge the rank of α -trimmed mean is bound by $\tilde{r}_{\beta_1} \geq (N+1)/2$ ($\tilde{r}_{\beta_1} < (N+1)/2$). The relation operator switches from \geq to $<$ within $N - 2(\alpha - 1)$ samples of the edge inflection point.

$< (N + 1)/2$ at a single point. Furthermore, the location of this transition point is bound to be within $N - 2(\alpha - 1)$ samples of the edge inflection point. This property of the α -trimmed mean rank is made exact in the following. The proof of the property follows closely that in [19], with the span of the window modified by trimming off the α smallest and largest samples, and so is not given.

Property 4.4 (α -trimmed mean rank bounds for increasing convex/concave edges) *For a size N window passing over an increasing convex/concave edge, there exists an integer m such that the rank of the α -trimmed mean, $F_1(\mathbf{x}(n))$, is bound below by $\tilde{r}_{\beta_1} \geq (N + 1)/2$ for $n \leq m$ and bound above by $\tilde{r}_{\beta_1} < (N + 1)/2$ for $n > m$. Moreover, the unique point m is within $N - 2(\alpha - 1)$ samples of the edge inflection point I , $|I - m| \leq N - 2(\alpha - 1)$. \square*

This property is illustrated in Fig. 5. The figure contains an increasing convex/concave edge and the $2(N - 2(\alpha - 1))$ sample wide region, centered at the edge inflection point, which contains the point m where the bound on the rank of the α -trimmed mean changes. This edge localizing property of the α -trimmed mean rank can be used by an EPRS filter to enhance edges. For an increasing edge, let T_1 and T_2 ($T_1 < T_2$) be two thresholds such that $x(n_1 - 1) < T_1 \leq x(n_1)$ and $x(n_2 - 1) < T_2 \leq x(n_2)$. Then $n_2 - n_1$ is the edge transition duration between levels T_1 and T_2 . The following property gives sufficient conditions on the selection rule $\mathcal{S}(\cdot)$ which result in an EPRS filter that reduces transition durations, or enhances edges.

Property 4.5 (Edge enhancement) *Let $F_{EPRS}(\cdot)$ be a window size N EPRS filter with $K = L = 1$ and $F_1(\cdot)$ the α -trimmed mean defined by (10). Restrict the selection rule $\mathcal{S} : \Omega_{\mathbf{z}} \mapsto \{1, 2, \dots, N +$*

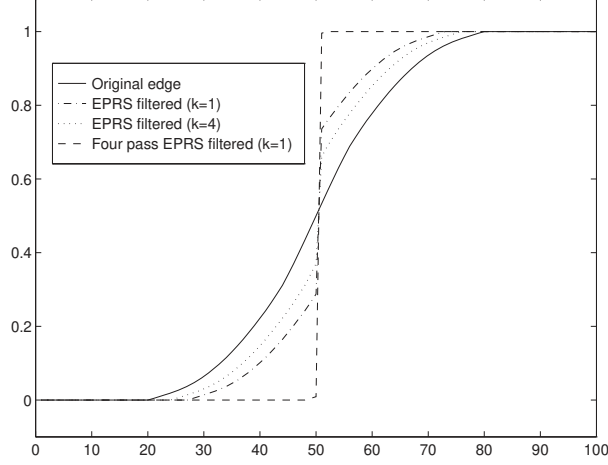


Figure 6: A convex/concave edge filtered by an EPRS filter meeting the conditions in Property 4.5. The selection rule is chosen to be symmetric, $k_1 = k_2 = k$. The window size is 15, and results for $k = 1$ and $k = 4$ are shown. For the $k = 1$ case, the results of a single and four passes are shown. After four passes the signal becomes a step edge and a root of the filter.

1} which defines $F_{EPRS}(\cdot)$ to be $\mathcal{S}(\tilde{\mathbf{r}}) = k_1$ when $\tilde{r}_{\beta_1} \geq (N+1)/2$ and $\mathcal{S}(\tilde{\mathbf{r}}) = N+2-k_2$ when $\tilde{r}_{\beta_1} < (N+1)/2$, $1 \leq k_1, k_2 < (N+1)/2$. Take $\{x\}$ to be an increasing convex/concave edge sequence with inflection point I . For any two thresholds $T_1 < x(I) < T_2$ with $|I-n_i| > (N-2(\alpha-1))$, $i = 1, 2$, the transition duration of $F_{EPRS}(\{x\})$ is less than that of $\{x\}$. If $|I-n_i| > ((3N+1)/2 - k_1 - 2(\alpha-1))$, $i = 1, 2$, then $F_{EPRS}(\cdot)$ reduces the transition duration by $N+1 - (k_1 + k_2)$ samples.

Proof: Note that since $\{x\}$ is increasing, time and rank order are identical, i.e., $\mathbf{x} = \mathbf{x}^r$. Thus, when $\tilde{r}_{\beta_1} \geq (N+1)/2$, $F_{EPRS}(\mathbf{x}) = \tilde{x}_{(k_1)} = x_{(k_1)} = x_{k_1}$ since $k_1 < (N+1)/2$. Similarly, when $\tilde{r}_{\beta_1} < (N+1)/2$, $F_{EPRS}(\mathbf{x}) = x_{N+1-k_2}$. Let m be the transition point where $\tilde{r}_{\beta_1} \geq (N+1)/2$ for windows centered at $n \leq m$ and $\tilde{r}_{\beta_1} < (N+1)/2$ for windows centered at $n > m$. Then, for observation windows centered at $n \leq m$, the sequence $\{x\}$ is retarded by $F_{EPRS}(\cdot)$, while the filter advances the sequence for windows centered at $n > m$. Since $T_1 < x(I) < T_2$ and $|I-n_i| > (N-2(\alpha-1))$, $i = 1, 2$, by property 4.4, the points n_1 and n_2 are guaranteed to be on opposite sides of m . Thus, $x(n_1)$ is among those points retarded (shifted right) and $x(n_2)$ is among those points advanced (shifted left). Consequently, the transition duration between T_1 and T_2 has been reduced. If $|I-n_i| > ((3N+1)/2 - k_1 - 2(\alpha-1))$, $i = 1, 2$, then $F_{EPRS}(\cdot)$ shifts $x(n_1)$ back by $(N+1)/2 - k_1$ samples and $x(n_2)$ ahead by $(N+1)/2 - k_2$ samples, reducing the transition by the sum of the shifts. \square

This edge enhancing property is illustrated in Fig. 6. The figure contains a convex/concave edge filtered by an EPRS filter meeting the above edge enhancing conditions. The results shown are for a symmetric selection rule, $k_1 = k_2 = k$. For this simple selection rule, the EPRS filter is equivalent to

the CS filter [19]. Such a basic selection rule can provide edge enhancement and allows for relatively simple analysis. However, in practice such a rule may perform poorly on signals with complex structures and edges (as will be seen in Section 5). By considering the ranks of the α -trimmed mean and M selected observation samples, the EPRS filter can use more sophisticated selection rules. This allows the filter to adapt to a wider variety of signal structures and edges. Moreover, by utilizing sample ranks, robust noise suppression and frequency selectivity, can be realized [17, 18].

Selecting a filter based solely on deterministic properties may be suitable for simple edge enhancement applications. A more practical solution for deblurring or deconvolution applications is to optimize over the filter class using training sequences that accurately account for the varied edge types present in the signal of interest. One such adaptive technique is discussed next. The procedure described is based on that in [17, 18] and is adapted for the EPRS filters. While the filters can be optimized under other criteria, such as the mean absolute error (MAE), we focus here on optimization under the sum of L_η normed error (LNE) criteria.

4.2 Filter Optimization

To develop and implement the optimization, the rank permutation vectors which comprise the permutation space must be indexed. By doing so, the permutation space can be expressed as

$$\Omega_{\mathbf{z}} = \{\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \dots, \tilde{\mathbf{r}}_{|\Omega_{\mathbf{z}}|}\}. \quad (24)$$

Also, let the observation vectors be written as a sequence, indexed in the order that they are utilized. Thus, the observation vectors can be written as $\mathbf{x}(\mathbf{n}_1), \mathbf{x}(\mathbf{n}_2), \dots, \mathbf{x}(\mathbf{n}_P)$, and the corresponding desired estimates as $d(\mathbf{n}_1), d(\mathbf{n}_2), \dots, d(\mathbf{n}_P)$.

For the EPRS filter defined by the decision rule $\mathcal{S}(\cdot)$, the LNE over the P element training sequence is

$$\sum_{i=1}^P |d(\mathbf{n}_i) - F_{EPRS}(\mathbf{x}(\mathbf{n}_i))|^\eta = \sum_{i=1}^P |d(\mathbf{n}_i) - \tilde{x}_{(\mathcal{S}(\tilde{\mathbf{r}}(\mathbf{n}_i)))}|^\eta. \quad (25)$$

The classifier that minimizes (25) is referred to as the optimal classifier and is denoted as $\mathcal{S}_{opt}(\cdot)$. In instances where more than one classifier satisfies the optimality criteria, a tie breaking rule must be employed to define a single optimal classifier.

The LNE in (25) can be partitioned according to the observation vectors with the same rank permutation. Let ξ_i be the index of rank permutation vector in $\Omega_{\mathbf{z}}$ corresponding to observation vector $\mathbf{x}(\mathbf{n}_i)$, such that $\tilde{\mathbf{r}}(\mathbf{n}_i) = \tilde{\mathbf{r}}_{\xi_i}$. Next, define $\Gamma_{j,P} = \{i \in \{1, 2, \dots, P\} : \xi_i = j\}$. The total LNE incurred over the training sequence by estimating the desired signal with the k^{th} order statistic, in those cases where the observation vector lies in the j^{th} partition can be written as

$$\mathcal{E}_j(k) = \sum_{i \in \Gamma_{j,P}} |d(\mathbf{n}_i) - \tilde{x}_{(k)}(\mathbf{n}_i)|^\eta. \quad (26)$$

If for some $j \in \{1, 2, \dots, |\Omega_{\mathbf{z}}|\}$, $\Gamma_{j,P} = \emptyset$, then define $\mathcal{E}_j(k) = 0$ for $k = 1, 2, \dots, N + 1$. The LNE of the EPRS filter defined by $\mathcal{S}(\cdot)$ can now be written as a sum of errors, partitioned according to permutation vector, yielding

$$\sum_{i=1}^P |d(\mathbf{n}_i) - F_{EPRS}(\mathbf{x}(\mathbf{n}_i))|^\eta = \sum_{j=1}^{|\Omega_{\mathbf{z}}|} \mathcal{E}_j(\mathcal{S}(\tilde{\mathbf{r}}_j)). \quad (27)$$

It is easy to show that the LNE in (27) is minimized if and only if each of the $\mathcal{E}_j(\mathcal{S}(\tilde{\mathbf{r}}_j))$ error sums is minimized. Thus, the optimal EPRS filter selection function is given by

$$\mathcal{S}_{opt}(\tilde{\mathbf{r}}_j) = k : \mathcal{E}_j(k) \leq \mathcal{E}_j(l) \forall l \neq k \quad (28)$$

for $j = 1, 2, \dots, |\Omega_{\mathbf{z}}|$. If there is not a unique minimum error for some j , then a tie breaking rule must be employed. In most practical cases, however, ties are unlikely given a sufficient number of training samples.

This training procedure always returns the globally optimal filter for the training set. Note that a low error norm, $\eta \leq 1$, may be useful for signals corrupted by heavy tailed noise. For such a choice, outliers do not dominate the sum of errors and filters that smooth excessively are not chosen. Moreover, for doubly exponential noise the $\eta = 1$ norm leads naturally to the maximum likelihood estimator. For other noise processes where outliers are less likely, e.g., Gaussian, higher order norms can be successfully used.

To implement this procedure the lookup table must be stored and the appropriate table index generated for each observation. The $N + K$ samples in the extended observation vector must be sorted and $M + L - 1$ (integer) multiplications performed to generate the appropriate index. The limiting factor for implementation, most often, is the size of the lookup table. This is clear from Theorem 3.1 which shows that the cardinality of the filter class grows rapidly for large values of $M + L$. Thus, for increasing values of $M + L$ a reduced lookup table may be required. This can be effectively accomplished by quantizing the extended rank vector using permutation colorings [24]. However, in many cases good performance can be achieved with low order filters that do not require coloring or prohibitively large lookup tables.

5 Experimental Results

The proposed filters can be used in a variety of signal and image restoration applications. Here we consider the application of these filters to the restoration of a blurred Markov signal and natural image corrupted by noise. Thus, the filters are performing deblurring or deconvolution in the presence of noise. Quantitative error results are presented and several filtered signals and images are shown for subjective evaluation. The EPRS filter are compared to other nonlinear edge enhancing filters.

5.1 Markov Signal Restoration

The first experiment involves the restoration of a blurred Markov sequence. A 1-dimensional signal is used in order to clearly illustrate the edge enhancement properties of the filters. The transition matrix \mathbf{P} characterizing the Markov signal model is a 5×5 matrix where

$$P_{ij} = \begin{cases} 0.80 & \text{for } i = j \\ 0.05 & \text{for } i \neq j \end{cases} . \quad (29)$$

The resulting signal is a 5 level sequence with relatively long constant regions connected by step edges. The blurring model is a 15 sample Gaussian point spread function (PSF) with variance of 4. The signal is further corrupted by additive contaminated Gaussian noise. We denote the contaminated Gaussian noise probability density function (pdf) as $\Phi(\sigma_1, \sigma_2, p)$. With probability $1 - p$, a noise sample is normally distributed with zero mean and variance σ_1^2 , and with probability p , a noise sample is normally distributed with zero mean and variance σ_2^2 . In general, $\sigma_1 < \sigma_2$ and p represents the ‘‘contamination’’ probability.

The mean absolute error (MAE) for the ERPS filters and others is shown in Fig. 7. Figure 7a shows the case where the noise has a $\Phi(0, 5, p)$ pdf. For the EPRS filters, $\alpha = 3$, and a window size of $N = 9$ is used for all filters. Also, for $M = 1$, $\gamma_1 = \delta$ which is the index of the center sample. For $M = 3$, γ_1 , γ_2 , and γ_3 are the indices of the three center samples. Each filter has been optimized under the L_1 or MAE criteria using signal and noise realizations not used for filtering. Notice that the EPRS filters outperform the other nonlinear filters. The order three filters provide the best results followed by the order one filters. Thus, improved performance can be gained by using high order EPRS filters. However, the filter selection lookup table grows rapidly and more training data is generally needed for higher order filters. The median yields the worst results because it has no edge enhancing capabilities and removes small signal structures. The results for the case where the blurred signal is corrupted by Gaussian noise are shown in Fig. 7b. In this case, $\alpha = 1$ and again the EPRS filters provide the best results.

To illustrate the performance of the various filters, a section of the filtered signals is shown in Fig. 8. The input signal is the blurred Markov signal which is corrupted by $\Phi(0, 5, .1)$ contaminated Gaussian noise. Each of the filter has been trained on a different signal and noise realization. The output of the order one EPRS filter where $N = 9$ and $\alpha = 3$ is shown in Fig. 8a. The MAE for the entire restored signal is 0.228. Notice that the impulses are suppressed and most edges are fully restored to step edges. The output of the same size order three EPRS filter is shown in Fig. 8b (MAE=0.204). This filter provides better detail preservation, as can be seen in the region $5 < n < 25$. The output of the order one RCRS filter, shown in Fig. 8c, does not show any gradient enhancement (MAE=0.310). However, the impulses are effectively suppressed. Thus, by

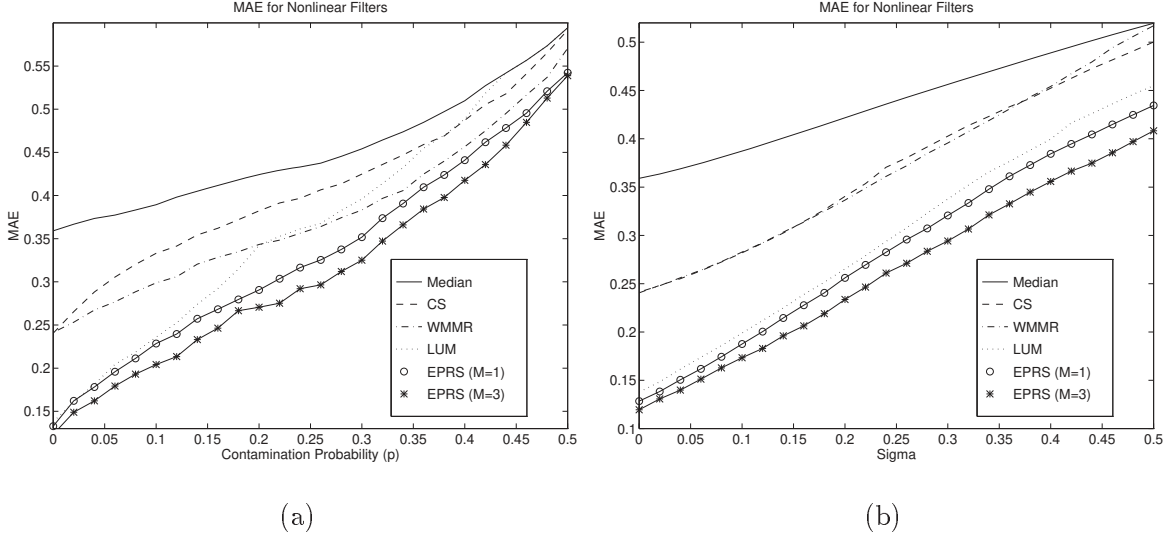


Figure 7: MAE for the Markov sequence restoration where the signal is blurred and corrupted by (a) $\Phi(0, 5, p)$ contaminated Gaussian noise (b) Gaussian noise. A window size of $N = 9$ is used for all filters.

incorporating the mean in the extended observation vector, the performance of the EPRS filters is significantly improved over that of the RCRS filters.

The outputs of the CS, LUM and WMMR filters are shown in Figs. 8d, 8e, and 8f with MAEs of 0.333, 0.236 and 0.299, respectively. These filters are also employing a window size of $N = 9$ and the CS and LUM filter parameters have been optimized by means of an exhaustive search using a different signal and noise realization. For the CS filter $j_{opt} = 3$ and $k_{opt} = l_{opt} = 3$ for the LUM. The CS and WMMR filters provide edge gradient enhancement and suppress the impulsive noise. However, significant loss of signal detail can be seen in the region $5 < n < 25$.

5.2 Image Restoration

In this section, the restoration of a blurred image corrupted by noise is examined. The original image, shown in Fig. 9a, is a 256×256 , 8 bit/pixel gray-scale image acquired from an airborne platform. The image is blurred using a 3×3 mean filter and $\Phi(0, 100, .02)$ contaminated Gaussian noise has been added. This corrupted image is shown in Fig. 9b (MAE=10.14). The nonlinear edge enhancing filters have been applied to this image. Each of the filters has been optimized under the L_1 or MAE criteria using the left half of the image in Fig. 9a with a different noise realization. In general, the EPRS filters should be optimized using imagery which is statistically representative of that to be restored. Note that the window sizes and parameters for each of the filters has been optimized using an exhaustive search.

The corrupted image has been filtered using a $5 \times 5 = N$ window size EPRS filter with $M = 1$, γ_1

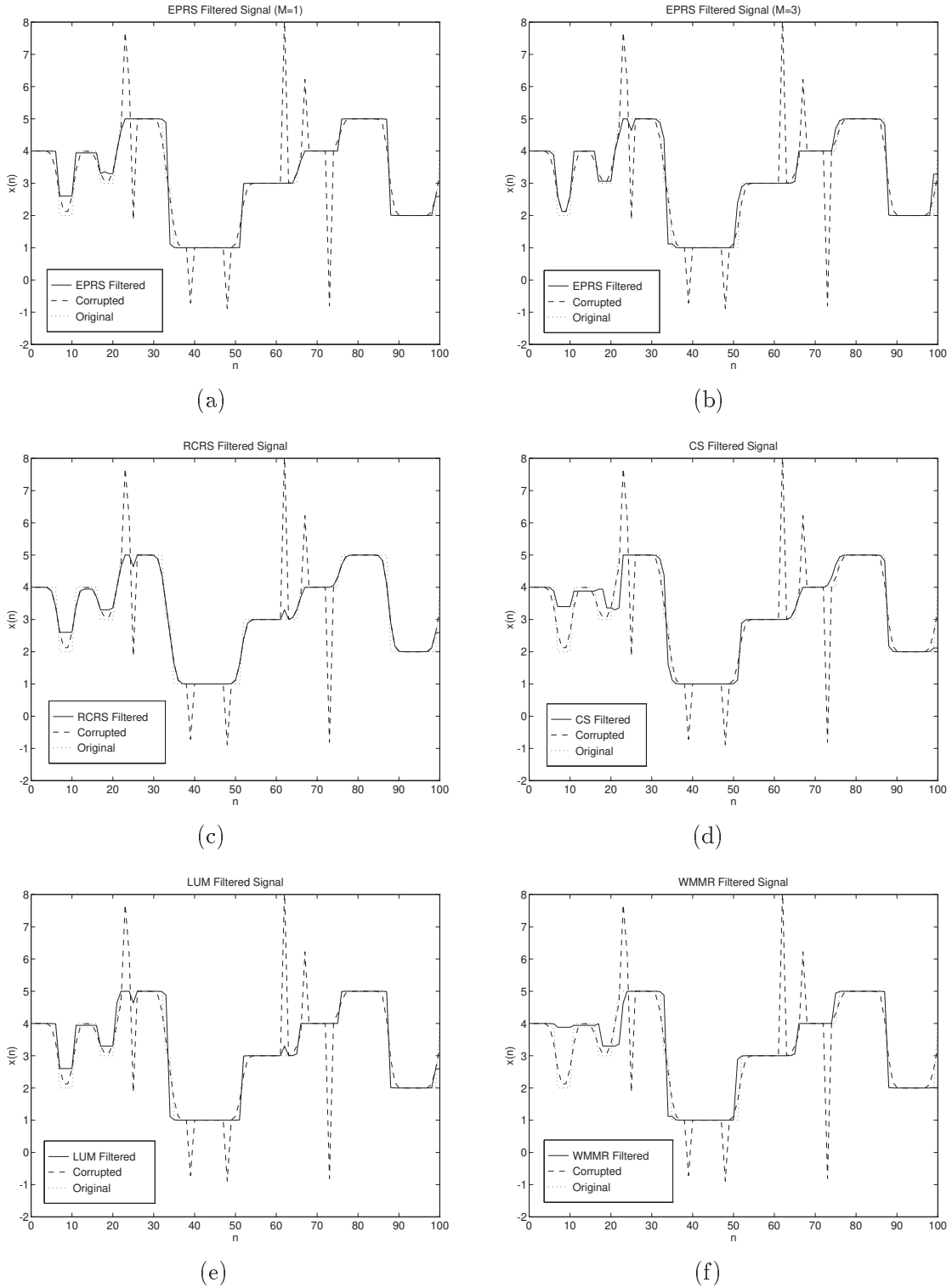


Figure 8: Filtered Markov signals. (a) Order one EPRS filter (MAE=0.228) (b) order three EPRS filter (MAE=0.204) (c) order one RCRS Filter (MAE=0.310) (d) CS filter (MAE=0.333, $j_{opt} = 3$) (e) LUM filter (MAE=0.236, $k_{opt} = l_{opt} = 3$) (f) WMMR filter (MAE=0.299).

is the index of the center sample, and $\alpha = 3$. The resulting image is shown in Fig. 9c (MAE=7.56). Notice that the impulses are suppressed and the edges are sharper in this image. The output of the optimal 3×3 CS filter is shown in Fig. 9d (MAE=9.44, $j_{opt} = 4$). Note that the 5×5 CS filter had a significantly higher error and is therefore not shown. While many edges are enhanced, some image detail is lost in this image. The output of the optimal 3×3 LUM filter is shown in Fig. 9e (MAE=8.71, $k_{opt} = 3$, $l_{opt} = 3$). A result similar to that of the CS is obtained with the 3×3 WMMR filter. This output image is shown in Fig. 9f (MAE=9.52).

6 Conclusions

The EPRS filters can be viewed as an extension of RCRS filters and permutation filters. They provide a broad framework in which many rank order based edge enhancing filters can be formulated including the CS, LUM, and WMMR filters. It has been shown that by using the rank of selected input samples and the rank of the mean, effective edge enhancement can be accomplished. The CS and LUM filters use partial information about the rank of the mean. However, it is demonstrated that by using the full mean rank information in addition to the full rank of selected input samples, superior results can be obtained. A deterministic optimization procedure is described here. This optimization guarantees the optimal EPRS filter for the given training data with any L_η normed error. The main advantage of EPRS filters over linear techniques is their ability to enhance edge in the presence of noise. In fact it is demonstrated that edge enhancement and noise suppression can be achieved simultaneously. Furthermore, no overshoot or undershoot is introduced by the EPRS filters.

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(a)

(b)



(c)

(d)

Figure 9: Image restoration example. (a) Original image (b) corrupted image (MAE=10.14) (c) 5×5 EPRS filtered image (MAE=7.56) (d) 3×3 CS filtered image (MAE=9.44, $j_{opt} = 4$) (e) 3×3 LUM filtered (MAE=8.71, $k_{opt} = 3$, $l_{opt} = 3$) (f) 3×3 WMMR filtered (MAE=9.52).



(e)



(f)

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