Araci and Özer *Advances in Difference Equations* (2015) 2015:272
DOI 10.1186/s13662-015-0610-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Extended q-Dedekind-type Daehee-Changhee sums associated with extended q-Euler polynomials

Serkan Araci[1*] and Özen Özer[2]

*Correspondence: mtsrkn@hotmail.com
[1]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, 27410, Turkey
Full list of author information is available at the end of the article

Abstract
In the present paper, we aim to specify a p-adic continuous function for an odd prime inside a p-adic q-analog of the extended Dedekind-type sums of higher order according to extended q-Euler polynomials (or weighted q-Euler polynomials) which is derived from a fermionic p-adic q-deformed integral on $\mathbb{Z}_p$.

MSC: 11S80; 11B68

Keywords: Dedekind sums; q-Dedekind-type sums; p-adic q-integral; extended q-Euler numbers and polynomials

1 Introduction

Let p be chosen as a fixed odd prime number. In this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$.

Let v_p be a normalized exponential valuation of $\mathbb{C}_p$ by

$$|p|_p = p^{-v_p(p)} = \frac{1}{p}.$$

When one talks of a q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$ (see, for details, [1–16]).

The following measure is defined by Kim: for any positive integer n and $0 \leq a < p^n$,

$$\mu_q\bigl(a + p^n\mathbb{Z}_p\bigr) = (-q)^a \frac{(1+q)}{1+q^{p^n}},$$

which can be extended to a measure on $\mathbb{Z}_p$ (for details, see [5–11]).

Extended q-Euler polynomials (also known as weighted q-Euler polynomials) are defined by

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} \left(\frac{1-q^{\alpha(x+\xi)}}{1-q^{\alpha}}\right)^n d\mu_q(\xi) \tag{1}$$

for $n \in \mathbb{Z}_+ := \{0, 1, 2, 3, \dots\}$. We note that

$$\lim_{q\to 1} \widetilde{E}_{n,q}^{(\alpha)}(x) = E_n(x),$$

where $E_n(x)$ are nth Euler polynomials, which are defined by the rule

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = e^{tx}\frac{2}{e^t+1}, \qquad |t| < \pi$$

(for details, see [13]). In the case $x = 0$ in (1), then we have $\widetilde{E}_{n,q}^{(\alpha)}(0) := \widetilde{E}_{n,q}^{(\alpha)}$, which are called extended q-Euler numbers (or weighted q-Euler numbers).

Extended q-Euler numbers and polynomials have the following explicit formulas:

$$\widetilde{E}_{n,q}^{(\alpha)} = \frac{1+q}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{\alpha l+1}}, \tag{2}$$

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \frac{1+q}{(1-q^{\alpha})^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{q^{\alpha lx}}{1+q^{\alpha l+1}}, \tag{3}$$

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n}\binom{n}{l}q^{\alpha lx}\widetilde{E}_{l,q}^{(\alpha)}\left(\frac{1-q^{\alpha x}}{1-q^{\alpha}}\right)^{n-l}. \tag{4}$$

Moreover, for $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$,

$$\widetilde{E}_{n,q}^{(\alpha)}(x) = \left(\frac{1+q}{1+q^d}\right)\left(\frac{1-q^{\alpha d}}{1-q^{\alpha}}\right)^n\sum_{a=0}^{d-1}(-1)^a\widetilde{E}_{n,q}^{(\alpha)}\left(\frac{x+a}{d}\right); \tag{5}$$

see [13].

For any positive integer h, k and m, Dedekind-type DC sums are given by Kim in [5, 6], and [7] as follows:

$$S_m(h,k) = \sum_{M=1}^{k-1}(-1)^{M-1}\frac{M}{k}\overline{E}_m\left(\frac{hM}{k}\right),$$

where $\overline{E}_m(x)$ are mth periodic Euler functions.

Kim [6] derived some interesting properties for Dedekind-type DC sums and considered a p-adic continuous function for an odd prime number to contain a p-adic q-analog of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$. Simsek [15] gave a q-analog of Dedekind-type sums and derived interesting properties. Furthermore, Araci *et al.* studied Dedekind-type sums in accordance with modified q-Euler polynomials with weight α [14], modified q-Genocchi polynomials with weight α [4], and weighted q-Genocchi polynomials [16].

Recently, weighted q-Bernoulli numbers and polynomials were first defined by Kim in [11]. Next, many mathematicians, by utilizing Kim's paper [11], have introduced various generalization of some known special polynomials such as Bernoulli polynomials, Euler polynomials, Genocchi polynomials, and so on, which are called weighted q-Bernoulli, weighted q-Euler, and weighted q-Genocchi polynomials in [1, 2, 11–13].

By the same motivation of the above knowledge, we give a weighted p-adic q-analog of the higher order Dedekind-type DC sums $k^m S_{m+1}(h,k)$ which are derived from a fermionic p-adic q-deformed integral on $\mathbb{Z}_p$.

2 Extended q-Dedekind-type sums associated with extended q-Euler polynomials

Let w be the Teichmüller character $(\operatorname{mod} p)$. For $x \in \mathbb{Z}_p^* := \mathbb{Z}_p / p\mathbb{Z}_p$, set

$$\langle x : q \rangle = w^{-1}(x)\left(\frac{1-q^x}{1-q}\right).$$

Let a and N be positive integers with $(p,a) = 1$ and $p \mid N$. We now consider

$$\widetilde{C}_q^{(\alpha)}\left(s,a,N:q^N\right) = w^{-1}(a)\left\langle a:q^\alpha\right\rangle^s \sum_{j=0}^{\infty}\binom{s}{j} q^{\alpha a j}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j \widetilde{E}_{j,q^N}^{(\alpha)}.$$

In particular, if $m+1 \equiv 0 \ (\operatorname{mod} p-1)$, then

$$\begin{aligned}\widetilde{C}_q^{(\alpha)}\left(m,a,N:q^N\right) &= \left(\frac{1-q^{\alpha a}}{1-q^\alpha}\right)^m \sum_{j=0}^{m}\binom{m}{j} q^{\alpha a j}\widetilde{E}_{j,q^N}^{(\alpha)}\left(\frac{1-q^{\alpha N}}{1-q^{\alpha a}}\right)^j \\ &= \left(\frac{1-q^{\alpha N}}{1-q^\alpha}\right)^m \int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha N(\xi+\frac{a}{N})}}{1-q^{\alpha N}}\right)^m d\mu_{q^N}(\xi).\end{aligned}$$

Thus, $\widetilde{C}_q^{(\alpha)}(m,a,N:q^N)$ is a continuous p-adic extension of

$$\left(\frac{1-q^{\alpha N}}{1-q^\alpha}\right)^m \widetilde{E}_{m,q^N}^{(\alpha)}\left(\frac{a}{N}\right).$$

Let $[\cdot]$ be the Gauss symbol and let $\{x\} = x - [x]$. Thus, we are now ready to introduce the q-analog of the higher order Dedekind-type DC sums $\widetilde{J}_{m,q}^{(\alpha)}(h,k:q^l)$ by the rule

$$\widetilde{J}_{m,q}^{(\alpha)}\left(h,k:q^l\right) = \sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right)\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha(l\xi+l\{\frac{hM}{k}\})}}{1-q^{\alpha l}}\right)^m d\mu_{q^l}(\xi).$$

If $m+1 \equiv 0 \ (\operatorname{mod} p-1)$,

$$\begin{aligned}&\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^{m+1}\sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha k}}\right)\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k(\xi+\frac{hM}{k})}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi) \\ &\quad= \sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^\alpha}\right)\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^m\int_{\mathbb{Z}_p}\left(\frac{1-q^{\alpha k(\xi+\frac{hM}{k})}}{1-q^{\alpha k}}\right)^m d\mu_{q^k}(\xi),\end{aligned}$$

where $p \mid k$, $(hM,p) = 1$ for each M. By (1), we easily state the following:

$$\begin{aligned}&\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^{m+1}\widetilde{J}_{m,q}^{(\alpha)}\left(h,k:q^k\right) \\ &\quad= \sum_{M=1}^{k-1}\left(\frac{1-q^{\alpha M}}{1-q^\alpha}\right)\left(\frac{1-q^{\alpha k}}{1-q^\alpha}\right)^m(-1)^{M-1}\end{aligned}$$

$$\times \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha k(\xi + \frac{hM}{k})}}{1 - q^{\alpha k}} \right)^m d\mu_{q^k}(\xi)$$

$$= \sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha}} \right) \widetilde{C}_q^{(\alpha)}\bigl(m, (hM)_k : q^k\bigr), \tag{6}$$

where $(hM)_k$ denotes the integer x such that $0 \leq x < n$ and $x \equiv \alpha \pmod{k}$.

It is not difficult to indicate the following:

$$\int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha(x+\xi)}}{1 - q^{\alpha}} \right)^k d\mu_q(\xi)$$

$$= \left(\frac{1 - q^{\alpha m}}{1 - q^{\alpha}} \right)^k \frac{1 + q}{1 + q^m} \sum_{i=0}^{m-1} (-1)^i \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha m(\xi + \frac{x+i}{m})}}{1 - q^{\alpha m}} \right)^k d\mu_{q^m}(\xi). \tag{7}$$

On account of (6) and (7), we easily see that

$$\left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi)$$

$$= \frac{1 + q^N}{1 + q^{Np}} \sum_{i=0}^{p-1} (-1)^i \left(\frac{1 - q^{\alpha Np}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha pN(\xi + \frac{a+iN}{pN})}}{1 - q^{\alpha pN}} \right)^m d\mu_{q^{pN}}(\xi). \tag{8}$$

Because of (6), (7), and (8), we develop the p-adic integration as follows:

$$\widetilde{C}_q^{(\alpha)}\bigl(s, a, N : q^N\bigr) = \frac{1 + q^N}{1 + q^{Np}} \sum_{\substack{0 \leq i \leq p-1 \\ a+iN \not\equiv 0 \pmod{p}}} (-1)^i \widetilde{C}_q^{(\alpha)}\bigl(s, (a + iN)_{pN}, p^N : q^{pN}\bigr).$$

So,

$$\widetilde{C}_q^{(\alpha)}\bigl(m, a, N : q^N\bigr) = \left(\frac{1 - q^{\alpha N}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha N(\xi + \frac{a}{N})}}{1 - q^{\alpha N}} \right)^m d\mu_{q^N}(\xi)$$

$$- \left(\frac{1 - q^{\alpha Np}}{1 - q^{\alpha}} \right)^m \int_{\mathbb{Z}_p} \left(\frac{1 - q^{\alpha pN(\xi + \frac{a+iN}{pN})}}{1 - q^{\alpha pN}} \right)^m d\mu_{q^{pN}}(\xi),$$

where $(p^{-1}a)_N$ denotes the integer x with $0 \leq x < N$, $px \equiv a \pmod{N}$ and m is integer with $m + 1 \equiv 0 \pmod{p - 1}$. Therefore, we have

$$\sum_{M=1}^{k-1} (-1)^{M-1} \left(\frac{1 - q^{\alpha M}}{1 - q^{\alpha}} \right) \widetilde{C}_q^{(\alpha)}\bigl(m, hM, k : q^k\bigr)$$

$$= \left(\frac{1 - q^{\alpha k}}{1 - q^{\alpha}} \right)^{m+1} \widetilde{J}_{m,q}^{(\alpha)}\bigl(h, k : q^k\bigr) - \left(\frac{1 - q^{\alpha k}}{1 - q^{\alpha}} \right)^{m+1}$$

$$\times \left(\frac{1 - q^{\alpha kp}}{1 - q^{\alpha k}} \right) \widetilde{J}_{m,q}^{(\alpha)}\bigl(\bigl(p^{-1}h\bigr), k : q^{pk}\bigr),$$

where $p \nmid k$ and $p \nmid hm$ for each M. Thus, we give the following definition, which seems interesting for further studying the theory of Dedekind sums.

Definition 1 Let h, k be positive integer with $(h,k) = 1$, $p \nmid k$. For $s \in \mathbb{Z}_p$, we define a p-adic Dedekind-type DC sums as follows:

$$\widetilde{\mathcal{J}}_{p,q}^{(\alpha)}\left(s:h,k:q^k\right) = \sum_{M=1}^{k-1}(-1)^{M-1}\left(\frac{1-q^{\alpha M}}{1-q^{\alpha}}\right)\widetilde{C}_q^{(\alpha)}\left(m,hM,k:q^k\right).$$

As a result of the above definition, we state the following theorem.

Theorem 2.1 *For $m+1 \equiv 0 \pmod{p-1}$ and $(p^{-1}a)_N$ denotes the integer x with $0 \le x < N$, $px \equiv a \pmod N$, then we have*

$$\begin{aligned}\widetilde{\mathcal{J}}_{p,q}^{(\alpha)}\left(s:h,k:q^k\right) &= \left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\widetilde{\mathcal{J}}_{m,q}^{(\alpha)}\left(h,k:q^k\right) \\ &\quad - \left(\frac{1-q^{\alpha k}}{1-q^{\alpha}}\right)^{m+1}\left(\frac{1-q^{\alpha kp}}{1-q^{\alpha k}}\right)\widetilde{\mathcal{J}}_{m,q}^{(\alpha)}\left(\left(p^{-1}h\right),k:q^{pk}\right).\end{aligned}$$

In the special case $\alpha = 1$, our applications in theory of Dedekind sums resemble Kim's results in [6]. These results seem to be interesting for further studies as in [5, 7] and [15].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the revised manuscript.

Author details
[1]Department of Economics, Faculty of Economics, Administrative and Social Sciences, Hasan Kalyoncu University, Gaziantep, 27410, Turkey. [2]Department of Mathematics, Faculty of Science and Arts, Kırklareli University, Kırklareli, 39000, Turkey.

Acknowledgements
The authors thank the reviewers for their helpful comments and suggestions, which have improved the quality of the paper.

Received: 22 June 2015 Accepted: 15 August 2015 Published online: 04 September 2015

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