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Extended Spinor Structure and Exotic Spinor Field in $\overline{\text{Poincaré}}$ Gauge Theory of Gravity

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In $\overline{\text{Poincaré}}$ gauge theory of gravity developed previously, we show that there is a structure, which is an extension of spinor structure, on the space-time manifold. An internal $\overline{\text{Poincaré}}$ gauge transformation induces a Poincaré transformation on the tangent affine space at each point of the space-time. Corresponding to the extended spinor structure, a spinor field φ having the "intrinsic" energy-momentum (=the quantum number associated with the internal translation) $P_k = K\gamma_k(1-s\gamma_5)$ is defined. We give an action I^M of φ on the Minkowski space-time, which leads to the Klein-Gordon equation. The action I^M is invariant both under the "internal $\overline{\text{Poincaré}}$" and the "Poincaré coordinate" transformations, and there are *four* conserved physical quantities correspondingly. They are the "intrinsic" and canonical energy-momenta and spin and orbital angular momenta. The "Poincaré coordinate" transformation can be regarded as a kind of internal transformation. For the interacting system of φ and $\overline{\text{Poincaré}}$ gauge fields, we have only *two* conserved quantities: one energy-momentum and one angular momentum. The "intrinsic" and canonical energy-momenta (the spin and orbital angular momenta) of the field φ are transformed into each other through the $\overline{\text{Poincaré}}$ gauge interaction. A brief comment on the second quantization of the free φ-field on the Minkowski space-time is given.

§ 1. Introduction

In a series of papers,[1)~6)] a theory of gravity, named a $\overline{\text{Poincaré}}$ gauge theory of gravity, has been developed. This theory has the covering group of the Poincaré group as the internal gauge group, and the possible existence of fields, named exotic fields, with non-vanishing "intrinsic" energy-momentum P_k has been pointed out.[1)] No particle of this kind has been observed, but we believe, notwithstanding, that the possibility of exotic fields is worth exploring because of the following reasons: (1) The internal translation functions effectively in our gauge theory. (2) Exotic fields definitely discriminate our theory from the other theories of gravity such as the Kibble-type Poincaré gauge theory.[7),8)]

In the present paper, we consider an extended spinor structure on the space-time M and an associated spinor field φ, named exotic spinor field, with the "intrinsic" energy-momentum $P_k = K\gamma_k(1-s\gamma_5)$. Also, we clarify the intuitive meaning of the internal $\overline{\text{Poincaré}}$*) gauge transformation. In § 2, we discuss a structure, which is an extension of spinor structure, and a spinor field associated with it is introduced. The transformation induced on an tangent affine space by the internal $\overline{\text{Poincaré}}$ gauge transformation is also discussed. In § 3, considering the field φ on the Minkowski space-time, we give an invariant action and the generators of the "*internal* $\overline{\text{Poincaré}}$" and "Poincaré coordinate" transformations. The discussion is given in an arbitrary coordinate system. In § 4, we discuss the interaction of φ with $\overline{\text{Poincaré}}$ gauge fields

*) The word (or rather symbol) $\overline{\text{Poincaré}}$ is used for the covering group of the Poincaré group.

$A^{kl}{}_\mu$ and $A^k{}_\mu$. In § 5, a summary and remarks are given.

§ 2. Extended spinor structure, exotic spinor field φ and transformation induced on tangent affine space

Our theory is formulated[1] on the basis of the principal fiber bundle

$$\mathcal{P}=\{P, M, \pi, \bar{P}_0, \bar{P}_0, U_\alpha, \varphi_\alpha\}, \tag{2·1}$$

over the space-time manifold M having the covering group $\bar{P}_0$ of the proper orthochronous Poincaré group P_0 as the structure group. Here P, π and φ_α are the bundle space, the projection and the coordinate function, respectively.

We consider the affine frame bundle[1]

$$\mathcal{A}(M)=\{A(M), M, \pi_A, GA(4, R), GA(4, R), V, \varphi_A\}. \tag{2·2}$$

This has a subbundle $\mathcal{P}(M)$ having the group P_0 as the structure group:

$$\mathcal{P}(M)=\{P(M), M, \pi_P, P_0, P_0, U_\alpha, \varphi_{P\alpha}\} \tag{2·3}$$

with

$$P(M)\overset{\text{def}}{=}\bigcup_\alpha\bigcup_{y\in U_\alpha}[\{(t-\psi_\alpha(y))^l e_{(\alpha)l}(y), L^l{}_k e_{(\alpha)l}(y)\}|(t, L)\in P_0], \quad k, l=0, 1, 2, 3, \tag{2·4}$$

where[1] $\psi_\alpha{}^k$ and $e_{(\alpha)k}$ are the Higgs type field ψ^k and the vierbein field e_k, respectively, corresponding to the local cross section[1] σ_α:

$$\sigma_\alpha(y)\overset{\text{def}}{=}\varphi_\alpha[y, (0, I_2)], \quad \forall y\in U_\alpha. \tag{2·5}$$

In Eq. (2·3), π_P is the projection, and the coordinate function $\varphi_{P\alpha}$ is defined by

$$\varphi_{P\alpha}[y, (t, L)]\overset{\text{def}}{=}\{(t-\psi_\alpha(y))^l e_{(\alpha)l}(y), L^l{}_k e_{(\alpha)l}(y)\}, \quad \forall y\in U_\alpha, \quad \forall(t, L)\in P_0. \tag{2·6}$$

There is a 2 to 1 bundle homomorphism[1] F: $\mathcal{P}\to\mathcal{A}(M)$ defined by

$$F(u)\overset{\text{def}}{=}\{(t_\alpha-\psi_\alpha(y))^k e_{(\alpha)k}(y), [\Lambda(a_\alpha)]^l{}_k e_{(\alpha)l}(y)\}, \quad \forall u=\varphi_\alpha[y, (t_\alpha, a_\alpha)]\in P \tag{2·7}$$

with Λ being the covering map from $SL(2, C)$ to the proper orthochronous Lorentz group. Also, the mapping F is a bundle homomorphism from $\mathcal{P}$ *onto* the subbundle $\mathcal{P}(M)$. Thus, we have a structure $\{F, \mathcal{P}, \mathcal{P}(M)\}$, which is an extension of the spinor structure[1] $\{F_0, \mathcal{S}, \mathcal{L}(M)\}$, and it will be called an extended spinor structure hereafter.

The internal Poincaré gauge transformation

$$\sigma'(y)=\sigma(y)\cdot(t(y), a(y)), \quad (t(y), a(y))\in\bar{P}_0 \tag{2·8}$$

induces the transformation

$$F(\sigma(y)\cdot(t(y), a(y)))=\{t^k(y)\boldsymbol{e}_k(y)+P(y), [\Lambda(a(y))]^l{}_k\boldsymbol{e}_l(y)\}, \qquad (2\cdot9)$$

where we have expressed

$$F(\sigma(y))=\{P(y), \boldsymbol{e}_k(y)\}. \qquad (2\cdot10)$$

Thus, *the internal* $\overline{\text{Poincaré}}$ *gauge transformation* $\sigma(y)\to\sigma'(y)$ *induces a Poincaré transformation on the tangent affine space at each point* y *of the space-time.*

Corresponding to the extended spinor structure $\{F, \mathscr{P}, \mathscr{P}(M)\}$, we can define a field φ which transforms according as

$$\varphi'(y)=\varphi(y)-it^k(y)P_k\varphi(y)-\frac{i}{2}\omega^{kl}(y)M_{kl}\varphi(y), \qquad (2\cdot11)$$

under the transformation (2·8) with

$$(\Lambda(a(y)))^k{}_l=\delta^k{}_l+\omega^k{}_l(y), \qquad (2\cdot12)$$

where t^k and $\omega^{kl}=-\omega^{lk}$ are infinitesimal functions.

In Eq. (2·11), we have defined*)

$$P_k\overset{\text{def}}{=}K\gamma_k(1-s\gamma_5), \qquad M_{kl}\overset{\text{def}}{=}-\frac{i}{4}[\gamma_k, \gamma_l] \qquad (2\cdot13)$$

with K being a constant complex c-number and $s^2=1$. The matrices P_k, M_{kl} satisfy the commutation relations of the Poincaré algebra:

$$[P_k, P_l]=0, \qquad [M_{kl}, P_m]=i\eta_{km}P_l-i\eta_{lm}P_k,$$

$$[M_{kl}, M_{mn}]=i\eta_{km}M_{ln}+i\eta_{ln}M_{km}-i\eta_{kn}M_{lm}-i\eta_{lm}M_{kn}. \qquad (2\cdot14)$$

We have the relations

$$P_kP_l=0, \qquad \Gamma_k\Gamma_l=0, \qquad (2\cdot15)$$

where Γ_k is the Pauli-Lubanski operator:

$$\Gamma_k\overset{\text{def}}{=}\frac{1}{2}\varepsilon_{klmn}M^{lm}P^n \qquad (2\cdot16)$$

with ε_{klmn} being the Levi-Civita symbol. Equation (2·15) implies that the eigenvalues of P_k and of Γ_k are all vanishing.

The field φ will be called an exotic spinor field.

§ 3. Field φ on the Minkowski space-time, invariant action and generators

3.1. *Invariant action and field equation*

The action

$$I^M\overset{\text{def}}{=}\int\boldsymbol{L}^M d^4y \qquad (3\cdot1)$$

*) The γ-matrices are defined in Appendix A.

with

$$\mathbf{L}^M=\sqrt{-g}\,L^M=\sqrt{-g}\,\bar{\varphi}(\Box-m^2)\varphi \tag{3·2}$$

leads to the Klein-Gordon equations

$$(\Box-m^2)\varphi=0\,,\qquad (\Box-m^2)\bar{\varphi}=0\,. \tag{3·3}$$

Here,

$$\bar{\varphi}\overset{\text{def}}{=}\varphi^*\eta\,,\qquad \eta\overset{\text{def}}{=}-i\gamma_0(p+iq\gamma_5) \tag{3·4}$$

with p and q being real constants satisfying the conditions

$$(K+K^*)p+is(K-K^*)q=0\,,\qquad p^2+q^2\neq 0\,. \tag{3·5}$$

In Eq. (3·2), $g\overset{\text{def}}{=}\det(g_{\mu\nu})$ with $g_{\mu\nu}$ being the component of the Minkowski metric in an arbitrary coordinate system $\{y^\mu,\ \mu=0,1,2,3\}$, and the d'Alembertian has the expression

$$\Box=\eta^{kl}\frac{\partial^2}{\partial x^k\partial x^l}=\eta^{kl}e^\mu{}_k\frac{\partial}{\partial y^\mu}\left(e^\nu{}_l\frac{\partial}{\partial y^\nu}\right) \tag{3·6}$$

with $\{x^k,\ k=0,1,2,3\}$ being a Minkowskian coordinate system and $e^\mu{}_k\overset{\text{def}}{=}\partial y^\mu/\partial x^k$.

The coordinate $\{x^k,\ k=0,1,2,3\}$ is regarded as a "field" on the space-time and the Euler derivative $\delta\mathbf{L}^M/\delta x^k$ is given by

$$\frac{\delta\mathbf{L}^M}{\delta x^k}=-e^\mu{}_k\left(\bar{\varphi}_{,\mu}\frac{\delta\mathbf{L}^M}{\delta\bar{\varphi}}+\frac{\delta\mathbf{L}^M}{\delta\varphi}\varphi_{,\mu}\right). \tag{3·7}$$

Thus, the field equation of x^k is automatically satisfied if Eq. (3·3) is satisfied, which is quite reasonable because x^k is not a dynamical variable. This corresponds to the statement A(iv) of section 9 of Ref. 1), and x^k is the special relativistic correspondent of the Higgs type field ψ^k.

The action I^M is invariant under the three kinds of transformations:

$$x'^k=x^k\,,\quad y'^\mu=y^\mu\,,\quad \varphi'(y')=\varphi(y)-ic^kP_k\varphi(y)-\frac{i}{2}d^{kl}M_{kl}\varphi(y)\,, \tag{3·8}$$

$$x'^k=x^k-C^k-\Omega^k{}_lx^l\,,\quad y'^\mu=y^\mu\,,\quad \varphi'(y')=\varphi(y)\,, \tag{3·9}$$

$$x'^k=x^k\,,\quad y'^\mu=y^\mu+\delta y^\mu\,,\quad \varphi'(y')=\varphi(y)\,. \tag{3·10}$$

Here, c^k, $d^{kl}=-d^{lk}$, C^k and $\Omega^{kl}\overset{\text{def}}{=}\eta^{lm}\Omega^k{}_m=-\Omega^{lk}$ are all infinitesimal real constants.

Neither of the transformations (3·8) and (3·9) is *not* the global correspondent of the transformation described by Eqs. (2·8), (2·11) and (2·12), but only their combination with the condition

$$c^k=C^k\,,\qquad d^{kl}=\Omega^{kl}\,, \tag{3·11}$$

corresponds*) to Eqs. (2·8), (2·11) and (2·12). In view of this, the transformations

*) See also Eq. (4·23) and its footnote.

(3·8) and (3·9) will be referred to as "internal $\overline{\text{Poincaré}}$" and "Poincaré coordinate" transformations, respectively.

The transformation (3·10) is a general coordinate transformation.

3.2. *Generators —— two energy-momenta and two angular momenta ——*

The generator $G^{(\varphi)1}(\sigma)$ of the "internal $\overline{\text{Poincaré}}$" transformation (3·8) is given by

$$G^{(\varphi)1}(\sigma)=c^k M^{(\varphi)}{}_k+\frac{1}{2}d^{kl}S^{(\varphi)}{}_{kl} \tag{3·12}$$

with

$$M^{(\varphi)}{}_k\overset{\text{def}}{=}\int_\sigma \boldsymbol{T}^{(\varphi)}{}_k{}^\mu d\sigma_\mu\,,\qquad S^{(\varphi)}{}_{kl}\overset{\text{def}}{=}\int_\sigma \boldsymbol{S}^{(\varphi)}{}_{kl}{}^\mu d\sigma_\mu\,, \tag{3·13}$$

where

$$\boldsymbol{T}^{(\varphi)}{}_k{}^\mu\overset{\text{def}}{=}i\sqrt{-g}\,g^{\mu\nu}(\bar{\varphi}_{,\nu}P_k\varphi-\bar{\varphi}P_k\varphi_{,\nu})\,, \tag{3·14}$$

$$\boldsymbol{S}^{(\varphi)}{}_{kl}{}^\mu\overset{\text{def}}{=}i\sqrt{-g}\,g^{\mu\nu}(\bar{\varphi}_{,\nu}S_{kl}\varphi-\bar{\varphi}S_{kl}\varphi_{,\nu})\,, \tag{3·15}$$

and $(g^{\mu\nu})$ is the inverse of the matrix $(g_{\mu\nu})$.

The generator $G^{(\varphi)2}(\sigma)$ of the "Poincaré coordinate" transformation (3·9) is given by

$$G^{(\varphi)2}(\sigma)=-C^k M^{(\varphi)c}{}_k+\frac{1}{2}\Omega^{kl}L^{(\varphi)}{}_{kl} \tag{3·16}$$

with

$$M^{(\varphi)c}{}_k\overset{\text{def}}{=}\int_\sigma \boldsymbol{T}^{(\varphi)c}{}_k{}^\mu d\sigma_\mu\,,\qquad L^{(\varphi)}{}_{kl}\overset{\text{def}}{=}\int_\sigma \boldsymbol{M}^{(\varphi)}{}_{kl}{}^\mu d\sigma_\mu\,, \tag{3·17}$$

where

$$\boldsymbol{T}^{(\varphi)c}{}_k{}^\mu\overset{\text{def}}{=}e^\mu{}_k\boldsymbol{L}^M+\sqrt{-g}\,g^{\mu\lambda}\{e^\nu{}_k\bar{\varphi}_{,\lambda}\varphi_{,\nu}-\bar{\varphi}(e^\nu{}_k\varphi_{,\nu})_{,\lambda}\}\,, \tag{3·18}$$

$$\boldsymbol{M}^{(\varphi)}{}_{kl}{}^\mu\overset{\text{def}}{=}2[x_{[k}\{e^\mu{}_{l]}\boldsymbol{L}^M+\sqrt{-g}\,g^{\mu\lambda}(e^\nu{}_{l]}\bar{\varphi}_{,\lambda}\varphi_{,\nu}-\bar{\varphi}(e^\nu{}_{l]}\varphi_{,\nu})_{,\lambda})\}-\sqrt{-g}\,e^\mu{}_{[k}e^\nu{}_{l]}\bar{\varphi}\varphi_{,\nu}]\,. \tag{3·19}$$

These generators are conserved,

$$G^{(\varphi)1}(\sigma_2)=G^{(\varphi)1}(\sigma_1)\,,\qquad G^{(\varphi)2}(\sigma_2)=G^{(\varphi)2}(\sigma_1)\,, \tag{3·20}$$

if

$$\int_C\{c^k\boldsymbol{T}^{(\varphi)}{}_k{}^\mu+\frac{1}{2}d^{kl}\boldsymbol{S}^{(\varphi)}{}_{kl}{}^\mu\}d\sigma_\mu=0 \tag{3·21}$$

and

$$\int_C\{-C^k\boldsymbol{T}^{(\varphi)c}{}_k{}^\mu+\frac{1}{2}\Omega^{kl}\boldsymbol{M}^{(\varphi)}{}_{kl}{}^\mu\}d\sigma_\mu=0\,, \tag{3·22}$$

respectively, where C is the hyper-cylinder at spatial infinity between the space-like surfaces σ_1 and σ_2. The conditions (3·21) and (3·22) are satisfied, if*)

$$\varphi=O\left(\frac{1}{r^{1/2+\varepsilon}}\right),\quad \partial_k\varphi=O\left(\frac{1}{r^{3/2+\varepsilon}}\right),\quad \partial_k\partial_l\varphi=O\left(\frac{1}{r^{5/2+\varepsilon}}\right),\quad (\varepsilon>0) \tag{3·23}$$

with $r\overset{\text{def}}{=}\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$ and ε being positive but otherwise arbitrary. The conservations of $G^{(\varphi)1}(\sigma)$ and of $G^{(\varphi)2}(\sigma)$ correspond to the invariance of I^M under the transformations (3·8) and (3·9), respectively.

The quantities $M^{(\varphi)}{}_k$ and $M^{(\varphi)c}{}_k$ are the "intrinsic" energy-momentum and the canonical energy-momentum,**) respectively, and also $S^{(\varphi)}{}_{kl}$ and $L^{(\varphi)}{}_{kl}$ are the spin angular momentum and the orbital angular momentum, respectively. It is remarkable that *these four are separately conserved.*

The generator $G^{(\varphi)\#}(\sigma)$ of the general coordinate transformation (3·10) vanishes identically

$$G^{(\varphi)\#}(\sigma)\equiv 0\,, \tag{3·24}$$

as is easily shown.

§ 4. Interaction of φ with the $\overline{\text{Poincaré}}$ gauge fields $A^{kl}{}_\mu$ and $A^k{}_\mu$ and conserved quantities

The interaction of φ with $\overline{\text{Poincaré}}$ gauge fields is introduced by the replacement***)

$$\partial_k\partial_l\varphi\longrightarrow \nabla_k\nabla_l\varphi=e^\mu{}_k\left[(\nabla_l\varphi)_{,\mu}+\frac{i}{2}A^{ij}{}_\mu M_{ij}\nabla_l\varphi+iA^i{}_\mu P_i\nabla_l\varphi+A_l{}^m{}_\mu\nabla_m\varphi\right] \tag{4·1}$$

with

$$\nabla_k\varphi=e^\mu{}_k\left[\varphi_{,\mu}+\frac{i}{2}A^{lm}{}_\mu M_{lm}\varphi+iA^l{}_\mu P_l\varphi\right], \tag{4·2}$$

in the Lagrangian L^M.

The total action for the interacting system of the fields φ, $A^{kl}{}_\mu$ and $A^k{}_\mu$ is given by****)

$$\boldsymbol{I}=\int \boldsymbol{L}d^4y \tag{4·3}$$

with

*) For Eq. (3·21), the condition for $\partial_k\partial_l\varphi$ in Eq. (3·23) is not necessary.

**) In this classical treatment, the canonical energy of φ is not positive definite. This point will be discussed in § 5(e).

***) In this section, $e^\mu{}_k$ and $g_{\mu\nu}$ are the components of the vierbein and the Lorentz metric tensor fields introduced in Ref. 1). They agree with the correspondents in § 3, if the space-time is Minkowskian. Also, *unless otherwise stated,* all the symbols, conventions, terminologies and the assumptions are the same as those in Ref. 3). Some of them are enumerated in Appendix B for convenience.

****) To denote a coordinate system, we use here $\{y^\mu, \mu=0,1,2,3\}$ instead of $\{x^\mu, \mu=0,1,2,3\}$ in Ref. 3). Also, it is assumed from the outset that Frobenius's condition for the unit time-like vector field[2),3)] N is satisfied. Hence, the space-time is sliced into a family $\mathfrak{F}$ of space-like surfaces.

$$\boldsymbol{L} \overset{\text{def}}{=} \boldsymbol{L}^G + \sqrt{-g}\,\bar{\varphi}(\eta^{kl}\nabla_k\nabla_l\varphi - m^2\varphi)\,, \tag{4·4}$$

where $\boldsymbol{L}^G$ is a kinematical Lagrangian of the fields $A^{kl}{}_\mu$ and $A^k{}_\mu$. In what follows, $\boldsymbol{L}^G$ is chosen to be that given by Eq. (4·8) of Ref. 3):

$$\boldsymbol{L}^G \overset{\text{def}}{=} \bar{\boldsymbol{L}}^G + 2a\partial_\nu(\sqrt{-g}\, e^\mu{}_k e^\nu{}_l A^{kl}{}_\mu)\,, \tag{4·5}$$

$$\bar{\boldsymbol{L}}^G \overset{\text{def}}{=} \sqrt{-g}\,[\alpha t^{klm}t_{klm} + \beta v^k v_k + \gamma a^k ak + a_1 A^{klmn}A_{klmn} + a_2 B^{klmn}B_{klmn}$$
$$+ a_3 C^{klmn}C_{klmn} + a_4 E^{kl}E_{kl} + a_5 I^{kl}I_{kl} + a_6 R^2 + aR]\,. \tag{4·6}$$

Here, α, β, γ, a_i $(i=1,2,\cdots,6)$ and a are all real constants, t_{klm}, v_k and a_k are the irreducible components[3] of the field strength

$$T_{klm} = 2e^\mu{}_l e^\nu{}_m(e_{k[\nu,\mu]} + A_{kn[\mu}e^n{}_{\nu]})\,, \tag{4·7}$$

and A_{klmn}, B_{klmn}, C_{klmn}, E_{kl}, I_{kl} and R are the irreducible components[3] of the field strength

$$R_{klmn} = 2e^\mu{}_m e^\nu{}_n(A_{kl[\nu,\mu]} + A_{kr[\mu}A^r{}_{l\nu]})\,. \tag{4·8}$$

By following the procedure developed in Refs. 3), 5) and 6), we can discuss the generators of $\overline{\text{Poincaré}}$ and coordinate transformations.

As has been pointed out in Ref. 5), generators depend critically on the choice of the set of independent field variables with which the generators are defined. When we choose $\{\psi^k, A^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ with *),3)

$$\psi^k = e^{(0)k}{}_\mu y^\mu + \psi^{(0)k} + O(1/r^\beta)\,, \qquad (\beta>0) \tag{4·9}$$

as the set of independent variables, we get the following results:

(i) The generator $G(\sigma)$ of the $\overline{\text{Poincaré}}$ gauge transformation

$$\sigma'(y) = \sigma(y)\cdot(t(y),\, a(y)) \tag{4·10}$$

with

$$(t(y))^k = c^k + b^k(y)\,, \qquad (\Lambda(a(y)))^k{}_l = \delta^k{}_l + d^k{}_l + \omega^k{}_l(y)\,, \tag{4·11}$$

has the expression

$$G(\sigma) = c^k M_k + \frac{1}{2}d^{kl}S_{kl}\,, \tag{4·12}$$

when the field equations of $A^k{}_\mu$ and of $A^{kl}{}_\mu$ are satisfied. Here,

$$M_k \overset{\text{def}}{=} \int_\sigma {}^{\text{tot}}\boldsymbol{T}_k{}^\mu d\sigma_\mu\,, \qquad S_{kl} \overset{\text{def}}{=} \int_\sigma {}^{\text{tot}}\boldsymbol{S}_{kl}{}^\mu d\sigma_\mu\,, \qquad \sigma\in\mathcal{F} \tag{4·13}$$

with**)

*) In Eq. (4·9), $e^{(0)k}{}_\mu \overset{\text{def}}{=} \lim_{r\to\infty} e^k{}_\mu$, $\psi^{(0)k}$ is a constant and β is positive but otherwise arbitrary. In Eq. (4·11), c^k and $d^{kl}=-d^{lk}$ are infinitesimal constants, and b^k and $\omega^{kl}=-\omega^{lk}$ are infinitesimal functions vanishing at spatial infinity.[3]

**) The signature of ${}^{\text{tot}}\boldsymbol{T}_k{}^\mu$ is opposite to that in Ref. 3).

$$ {}^{\rm tot}T_k{}^\mu \stackrel{\rm def}{=} \Big[\partial_\nu\Big(\frac{\partial L}{\partial \psi^k{}_{,\mu\nu}}\Big) - \frac{\partial L}{\partial \psi^k{}_{,\mu}} - A_k{}^l{}_\nu \frac{\partial L}{\partial A^l{}_{\nu,\mu}} $$

$$ + i\Big(\partial_\nu\Big(\frac{\partial L}{\partial \varphi_{,\mu\nu}}\Big) - \frac{\partial L}{\partial \varphi_{,\mu}}\Big) P_k \varphi - i\frac{\partial L}{\partial \varphi_{,\mu\nu}} P_k \varphi_{,\nu} \Big], \tag{4·14} $$

$$ {}^{\rm tot}S_{kl}{}^\mu \stackrel{\rm def}{=} 2\Big[\psi_{[k}\Big(\frac{\partial L}{\partial \psi^{l]}{}_{,\mu}} - \partial_\nu\Big(\frac{\partial L}{\partial \psi^{l]}{}_{,\mu\nu}}\Big)\Big) + \psi_{[k,\nu}\frac{\partial L}{\partial \psi^{l]}{}_{,\mu\nu}} + A_{[k\nu}\frac{\partial L}{\partial A^{l]}{}_{\nu,\mu}} + 2A^m{}_{[k\nu}\frac{\partial L}{\partial A^{l]m}{}_{\nu,\mu}} $$

$$ + \frac{i}{2}\Big(\partial_\nu\Big(\frac{\partial L}{\partial \varphi_{,\mu\nu}}\Big) - \frac{\partial L}{\partial \varphi_{,\mu}}\Big) M_{kl}\varphi - \frac{i}{2}\frac{\partial L}{\partial \varphi_{,\mu\nu}} M_{kl}\varphi_{,\nu} \Big]. \tag{4·15} $$

The dynamical energy-momentum M_k and the "spin" angular momentum S_{kl} are both conserved. Also, the flux integral representation for M_k is given by

$$ M_k = 2ae^{(0)}{}_{k\lambda}\int_\sigma \partial_\rho\partial_\sigma\{gg^{\lambda[\nu}g^{\rho]\sigma}\}d\sigma_\nu\,, \tag{4·16} $$

while for S_{kl} we have

$$ S_{kl} = e^{(0)}{}_{k\mu}e^{(0)}{}_{l\nu}\int_\sigma \partial_\rho K^{\mu\nu\lambda\rho}d\sigma_\lambda - 2\psi^{(0)}{}_{[k}M_{l]} \tag{4·17} $$

with

$$ K^{\mu\nu\lambda\rho} \stackrel{\rm def}{=} 2a[y^\mu\partial_\sigma\{(-g)g^{\nu[\lambda}g^{\rho]\sigma}\} - y^\nu\partial_\sigma\{(-g)g^{\mu[\lambda}g^{\rho]\sigma}\} + (-g)g^{\mu[\lambda}g^{\nu\rho]} - \eta^{\mu[\lambda}\eta^{\nu\rho]}] \tag{4·18} $$

for the asymptotically flat space-time. The quantities M_k and S_{kl} are the total energy-momentum and *total* (=*spin*+*orbital*) angular momentum, respectively.*)
(ii) The generator $G^\#(\sigma)$ of the transformation

$$ y'^\mu = y^\mu + \delta y^\mu \tag{4·19} $$

with

$$ \delta y^\mu = O(r^{1-\zeta})\,, \qquad (\delta y^\mu)_{,\nu} = O(r^{-\zeta})\,, \qquad (\zeta \geq 0) \tag{4·20} $$

vanishes,

$$ G^\#(\sigma) = 0\,, \tag{4·21} $$

when the field equations of $A^k{}_\mu$ and of $A^{kl}{}_\mu$ are satisfied. In Eq. (4·20), ζ is non-negative but otherwise arbitrary.

For the case when Eq. (4·9) is replaced with the more general form[5)]

$$ \psi^k = \rho e^{(0)k}{}_\mu y^\mu + \psi^{(0)k} + O(1/r^\beta)\,, \qquad (\beta > 0) \tag{4·22} $$

and for the case in which $\{\psi, e^k{}_\mu, A^{kl}{}_\mu, \varphi\}$ is chosen as the set of independent field

*) The expression (4·16) is different from the corresponding expression (5·3) of Ref. 3) by a factor of −1. The expression (4·17) is identical with Eq. (5·8) of Ref. 3). They essentially agree with the corresponding expressions for the total energy-momentum and for the total angular momentum in general relativity, respectively. For the asymptotically "non-flat" space-time, S_{kl} has the expression given by Eq. (6·4) of Ref. 3).

variables, we can easily discuss the generators by following the procedure given in Refs. 5) and 6).

The following should be noted: There are *four* conserved quantities for the free φ-field on the Minkowski space-time, while we have only *two* for the interacting system of the fields φ, $A^{kl}{}_\mu$ and $A^k{}_\mu$. This corresponds to the fact that the action $\boldsymbol{I}$ is *not invariant* under the transformation*)

$$\psi'^k=\psi^k-C^k-\Omega^k{}_l\psi^l\,,\quad A'^k{}_\mu=A^k{}_\mu+A^k{}_{l\mu}C^l-\Omega^k{}_lA^l{}_\mu\,,$$

$$A'^{kl}{}_\mu=A^{kl}{}_\mu-\Omega^k{}_mA^{ml}{}_\mu-\Omega^l{}_mA^{km}{}_\mu\,,\quad \varphi'=\varphi-ic^kP_k\varphi-\frac{i}{2}d^{kl}M_{kl}\varphi \qquad (4\cdot23)$$

with infinitesimal constants C^k, $\Omega^{kl}=-\Omega^{lk}$, c^k and $d^{kl}=-d^{lk}$, unless

$$C^k=c^k \qquad \text{and} \qquad \Omega^{kl}=d^{kl}\,. \qquad (4\cdot24)$$

§ 5. Summary and remarks

The results obtained in the above can be summarized as follows:

(i) There is the extended spinor structure $\{F, \mathcal{P}, \mathcal{P}(M)\}$ on the space-time M and the exotic spinor field φ is defined correspondingly.

(ii) The internal $\overline{\text{Poincaré}}$ gauge transformation $\sigma(y)\to\sigma'(y)$ induces a Poincaré transformation on the tangent affine space at each point y of M. This gives an intuitive interpretation for the internal transformations in the present theory.

(iii) For the field φ on the Minkowski space-time, the action I^M of free field has been given. It leads to the Klein-Gordon equation and is invariant under the "internal $\overline{\text{Poincaré}}$", the "Poincaré coordinate" and the general coordinate transformations. There are *four* conserved quantities $M^{(\varphi)}{}_k$, $S^{(\varphi)}{}_{kl}$, $M^{(\varphi)c}{}_k$ and $L^{(\varphi)}{}_{kl}$. The first two (the second two) constitute the generator of the "internal $\overline{\text{Poincaré}}$" transformation (3·8) (the "Poincaré coordinate" transformation (3·9)), and their conservations correspond to the invariance of I^M under the corresponding transformations.

The generator $G^{(\varphi)\#}(\sigma)$ of the general coordinate transformation (3·10) vanishes identically and thus carries no dynamical information at all. The field equation of the Minkowskian coordinate $\{x^k,\ k=0, 1, 2, 3\}$, when x^k is regarded as a "field", is automatically satisfied if the field equation of φ is satisfied. The "field" x^k is the special relativistic correspondent of the field ψ^k.

(iv) For the interacting system of the fields φ, $A^{kl}{}_\mu$ and $A^k{}_\mu$, there are *two* conserved quantities M_k and S_{kl}, which are the total energy-momentum and the *total* (=*spin* +*orbital*) angular momentum of the system. They constitute the generator of the $\overline{\text{Poincaré}}$ transformation, while the generator $G^\#(\sigma)$ of the coordinate transformation (4·19) vanishes. That we do not have *four* conserved quantities now corresponds to the fact that the action $\boldsymbol{I}$ is *not invariant* under the transformation (4·23). The "intrinsic" energy-momentum and canonical energy-momentum of the field φ are transformed into each other through the $\overline{\text{Poincaré}}$ (gravitational) gauge interaction.

*) This transformation corresponds to the product of the transformations (3·8) and (3·9), and *it is a constant $\overline{\text{Poincaré}}$ gauge transformation when the relation (4·24) holds.*

The same holds true also for the spin and orbital angular momenta.

The following is worth mentioning:

(a) The "Poincaré coordinate" transformation (3·9) is a kind of "internal" transformation. Two kinds of "internal" transformations leave the action I^M invariant, which, however, is *not inherited* when the $\overline{\text{Poincaré}}$ gauge interaction takes part in. The field φ is quite distinctive also in this respect. There is no corresponding situation for the Dirac field.[6] It should be minded here that these "internal" transformations are qualitatively different from "genuine" internal transformations in theories of strong and electro-weak interactions. The transformations (3·8) and (3·9) (and also the $\overline{\text{Poincaré}}$ gauge interaction) are rather suitable to be called soldered-internal (or semi-internal or quasi-internal) transformations.

(b) For the interacting system of φ and the Dirac field φ^D on the Minkowski space-time, the Lagrangian

$$L^{(e-D)} \overset{\text{def}}{=} \bar{\varphi}(\Box - m^2)\varphi - \frac{1}{2}(\bar{\varphi}^D \gamma^k \partial_k \varphi^D - \partial_k \bar{\varphi}^D \gamma^k \varphi^D) - m^D \bar{\varphi}^D \varphi^D + f\bar{\varphi}\varphi\bar{\varphi}^D\varphi^D \tag{5·1}$$

with m^D and f being the mass of φ^D and the coupling constant, respectively, is invariant under the transformations

$$\varphi'(x') = \varphi(x) - ic^k P_k \varphi(x) - \frac{i}{2} d^{kl} M_{kl} \varphi(x),$$
$$x'^k = x^k, \qquad \varphi'^D(x') = \varphi^D(x), \tag{5·2}$$

$$\varphi'^D(x') = \varphi^D(x) - \frac{i}{2} \Omega^{kl} M_{kl} \varphi^D(x),$$
$$x'^k = x^k - C^k - \Omega^k{}_l x^l, \qquad \varphi'(x') = \varphi(x). \tag{5·3}$$

There are *four* conserved quantities correspondingly. The "intrinsic" energy momentum and the spin angular momentum of the field φ are both conserved by themselves also for this system. Also, they are qualitatively different from the spin angular momentum of φ^D, as is easily seen from Eqs. (5·2) and (5·3): The "internal $\overline{\text{Poincaré}}$" transformation of φ is qualitatively different from the internal $\overline{\text{Poincaré}}$ transformation of φ^D.

(c) For fields on the Minkowski space-time, there are *four* conserved quantities also for the following cases:

(1) The field φ has (non-gravitational) gauge interactions.

(2) The field φ has non-gauge interactions such as $\bar{\varphi}\varphi\varphi^s$, where φ^s is a scalar field.

Presumably, for all the interactions but for the $\overline{\text{Poincaré}}$ gauge interaction, there are *four* conserved quantities for the field φ, and it is very likely that the mutual transformations between the "intrinsic" and canonical energy-momenta and between the spin and orbital angular momenta can take place *only through* the $\overline{\text{Poincaré}}$ gauge interaction.

(d) Although the field φ is a spinor field, φ does not have the Dirac magnetic moment with the magnitude $Q\hbar/2mc$, even if it has the electric charge Q.

(e) The canonical energy $M^{(\varphi)c0}$ is not positive definite, as has been mentioned in § 3. This can be seen by examining the structure of $\boldsymbol{L}^M$ given by Eq. (3·2). Consider the case with $p=1$ and $q=0$, for simplicity, and express φ as

$$\varphi=\begin{pmatrix}\xi\\ \zeta\end{pmatrix} \tag{5·4}$$

with ξ and ζ being two component spinor fields. Then, $\boldsymbol{L}^M$ takes the form

$$\boldsymbol{L}^M=\sqrt{-g}\{-\xi^*(\Box-m^2)\xi+\zeta^*(\Box-m^2)\zeta\}, \tag{5·5}$$

which implies that $M^{(\varphi)c0}$ is not positive definite. The positive definiteness of the canonical energy of the classical free φ-field may possibly be realized by imposing some subsidiary conditions.

Also, naive methods[9] of the second quantization does not work well for the free φ-field. This is deeply related with the structure of $\boldsymbol{L}^M$ mentioned above. Presumably, for the quantization of φ, some techniques utilizing an indefinite metric Hilbert space and subsidiary conditions will be necessary. It is very likely that the free φ-field itself (or a part of it) is a ghost.

(f) All the observed fields are not exotic fields.[1] (1) It is possible that exotic fields exist and they interact with usual fields only through the $\overline{\text{Poincaré}}$ gauge interactions. If this is actually the case, it is natural that exotic fields have not been observed as yet. One may speculate that the dark matter is composed of particles of this kind. (2) Also, there is a possibility that all the exotic fields are ghosts.

(g) As for the other theoretical possibilities of exotic fields, the following can be known quite easily:

(1) The scalar and vector fields cannot be exotic fields.

(2) For the spinor fields $\varphi^{(s)}$ $(s=1, -1)$ with "intrinsic" energy-momentum $P^{(s)}{}_k$ $=K\gamma_k(1-s\gamma_5)$, we have the $\overline{\text{Poincaré}}$ gauge invariant Lagrangian

$$\begin{aligned}\boldsymbol{L}'^M\overset{\text{def}}{=}\sqrt{-g}\Big[\frac{1}{2}(&\bar{\nabla}_k\bar{\varphi}^{(1)}(1+\gamma_5)\gamma^k\varphi^{(1)}-\bar{\varphi}^{(1)}\gamma^k(1-\gamma_5)\bar{\nabla}_k\varphi^{(1)}\\ &+\bar{\nabla}_k\bar{\varphi}^{(-1)}(1-\gamma_5)\gamma^k\varphi^{(-1)}-\bar{\varphi}^{(-1)}\gamma^k(1+\gamma_5)\bar{\nabla}_k\varphi^{(-1)})\\ &-m\bar{\varphi}^{(1)}(1+\gamma_5)\varphi^{(-1)}-m^*\bar{\varphi}^{(-1)}(1-\gamma_5)\varphi^{(1)}\Big]\end{aligned} \tag{5·6}$$

with m being a constant and

$$\bar{\nabla}_k\bar{\varphi}^{(s)}=e^\mu{}_k\Big[\bar{\varphi}^{(s)}{}_{,\mu}-\frac{i}{2}A^{lm}{}_\mu\bar{\varphi}^{(s)}M_{lm}-iA^l{}_\mu\bar{\varphi}^{(s)}P^{(s)}{}_l\Big]. \tag{5·7}$$

But, the energy-momentum $P^{(s)}{}_k$ does not play any role in the $\overline{\text{Poincaré}}$ gauge interaction. Namely, $P^{(s)}{}_k$ in the covariant derivatives gives vanishing contributions to the Lagrangian $\boldsymbol{L}'^M$, and also, the expressions of the energy-momentum and of angular momentum of this system do not depend on $P^{(s)}{}_k$ explicitly. The "intrinsic" energy-momentum in this case seems "void".

(3) A spinor-tensor field $\varphi_{ijk\cdots}$ can have "intrinsic" energy-momentum

$$(P_k)_{\alpha ijk\cdots\beta}{}^{lmn\cdots}=K(\gamma_k(1-s\gamma_5))_{\alpha\beta}\delta_i{}^l\delta_j{}^m\delta_k{}^n\cdots . \tag{5·8}$$

(4) There are representations of $\overline{\text{Poincaré}}$ algebra with the generator of the internal translation having continuous eigenvalues, but this case seems not to be easily dealt with by standard methods of field theory, because it is too exotic.

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Appendix A

The γ-matrices used in the text are defined by

$$\gamma_k\overset{\text{def}}{=}i\beta\alpha_k\,,\quad k=1,2,3\,,\quad \gamma_0\overset{\text{def}}{=}-i\beta\,,\quad \gamma_5\overset{\text{def}}{=}i\gamma_1\gamma_2\gamma_3\gamma_0 \tag{A·1}$$

with the matrices

$$\alpha_k\overset{\text{def}}{=}\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad k=1,2,3\,,\quad \beta\overset{\text{def}}{=}\begin{pmatrix}I_2 & 0\\ 0 & -I_2\end{pmatrix}, \tag{A·2}$$

where σ_k is the Pauli matrix and I_2 is the 2×2 unit matrix. We have the relation

$$\{\gamma_k,\gamma_l\}=2\eta_{kl} \tag{A·3}$$

with $(\eta_{kl})\overset{\text{def}}{=}\text{diag}\,(-1,1,1,1)$, and γ_0 and γ_a $(a=1,2,3,5)$ are anti-hermite and hermite, respectively.

Appendix B

We enumerate here the symbols, terminologies and assumptions in Ref. 3) which are presupposed in § 4.

(i) First, $A^{kl}{}_\mu$ and $A^k{}_\mu$ are the Lorentz and translational gauge potentials, respectively. They, together with the Higgs type field ψ^k, constitute the components $e^k{}_\mu$ of the duals of the vierbein fields:

$$e^k{}_\mu\overset{\text{def}}{=}\nabla_\mu\psi^k=\psi^k{}_{,\mu}+A^k{}_{l\mu}\psi^l+A^k{}_\mu\,. \tag{B·1}$$

They transform according as

$$\psi'^k=(\Lambda(a^{-1}))^k{}_l(\psi^l-t^l)\,,\quad A'^k{}_\mu=(\Lambda(a^{-1}))^k{}_l(A^l{}_\mu+t^l{}_{,\mu}+A^l{}_{m\mu}t^m)\,,$$

$$A'^k{}_{l\mu}=(\Lambda(a^{-1}))^k{}_mA^m{}_{n\mu}(\Lambda(a))^n{}_l+(\Lambda(a^{-1}))^k{}_m(\Lambda(a))^m{}_{l,\mu}\,,$$

$$e'^{k}{}_{\mu}=(\Lambda(a^{-1}))^{k}{}_{l}e^{l}{}_{\mu}\,, \tag{B·2}$$

under the Poincaré gauge transformation

$$\sigma'(y)=\sigma(y)\cdot(t(y),a(y))\,,\quad t(y)\in T^4\,,\quad a(y)\in SL(2,C)\,. \tag{B·3}$$

(ii) By asymptotically flat space-time, we mean the space-time satisfying the following conditions:

⟨1⟩ The metric tensor

$$g=g_{\mu\nu}dy^{\mu}\otimes dy^{\nu} \tag{B·4}$$

with $g_{\mu\nu}\stackrel{\text{def}}{=}e^{k}{}_{\mu}\eta_{kl}e^{l}{}_{\nu}$ has the asymptotic property

$$h_{\mu\nu,(m)}=O(1/r^{1+m})\,,\quad m=0,1,2,3,4 \tag{B·5}$$

with $r\stackrel{\text{def}}{=}\sqrt{(y^1)^2+(y^2)^2+(y^3)^2}$ and $h_{\mu\nu,(m)}$ being the m-th order partial derivatives of $h_{\mu\nu}\stackrel{\text{def}}{=}g_{\mu\nu}-\eta_{\mu\nu}$ with respect to y^{λ}, where $(\eta_{\mu\nu})\stackrel{\text{def}}{=}\text{diag}\,(-1,1,1,1)$.

⟨2⟩ The field strength $R^{kl}{}_{\mu\nu}$ has the asymptotic property

$$R^{kl}{}_{\mu\nu}=O(1/r^{2+\alpha})\,,\quad (\alpha>0) \tag{B·6}$$

with α being positive but otherwise arbitrary. The condition (B·6) is ensured if

$$A^{kl}{}_{\mu}=O(1/r^{1+\alpha})\,,\quad A^{kl}{}_{\mu,(m)}=O(1/r^{2+\alpha})\,,\quad m=1,2\,, \tag{B·7}$$

which shall be assumed in discussion for the case of the asymptotically flat space-time.

By asymptotically "non-flat" space-time, we mean the space-time satisfying the above ⟨1⟩ and the condition:

$$\langle 2\rangle' \qquad A^{kl}{}_{\mu}=K^{kl}{}_{\mu}/r+L^{kl}{}_{\mu} \tag{B·8}$$

with $K^{kl}{}_{\mu}$ being a constant and

$$L^{kl}{}_{\mu}=O(1/r^2)\,,\quad L^{kl}{}_{\mu,(m)}=O(1/r^3)\,,\quad m=1,2\,. \tag{B·9}$$

(iii) In the discussion of angular momenta, we assume the following:

$$\text{(a)}\qquad \left(\alpha+\frac{2}{3}a\right)\left(\beta-\frac{2}{3}a\right)\neq 0 \qquad \text{or} \qquad \alpha=-\beta=-\frac{2}{3}a\,. \tag{B·10}$$

(b) The dynamical system is at rest as a whole with respect to a frame defined by the family $\mathcal{F}$ of space-like surfaces.

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