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Extended theory of harmonic maps connects general relativity to chaos and quantum mechanism

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ABSTRACT

General relativity and quantum mechanism are two separate rules of modern physics explaining how nature works. Both theories are accurate, but the direct connection between two theories was not yet clarified. Recently, researchers blur the line between classical and quantum physics by connecting chaos and entanglement equation. Here, we showed the Duan's extended HM theory, which has the solution of the general relativity, can also have the solutions of the classic chaos equations and even the solution of Schrödinger equation in quantum physics, suggesting the extended theory of harmonic maps may act as a universal theory of physics.

1. Introduction

Harmonic map (HM) has been used to recover the Ernst formulation of Einstein's equations in general relativity [1–4], in which the solution of Euler-Lagrange equations of the harmonic maps could be obtained through solving the Laplace-Beltrami's equation and the geodesic equations [3]. From the viewpoint of physics, the traditional theory of harmonic maps contains only the kinetic energy term. For a wider application of this method, the Lagrangian of the traditional HM theory was supplemented with a potential energy term by Duan [5]. This extended HM theory is helpful for studying the travelling wave solutions of some types of nonlinear partial differential equations. Recently, researchers discovered the classical chaos showed a connection to quantum physics via entanglement [6,7]. Here, we asked whether the extended HM theory can recover the classical chaos questions even the Schrödinger equation in quantum physics.

In classic physics, all physical processes are described by differential equations. Over the last several decades, many nonlinear ordinary differential equations possessing chaotic behaviours have been discovered [8], including the nonlinear equation for anharmonic system in periodic fields [9] given by

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}-\beta x+\alpha x^3=b\mathrm{Cos}(\varpi t), \tag{1}$$

where k, α and β are control parameters, and equation for a parametrically excited pendulum given by [10]

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+[\alpha+\beta\mathrm{Cos}(\varpi t)]\mathrm{Sin}(x)=0, \tag{2}$$

where k, α and β are control parameters. To date, the study of chaotic behaviours using the space distribution of nonlinear partial differential equations has been rare and is currently only in its initial stage.

Here, we used the extended theory of HM, as the express of the space distribution of nonlinear partial differential equations, derivate the classical chaos Eqs. (1) and (2). Considering the chaotic behaviours connects to quantum mechanism via quantum engagement [6,7], we further found the extended theory of HM has the solution of the Schrödinger equation for a one-dimensional harmonic oscillator. Considering the extended theory of HM has solutions of the Einstein's equations, classic chaotic solutions and Schrödinger equation, suggesting the extended theory of HM may function as a fundamental rule our universe of physics.

2. The extended theory of harmonic maps (HM)

The theory of HM [11–13] became an important branch of mathematical physics decades ago and has been applied to a wide variety of problems in mathematics and theoretical physics. In this section, we re-introduce the formulation of the extended theory of HM that has been reported by Duan [5].

Let M and N be two Riemannian manifolds with local coordinates x^μ $(\mu=1,2,\dots,m)$ on M and local coordinates

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Φ^A (A = 1, 2, ..., n) on N. The metrics on M and N are denoted by

$$dl^2 = g_{\mu\nu}(x)dx^\mu dy^\nu; \quad \dim(M) = m$$
$$dL^2 = G_{AB}(\Phi)d\Phi^A d\Phi^B; \quad \dim(N) = n \tag{3}$$

respectively. A mapping

$$\Phi : M \to N$$
$$x \to \Phi(x)$$

is called an extended harmonic map if it satisfies the Euler–Lagrange equation resulting from the variational principle $\delta I = 0$, using the action

$$I = \int d^n x\sqrt{g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi^A\partial_\nu\Phi^B G_{AB}(\Phi) + V(\Phi)\right] \tag{4}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $V(\Phi)$ is the potential function of Φ^A, and $V(\Phi) = V(\Phi^1, \Phi^1, \ldots, \Phi^n)$.

It is obvious that the traditional HM corresponds to the case of $V(\Phi) = 0$ in Eq. (4). It has been showed that, in two dimensional case, the solutions of the Ernst equation in Einstein's general relativity can be recovered from the Euler-Lagrange's equations of traditional HM [3,14].

The conditions for a map to be harmonic are given by the Euler–Lagrange equations

$$\frac{\partial L}{\partial\Phi^A} - \partial_\mu\frac{\partial L}{\partial\partial_\mu x\Phi^A} = 0; \ A = 1, 2, \ldots, n \tag{5}$$

where

$$L = -\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi^A\partial_\nu\Phi^B G_{AB}(\Phi)\sqrt{g} + V(\Phi)\sqrt{g} \tag{6}$$

The Eq. (5) follows the notational conventions of Wald's equation [15]. This Eq. (5) appeared in the original paper of Duan (as Eq. 2) [5].

By substituting (6) into (5), we can obtain the Euler–Lagrange equations of extended HM given by

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\Phi^A\right) + \Gamma^A_{BC}\partial_\mu\Phi^B\partial_\nu\Phi^C g^{\mu\nu} + G^{AB}\frac{\partial V(\Phi)}{\partial\Phi^B} = 0, \tag{7}$$

where Γ^A_{BC} are the Christoffels symbols on manifold N given by

$$\Gamma^A_{BC} = \frac{1}{2}G^{AD}\left[\frac{\partial G_{BD}}{\partial\Phi^C} + \frac{\partial G_{CD}}{\partial\Phi^B} - \frac{\partial G_{BC}}{\partial\Phi^D}\right]. \tag{8}$$

In order to obtain a special type of solution of partial differential Eq. (7), Duan used his published approach [3], which could be especial convenient to be used to study the solution of the partial differential Eq. (7). In brief, in the case of Φ^A (A = 1, 2, ..., n) are functions solely of the argument σ, and σ is a function of x^μ on the manifold M:

$$\Phi^A = \Phi^A(\sigma); \tag{9}$$

$$\sigma = \sigma(x). \tag{10}$$

The Eq. (7) can be written as following,

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\sigma\right)\frac{\partial\Phi^A}{d\sigma} + G^{AB}\frac{\partial V(\Phi)}{\partial\Phi^B}$$
$$+\left(\frac{d^2\Phi^A}{d\sigma^2} + \Gamma^A_{BC}\frac{d\Phi^B}{d\sigma}\frac{d\Phi^C}{d\sigma}\right)g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma = 0. \tag{11}$$

This important Eq. (11) appear in the original paper of Duan (as Eq. 7) [5].

The solution, such as the solution of Sine-Gorden equation, and the travelling wave solution can be found in certain cases of the metrics and potential function in the extended Euler–Lagrange Eq. (11).

If the function $\sigma = \sigma(x)$ satisfies the Laplace–Beltrami equations

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\sigma\right) = 0, \tag{12}$$

the Eq. (11) will take the form

$$\frac{d^2\Phi^A}{d\sigma^2} + \Gamma^A_{BC}\frac{d\Phi^B}{d\sigma}\frac{d\Phi^C}{d\sigma} = -\frac{1}{f}G^{AB}\frac{\partial V(\Phi)}{\partial\Phi^B}, \tag{13}$$

where

$$f = g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \tag{14}$$

is a scalar function on the manifold M.

Because f is not the function of Φ^A, a rescaling of $V(\Phi)$ as $V(\Phi) = fU(\Phi)$ re-expresses the Eq. (13) as

$$\frac{d^2\Phi^A}{d\sigma^2} + \Gamma^A_{BC}\frac{d\Phi^B}{d\sigma}\frac{d\Phi^C}{d\sigma} = -G^{AB}\frac{\partial U(\Phi)}{\partial\Phi^B}. \tag{15}$$

Eq. (15) can be represented as the geodesic equation of a particle on the Riemannian manifold N subjected to an external force of

$$F^A = -G^{AB}\frac{\partial U(\Phi)}{\partial\Phi^B}. \tag{16}$$

This Eq. (15) will become equivalent to the usual geodesic equations on the manifold N if the external force $F^A = 0$ $(A = 1, 2, \ldots, n)$. Eq. (15) is physical dynamical equation.

3. Chaotic solutions of the extended HM equations

If the chaos equation could be derivate from the Eq. (15) under certain cases of metrics and potential function, the chaotical behaviour satisfied the extended HM equations. In order to evidence whether the chaotic behaviours exist in the partial differential Eq. (15), we simplified the case as that the M is the pseudo-Euclidean space-time and N is a 2-dimensional manifold with coordinates $\{\Phi^1, \Phi^2\}$. We suppose that Φ^1 and Φ^2 are functions of an argument $\sigma = \sigma(x) = k_\mu x^\mu$ and further assume that $\Phi^2 = \sigma$, *i.e.*,

$$\Phi^1 = \Phi(\sigma); \ \Phi^2 = \sigma; \ \sigma = k_\mu x^\mu. \tag{17}$$

Under this simplified case, the Eq. (15) can be expressed as

$$\frac{d^2\Phi}{d\sigma^2} + \Gamma^1_{11}\left(\frac{d\Phi}{d\sigma}\right)^2 + 2\Gamma^1_{12}\frac{d\Phi}{d\sigma} + \Gamma^1_{22}$$
$$= -\left(G^{11}\frac{\partial U(\Phi,\sigma)}{\partial\Phi} + G^{12}\frac{\partial U(\Phi,\sigma)}{\partial\sigma}\right); \tag{18}$$

$$\Gamma^2_{11}\left(\frac{d\Phi}{d\sigma}\right)^2 + 2\Gamma^2_{12}\frac{d\Phi}{d\sigma} + \Gamma^2_{22}$$
$$= -\left(G^{21}\frac{\partial U(\Phi,\sigma)}{\partial\Phi} + G^{22}\frac{\partial U(\Phi,\sigma)}{\partial\sigma}\right). \tag{19}$$

Eliminating the term $(\frac{d\Phi}{d\sigma})^2$ from the differential Eqs. (18) and (19), we obtain

$$\frac{d^2\Phi}{d\sigma^2} + 2\left(\Gamma^1_{12} - \frac{\Gamma^1_{11}\Gamma^2_{12}}{\Gamma^2_{11}}\right)\frac{d\Phi}{d\sigma} + \left(\Gamma^1_{22} - \frac{\Gamma^1_{11}\Gamma^2_{22}}{\Gamma^2_{11}}\right)$$
$$+\left[G^{11}\frac{\partial U(\Phi,\sigma)}{\partial\Phi} + G^{12}\frac{\partial U(\Phi,\sigma)}{\partial\sigma}\right.$$
$$\left.-\frac{\Gamma^1_{11}}{\Gamma^2_{11}}\left(G^{21}\frac{\partial U(\Phi,\sigma)}{\partial\Phi} + G^{22}\frac{\partial U(\Phi,\sigma)}{\partial\sigma}\right)\right] = 0 \tag{20}$$

If the chaos equation can be derivate from these geodesical ordinary differential equations (20), the partial differential equation (15) must contain the chaotic behaviours.

By given the metrics on manifold N are diagonal and take the following form:

$$G_{11} = e^{\Phi+\sigma};\ G_{12} = 0;\ G_{21} = 0;\ G_{22} = (k-1)e^{\Phi+\sigma}, \tag{21}$$

where k is a constant.

The Christoffels symbols (8) on the 2-dimensional manifold are calculated as following,

$$\Gamma^1_{11} = \frac{1}{2};\ \Gamma^1_{12} = \frac{1}{2};\ \Gamma^1_{22} = -\frac{1}{2}(k-1);$$
$$\Gamma^2_{11} = -\frac{1}{2(k-1)};\ \Gamma^2_{12} = \frac{1}{2};\ \Gamma^2_{22} = \frac{1}{2}. \tag{22}$$

The detailed procedures of calculation showed in the supplementary information.

By substituting (21) and (22) into (20), the differential Eq. (20) can be written as following,

$$\frac{d^2\Phi}{d\sigma^2} + k\frac{d\Phi}{d\sigma} + \left(e^{-\Phi-\sigma}\right)\left[\frac{\partial U(\Phi,\sigma)}{\partial\Phi} + \frac{\partial U(\Phi,\sigma)}{\partial\sigma}\right] = 0. \tag{23}$$

We found, under following given potential function U(Φ, σ), *i.e.*

$$U(\Phi,\sigma) = e^{\Phi+\sigma}\left[\frac{1}{2}\alpha\Phi^3 - \frac{3}{4}\alpha\Phi^2 - \frac{1}{4}(2\beta-3\alpha)\Phi + \frac{1}{8}(2\beta-3\alpha) - \frac{2b}{\omega^2+4}\cos(\omega\sigma) - \frac{\omega b}{\omega^2+4}\sin(\omega\sigma)\right], \tag{24}$$

Substituting (24) into Eq. (23), we obtain

$$\frac{d^2\Phi}{d\sigma^2} + k\frac{d\Phi}{d\sigma} - \beta\Phi + \alpha\Phi^3 = b\cos(\omega\sigma);\ \sigma = k_\mu x^\mu \tag{25}$$

The detailed procedures of calculation showed in the supplementary information.

Comparing the Eq. (25) with (1), *i.e.*, the equation of anharmonic system in a periodic field, we find that they are of the same form. Since the Eq. (1) possesses chaotic states that are characterized by the existence of a strange attractor in the phase space, this implies that Eq. (23) for $\Phi = \Phi(\sigma)$ should possess the same chaotic profile.

By giving a new potential function $U = U(\Phi,\sigma)$, *i.e.*

$$U(\Phi,\sigma) = e^{\Phi+\sigma}\left\{\frac{2}{5}\alpha\sin(\Phi) - \frac{1}{5}\alpha\cos(\Phi) + \frac{\beta}{25+6\omega^2+\omega^4}\left[\omega\left(\omega^2+3\right)\sin(\omega\sigma)\sin(\Phi) - 4\omega\sin(\omega\sigma)\cos(\Phi) + 2\left(\omega^2+5\right)\cos(\omega\sigma)\sin(\Phi) + \left(\omega^2-5\right)\cos(\omega\sigma)\cos(\Phi)\right]\right\}, \tag{26}$$

in which, α and β are constants, the Eq. (23) become following,

$$\frac{d^2\Phi}{d\sigma^2} + k\frac{d\Phi}{d\sigma} + [\alpha + \beta\cos(\omega\sigma)]\sin(\Phi) = 0;\ \sigma = k_\mu x^\mu \tag{27}$$

The detailed procedures of calculation showed in the supplementary information.

Comparing the Eq. (27) with (2), i.e., the nonlinear equation of a parametrically excited damped pendulum, we find that they are of the same form. Since the Eq. (2) possesses chaotic states that are characterized by the existence of a strange attractor in the phase space, this implies that Eq. (27) for $\Phi(\sigma)$ should possess the same chaotic profile.

4. Quantum mechanism solutions of the extended HM equations

The extended equation of HM contains the chaotic solution, which recently found with connection to quantum physics via entanglement [1,2]. It would be interested to test whether the Schrodinger equation in quantum physics can be derived. As a simple case, we studied the Schrödinger equation for a one-dimensional harmonic oscillator as below,

$$\frac{d^2\Psi}{dx^2} = -\frac{2m}{\hbar^2}\left(E - \frac{1}{2}Kx^2\right)\Psi(x), \tag{28}$$

where $\Psi(x)$ is the wave function, E is the total energy (constant), and K is the force constant (the force on the mass being $F = -Kx$, proportional to the displacement x and directed towards the origin.

By given the metrics on manifold N are diagonal and take the following form:

$$G_{11} = e^{\Phi+\sigma};\ G_{12} = 0;\ G_{21} = 0;\ G_{22} = -e^{\Phi+\sigma}, \tag{29}$$

This Eq. (29) is a special case of Eq. (21), *i.e.* $k = 0$. Thus, the differential Eq. (20) can be written as following,

$$\frac{d^2\Phi}{d\sigma^2} + \left(e^{-\Phi-\sigma}\right)\left[\frac{\partial U(\Phi,\sigma)}{\partial\Phi} + \frac{\partial U(\Phi,\sigma)}{\partial\sigma}\right] = 0. \tag{30}$$

By given the potential function U(Φ, σ) as following, *i.e.*

$$U(\Phi,\sigma) = e^{\Phi+\sigma}\left[-\frac{m}{2\hbar^2}K\Phi\sigma^2 - \frac{m}{4\hbar^2}K\sigma^2 - \frac{m}{2\hbar^2}K\Phi\sigma + \frac{2m}{\hbar^2}(E-K)\Phi - \frac{m}{2\hbar^2}K\sigma + \frac{m}{\hbar^2}\left(\frac{3}{4}K - E\right)\right], \tag{31}$$

where E, K, m, $\hbar$ are the constants,

Substituting (31) into Eq. (30), we obtain

$$\frac{d^2\Phi}{d\sigma^2} = -\frac{2m}{\hbar^2}\left(E - \frac{1}{2}K\sigma^2\right)\Phi(\sigma), \tag{32}$$

The detailed procedures of calculation showed in the supplementary information.

Comparing the Eq. (32) with Eq. (28), we find that they are of the same form when $\Phi(\sigma) = \Psi(x)$ *and* $\sigma = x$. Since the Eq. (28) is one of typical equation of Schrodinger equation in quantum physics, this implies that the Eq. (15) contains the Schrodinger equation in quantum physics.

Since the extended HM (when the potential function $V(\Phi) = 0$ in the Eq. (4)) can be recovered the Einstein's equations in general relativity [14], and the same theory can also recover the Schrodinger equation in quantum physics, we propose here the extended equation of HM as a single and universal theory for our universe.

Author contributions

This project was initiated by GR and YD., GR calculated the solution, and YS validated the solution. GR drafted the initial manuscript, which was revised by YD.

Competing financial interests

The author(s) declare no competing financial interests.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.chaos.2017.07.004.

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