Extending Brauer's Height Zero Conjecture to blocks with nonabelian defect groups

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Abstract

We propose a generalization of Brauer's Height Zero Conjecture that considers positive heights. We give strong evidence supporting one half of the generalization and obtain some partial results regarding the other half.

1 Introduction

One of the main problems in the representation theory of finite groups is Brauer's Height Zero Conjecture, appearing as Problem 23 in [4]. Much work has been done in the last half century regarding this conjecture but it still remains open. It asserts that the irreducible characters in a block B of a finite group G all have height zero if and only if the defect group D is abelian. A common feature of the height zero conjecture with many of the main problems in representation theory (such as the Alperin-McKay Conjecture or Broué's Abelian Defect Group Conjecture, which gives a much stronger description of the blocks with abelian defect group than Brauer's Conjecture) is that it does not say a great deal about characters of positive height or blocks with nonabelian defect group. Our hope is that this paper will help to increase our understanding of blocks with nonabelian defect groups.

Let p be a prime number and B a p-block of a finite group G, with defect group D of order p^d. Recall that the *defect* of an irreducible character χ in B is the integer $d(\chi)$ such that $p^{d(\chi)}\chi(1)_p = |G|_p$ and the *height* of χ is $d - d(\chi)$, a non-negative integer. Write $\mathrm{Irr}(B)$ for the set of irreducible characters in B, and $\mathrm{Irr}_a(B)$ for the set of those with defect a. Note that a p-group has only one p-block, so the height of an irreducible character is

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in this case simply the exponent of p in its degree. Brauer's Height Zero Conjecture, therefore, predicts that $\mathrm{Irr}(B) = \mathrm{Irr}_d(B)$ if and only if D is abelian. Hence we expect a block to possess irreducible characters of positive height if and only if a defect group does. In this article, we compare the smallest non-zero height $\mathrm{mh}(B)$ in $\mathrm{Irr}(B)$ with the corresponding number $\mathrm{mh}(D)$ in $\mathrm{Irr}(D)$. If P is a p-group, write $\mathrm{m}(P)$ for the minimal degree of a non-linear character of P, so $\mathrm{m}(P) = p^{\mathrm{mh}(P)}$. We set $\mathrm{mh}(B) = \infty$ (and similarly $\mathrm{mh}(P) = \mathrm{m}(P) = \infty$) if there is no irreducible character of positive height, and use the convention that $n < \infty$ for all $n \in \mathbb{Z}$.

For the sake of discussion, we state the following generalization of Brauer's Height Zero Conjecture. (As discussed later, conjecturing inequality in one direction is uncontroversial whilst we regard inequality in the other as a question.)

Conjecture A. *Let B be a p-block of a finite group G with defect group D. Then* $\mathrm{mh}(B) = \mathrm{mh}(D)$*.*

Our first main result shows that one inequality holds for p-solvable groups.

Theorem B. *Let B be a block of a p-solvable group with defect group D. Then* $\mathrm{mh}(D) \leq \mathrm{mh}(B)$*.*

The investigations in this paper were originally motivated by Conjecture 1.1 of [6]. This conjecture asserts that for a block B of any finite group G with defect group D of order p^d, and for each (non-negative) integer n, we should have

$$\sum_{0 \leq h \leq n} k_{d-h}(B) \leq \sum_{0 \leq h \leq n} p^{2h} k_{d-h}(D),$$

where $k_a(B) = |\,\mathrm{Irr}_a(B)|$. By [18], if G is p-solvable, $O_{p'}(G) = 1$ and $P = N_G(P)$ for $P \in \mathrm{Syl}_p(G)$, then $k_d(B_0(G)) = k_d(P)$, where $|P| = p^d$ and $B_0(G)$ is the principal block of G. Hence this particular case of Theorem B is a, perhaps, surprising consequence of Conjecture 1.1 of [6].

Another outstanding conjecture regarding defects of irreducible characters in blocks is Dade's Conjecture, which we state later (in the form given by Robinson). We prove the following.

Theorem C. *Let B be a p-block of a finite group G with defect group D. If Dade's Projective Conjecture holds for every group involved in G, then* $\mathrm{mh}(D) \leq \mathrm{mh}(B)$ *.*

This provides a second proof that $\mathrm{mh}(D) \leq \mathrm{mh}(B)$ when B is a block of a p-solvable group, since Dade's Conjecture holds for such blocks by [23].

In view of this result, the inequality $\mathrm{mh}(D) \leq \mathrm{mh}(B)$ should be expected to be true for any block of any finite group. We prove several results regarding the converse inequality but, unfortunately, we have far less evidence to believe that this inequality also holds. The reason for this is that we have been unable to prove it for p-solvable groups which, if true, might be very difficult (it is worth remarking that the proof by D. Gluck and T. Wolf [9] of one half of Brauer's Conjecture for p-solvable groups was already exceptionally hard.) However, it is not difficult to deduce from known results that Conjecture A holds for the most representative nonsolvable groups.

Theorem D. *Let B be a p-block of a general linear group, a symmetric group (for some $p \geq 5$) or a sporadic group. Then Conjecture A holds for G.*

In order to prove some of the partial results that we obtain for p-solvable groups we need the following theorem of independent interest.

Theorem E. *Let G be a finite solvable group, and $P \in \mathrm{Syl}_p(G)$, where p is odd. If the set of degrees of the irreducible characters of P is a subset of $\{1, p^a, \ldots, p^{2a}\}$ containing p^a for some $a \geq 2$, then the p-length of G is 1.*

All blocks are with respect to a complete local discrete valuation ring $\mathcal{O}$ with residue field $\mathcal{O}/J(\mathcal{O})$ of characteristic p and field of fractions K of characteristic zero. We assume that K is 'large enough,' which here means that it contains a primitive $|G|^3$th root of unity. Write $\mathrm{cd}(B)$ for the set of degrees of the irreducible characters in B.

The layout of the paper is as follows. In Section 2 we prove Theorem B. We give the conjectures of Dade and Robinson in Section 3, and show Theorem C. In Section 4 we prove Theorem D and in Section 5 we prove Theorem E. Finally, in Section 6 we obtain some partial results for p-solvable groups examining when the equality $\mathrm{mh}(B) = \mathrm{mh}(D)$ occurs.

2 p-solvable groups

We first collect together some useful results.

Proposition 2.1 ([17]). *Suppose that a p-group P acts on a p-solvable group M and that Q is a P-invariant Sylow p-subgroup of M. Write $|Q| = p^a$. If P fixes every element of $\mathrm{Irr}_a(N_M(Q))$, then P fixes every element of $\mathrm{Irr}_a(M)$.*

Proof. This is [17, Lemma 2.1]. □

The next lemma is immediate.

Lemma 2.2. *Let Q be a maximal subgroup of a p-group P. If Q is not abelian, then*

$$\mathrm{mh}(P) \leq \mathrm{mh}(Q) + 1.$$

We also need the following deep result.

Lemma 2.3 ([17])**.** *Let G be a p-solvable group and $\chi \in \mathrm{Irr}(G)$. Write $\overline{G} = G/O_{p'}(G)$. Assume that U is a subgroup of G with a primitive character $\gamma \in \mathrm{Irr}(U)$ such that $\gamma^G = \chi$. Let $Q \in \mathrm{Syl}_p(U)$. If $\chi_{O_{p'}(G)}$ is homogeneous, then*

$$C_{\overline{G}}(\overline{Q}) = \overline{Z(Q)}.$$

Proof. This follows from Corollary B of [17]. □

Lemma 2.4. *Let $N \lhd G$ and let B be a p-block of G covering a block b of N, where B has defect group D such that $G = ND$. Then B is the unique block of G covering b and b is the unique block of N covered by B. Further, $D \cap N$ is a defect group for b.*

Proof. That B is the unique block of G covering b is [8, V.3.5]. By [1, 15.1], a defect group for B stabilizes b, hence b is G-stable and b is the unique block covered by B. That $D \cap N$ is a defect group for b then also follows from [1, 15.1]. □

Now, we prove Theorem B, which we restate.

Theorem 2.5. *Let G be a p-solvable group and B a p-block of G with defect group D. If $\chi \in \mathrm{Irr}(B)$ has positive height h, then*

$$h \geq \mathrm{mh}(D).$$

Remark. We note that the proof of this result uses [17, Lemma 2.1], which relies on the fact that the McKay Conjecture holds for p-solvable groups (for which see [19]). We will see later that our result is also a consequence of the fact that Dade's Projective Conjecture holds for p-solvable groups. However, given the relative simplicity of the proof of the McKay Conjecture for p-solvable groups, compared to the (very difficult) proof of Dade's conjecture in this case, we regard the proof below to be a great simplification.

Proof. Let B be a block of G which is a counterexample with $[G : Z(G)]$ minimized, and choose $\chi \in \mathrm{Irr}(B)$ of minimal positive height h in B. In particular, $h < \mathrm{mh}(D)$.

Let b be a block of $O_{p'}(G)$ covered by B, and write T for the stabilizer of b under the action of G. By [8, X.1.2] there is a block $\tilde{B}$ of a group $\tilde{G}$

with central subgroup $\tilde{Z}$ of order prime to p such that $\tilde{G}/\tilde{Z} \cong T/N$, and $\tilde{B}$ has defect group $\tilde{D}$ isomorphic to D. Further $\tilde{B}$ possesses an irreducible character of height h. Note that $\tilde{G}$ is p-solvable. Note also that $Z(G) \leq T$.

If $O_{p'}(G) \not\leq Z(G)$, then $[\tilde{G} : Z(\tilde{G})] < [G : Z(G)]$, in which case by minimality $\mathrm{mh}(D) = \mathrm{mh}(\tilde{D}) \leq h$, a contradiction. Hence $O_{p'}(G) \leq Z(G)$. Consequently, since G is p-solvable, $C_G(O_p(G)) = Z(Q)O_{p'}(G)$, so the p-blocks of G all have maximal defect and each is the unique block covering a given block of $O_{p'}(G)$. In particular $D \in \mathrm{Syl}_p(G)$. Note that $\chi(1)_p = p^h$.

Now every block of $O^{p'}(G)$ covered by B has defect group D, and so χ covers an irreducible character of height h in a block of $O^{p'}(G)$ with defect group D. Hence by minimality $O^{p'}(G) = G$.

The result is trivial if $p \in \mathrm{cd}(D)$, so D does not have any irreducible character of degree p. In particular, D does not have any abelian subgroup of index p.

Since $O^{p'}(G) = G$ and G is p-solvable, each maximal normal subgroup has index p, and so each irreducible character of a maximal normal subgroup either extends to G or induces to an irreducible character.

We claim that χ restricts irreducibly to every maximal normal subgroup of G. Suppose not. Then there exists $M \trianglelefteq G$ of index p and $\psi \in \mathrm{Irr}(M)$ such that $\psi^G = \chi$. Notice that $G = MD$. Now by Lemma 2.4 ψ lies in the unique block of M covered by B, and this has defect group $Q = D \cap M \in \mathrm{Syl}_p(M)$. Since $[D : Q] = p$, by the previous paragraph Q is not abelian. If $p \mid \psi(1)$, then by minimality and Lemma 2.2, we have

$$\chi(1)_p = p\psi(1)_p \geq p^{\mathrm{mh}(Q)+1} \geq p^{\mathrm{mh}(D)},$$

a contradiction, so we may assume that ψ has p'-degree.

Suppose that $N_M(Q)$ does not possess an irreducible character of p'-degree which induces irreducibly to $N_G(Q)$. Then, since $[N_G(Q) : N_M(Q)] = p$ and $N_G(Q) = DN_M(Q)$, every irreducible character of $N_M(Q)$ of p'-degree is fixed by D. But then by Proposition 2.1 every irreducible character of M of degree prime to p is fixed by D, contradicting the existence of ψ. We conclude that there exists $\beta \in \mathrm{Irr}(N_M(Q))$ of p'-degree which induces irreducibly to $N_G(Q)$. Notice that D is also a Sylow p-subgroup of $N_G(Q)$ and that $\beta^{N_G(Q)}(1)_p = p = \chi(1)_p$. We want to show that $p \in \mathrm{cd}(D)$. Since ψ has p'-degree, the character χ is induced from some primitive character of some subgroup U such that some conjugate of Q is a Sylow p-subgroup of U. Write $Z = O_{p'}(G)$. By Lemma 2.3 $C_{G/Z}(QZ/Z) = Z(Q)Z/Z$. Since $QZ/Z \leq O_p(N_G(Q)/Z)$, this implies that $O_{p'}(N_G(Q)/Z) = 1$, i.e., that $O_{p'}(N_G(Q)) = Z$. Hence every p-block of $N_G(Q)$ has defect group D, and $\beta^{N_G(Q)}$ has positive height. By minimality, we have $G = N_G(Q)$. Also, since

G is a counterexample, D is not normal in G. This implies that $Q = O_p(G)$ and $F(G) = Z(G)O_p(G)$. Hence G/Q acts faithfully on $Q/\Phi(Q)$. Since $|G/Q|_p = p$, we deduce that $p \in \mathrm{cd}(D/\Phi(Q)) \subseteq \mathrm{cd}(D)$. This contradiction implies our claim.

Now let $K = O^p(G)$. Since G/K is a p-group, the previous claim implies that χ is not induced from any proper subgroup of G that contains K. Now we want to prove that $\chi_K \in \mathrm{Irr}(K)$. In order to see this, let $\alpha \in \mathrm{Irr}(K)$ lying under χ and let $A_1/K = Z(G/K)$. If α does not extend to A_1, we choose a subgroup $K < B_1 < A_1$ normal in G and maximal such that α extends to $\alpha_1 \in \mathrm{Irr}(B_1)$. We may take an extension α_1 lying under χ (because B_1/K is central in G/K). We have that α_1 is not G-invariant (otherwise α would extend to A_1). Now Clifford theory implies that χ is induced from some proper subgroup of G that contains K. This is a contradiction. Hence, we may assume that α extends to A_1. Furthermore, we can take an extension of α to A_1 lying under χ. Now, let A_2/A_1 be a maximal normal abelian subgroup of G/A_1. Arguing in the same way, we can find an extension of α to A_2 lying under χ. Repeating this procedure yields that $\chi_K = \alpha$, as desired.

Now put $N = O^{p'}(K)$. Since G is a counterexample, it cannot be a p-group, so $N < K$. Let R/N be a Sylow p-subgroup of G/N. Observe that $G = KR$ and $K \cap R = N$. Let $\theta \in \mathrm{Irr}(N)$ lying under χ. Since K/N is a p'-group,

$$\chi(1)_p = \chi_N(1)_p = \theta(1)_p,$$

Assume that θ does not extend to R^g/N for any $g \in G$. Then for any $g \in G$ the p-part of the degree of any irreducible character of R lying over θ^g is at least $p\theta(1)_p$. However, χ_R is a sum of irreducible characters that lie over some conjugate of θ, i.e., it is a sum of irreducible characters whose degree has p-part at least $p\theta(1)_p = p\chi(1)_p$. We conclude that $\chi(1)_p \geq p\chi(1)_p$. This contradiction implies that θ extends to some conjugate of R. Without loss of generality, we may assume that θ extends to an irreducible character τ of R. We have $O_{p'}(R) \leq O_{p'}(N) \leq Z \leq Z(G)$. Hence every block of R has maximal defect, in particular τ lies in a block with defect group D. By the minimality of G, $\chi(1)_p = \theta(1)_p \geq p^{\mathrm{mh}(D)}$. This is the final contradiction. $\square$

3 Consequences of conjectures of Dade and Robinson

In this section we show that the conclusion of Theorem 2.5 (for arbitrary groups) is a consequence of Dade's Projective Conjecture, under the assumption that the conjecture holds in all sections of G.

Dade's Projective Conjecture is equivalent to the following conjecture of Robinson (see [22] and [7]), which, for ease of notation, we state in a slightly weakened form. We first give the relevant notation.

A p-subgroup Q of G is *radical* if $Q = O_p(N_G(Q))$. A chain

$$Q_0 < \cdots < Q_n$$

of p-subgroups of G is a *radical p-chain* if for each i we have

$$Q_i = O_p(N_G(Q_0) \cap \cdots \cap N_G(Q_i)).$$

Write $|\sigma| = n$, and $G_\sigma = N_G(Q_0) \cap \cdots \cap N_G(Q_n)$. Also write $V_\sigma = Q_0$ and $V^\sigma = Q_n$. Let $\mathcal{R}(G)$ be the set of radical p-chains of G, and let $\mathcal{R}(G)/G$ be a set of representatives of the G-conjugacy classes of elements of $\mathcal{R}(G)$. Notice that $V^\sigma \leq N_G(\sigma)$ whenever $\sigma \in \mathcal{R}(G)$.

If Q is a normal p-subgroup of a subgroup H of G, B is a block of G and a is an integer, write $w_a(H, B, Q)$ for the number of irreducible characters χ in blocks of H with Brauer correspondent B such that $d(\chi) = a$ and χ is *Q-projective*, i.e., $\chi(1)_p = [H : Q]_p \mu(1)$ whenever $\mu \in \mathrm{Irr}(Q)$ is covered by χ. We also write $k_a(G, B)$ for the number of irreducible characters in B of defect a.

Conjecture 3.1 (Robinson)**.** *Let B be a p-block of G and a an integer. Then*

$$k_a(G, B) = \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} w_a(G_\sigma, B, V_\sigma).$$

It is important to consider this conjecture rather than Dade's Projective Conjecture, since it allows the possibility that $O_p(G) \not\leq Z(G)$.

Note that if $\sigma \in \mathcal{R}(G)$ and $w_a(G_\sigma, B, V_\sigma) \neq 0$, then we have $|V_\sigma| \geq p^a$ and we may take $V_\sigma \leq D$, where D is a fixed defect group for B.

Lemma 3.2. *Let B be a p-block of a finite group G with defect group D such that $D/O_p(G)$ is cyclic. Then Conjecture 3.1 holds for B.*

Proof. Note that the proof does not rely on the classification of finite simple groups, instead following from the theory of blocks with cyclic defect groups. By the proof of [7, 1.8] it suffices to prove the conjecture in the case $O_p(G) \leq Z(G)$. In this case Conjecture 3.1 is implied by Conjecture 5.1 of [2]. Conjecture 5.1 of [2] holds in our case by [2, 5.2].

□

The following is an easy application of Frobenius reciprocity:

Lemma 3.3. *Let P be a finite p-group and $Q \leq P$. Let $\mu \in \mathrm{Irr}(Q)$. Then there is $\theta \in \mathrm{Irr}(P)$ such that $\mu(1)|\theta(1)$ and $\theta(1)|[P : Q]\mu(1)$.*

The following includes Theorem C.

Theorem 3.4. *Let B be a block of a finite group G with defect group D, and suppose that Conjecture 3.1 holds for every factor group of every subgroup of G. Then* $\mathrm{mh}(B) \geq \mathrm{mh}(D)$.

Proof. If $\mathrm{mh}(B) = \infty$ (i.e., B has no irreducible character of positive height), then we are done.

Let a be the smallest positive integer such that there is a block B satisfying the hypotheses regarding Conjecture 3.1 with $\mathrm{mh}(B) = a$ but $\mathrm{mh}(B) < \mathrm{mh}(D)$ for D a defect group of B. Write $|D| = p^d$.

We first claim that there exists a radical p-subgroup Q of order p^{d-a} contained in D such that $w_{d-a}(N_G(Q), B, Q) \neq 0$.

Since Conjecture 3.1 holds for B, we have

$$0 \neq k_{d-a}(G, B) = \sum_{\sigma \in \mathcal{R}(G)/G, |V_\sigma| \geq p^{d-a}, V^\sigma \leq D} (-1)^{|\sigma|} w_{d-a}(N_G(\sigma), B, V_\sigma)$$

Suppose there is $\sigma \in \mathcal{R}(G)/G$ with $p^{d-a} < |V^\sigma|$ and $V^\sigma \leq D$, and that $w_{d-a}(N_G(\sigma), B, V_\sigma) \neq 0$. So there is $\chi \in \mathrm{Irr}_{d-a}(N_G(\sigma), B)$ which is V_σ-projective, and so V^σ-projective (noting that since σ is radical, we have $V^\sigma \lhd N_G(\sigma)$). Write $|V^\sigma| = p^b$. Let $\mu \in \mathrm{Irr}(V^\sigma)$ be covered by χ. Then $\chi(1)_p = [N_G(\sigma) : V^\sigma]_p \mu(1)$. But $|N_G(\sigma)|_p = p^{d-a}\chi(1)_p$, so $|V^\sigma| = p^{d-a}\mu(1)$. Hence $\mu(1) = p^{b-(d-a)} \neq 1$, and by Lemma 3.3 there is a non-linear $\theta \in \mathrm{Irr}(D)$ such that $\theta(1)|p^a$, a contradiction. Hence the only chains contributing are those with $|V_\sigma| = |V^\sigma| = p^{d-a}$, and the claim follows.

Choose Q a radical p-subgroup of order p^{d-a} contained in D with $w_{d-a}(N_G(Q), B, Q) \neq 0$, and write $H = N_G(Q)$. Then there is $\chi \in \mathrm{Irr}_{d-a}(H, B)$ which is Q-projective. Let b be the block of H containing χ, so that $b^G = B$. We claim next that b has a defect group G-conjugate to D.

If b has defect group Q, then $b^G = B$ also has defect group Q by Brauer's first main theorem. So $Q = D$, a contradiction. Without loss of generality, we may choose D and a defect group P of b such that $P \leq D$ (this is a property of the Brauer correspondence). Suppose that $P \neq D$, and write $|P| = p^e$. Then χ has height $e - (d - a) = a - (d - e) < a$ in b. Note that $e > d - a$ since Q is not a defect group for b, i.e., χ has positive height. Since Conjecture 3.1 holds for all blocks of factor groups of subgroups of H, by minimality $\mathrm{mh}(P) \leq \mathrm{mh}(b)$, i.e., there is $\tau \in \mathrm{Irr}(P)$ with $1 < \tau(1) \leq$

$p^{a-(d-e)}$ (note for reference later that we only mention Conjecture 3.1 at this point since it is necessary for the inductive hypotheses, and that we omit explicit mention of this in later applications of minimality). Hence by Lemma 3.3 there is non-linear $\theta \in \mathrm{Irr}(D)$ such that $\theta(1)|p^{a-(d-e)}p^{d-e} = p^a$, a contradiction. Hence b may be taken to have defect group D as claimed.

Now note that since χ is Q-projective and $p^{d-a}\chi(1)_p = |H|_p$, we have that χ covers a linear character of Q, and so $Q' \leq \ker(\chi)$. Let $\bar{b}$ be the block of $\overline{H} = H/Q'$ containing χ regarded as a character of $\overline{H}$.

By [8, V.4] each block of $\overline{H}$ is contained in a unique block of H in the sense that the inflations of its irreducible characters all lie in the same block, and a defect group of any block of $\overline{H}$ is contained in $\overline{P}$ for P a defect group of the block of H containing it.

We claim that $\bar{b}$ has defect group D/Q'. Suppose first that $\bar{b}$ has defect group Q/Q'. Since blocks of H with defect group Q must possess an irreducible character of height zero, they must possess an irreducible character with Q' in the kernel. Hence each such block contains a block of H/Q' with defect group Q/Q' in the above sense. By [21, Corollary 7] the number of blocks of H with defect group Q is equal to the number of blocks of H/Q' with defect group Q/Q', and so the inflation of every irreducible character in a block of H/Q' with defect group Q/Q' lies in a block with defect group Q. But then χ lies in a block of H with defect group Q, a contradiction. Hence there is $P \leq D$ with $Q < P$ such that $\bar{b}$ has defect group P/Q'.

Suppose that $P \neq D$. Write $|P| = p^c$. Then χ has height $a-(d-c) > 0$ in $\bar{b}$, so by minimality, there is a non-linear $\mu \in \mathrm{Irr}(P)$ with $\mu(1)|p^{a-(d-c)}$. Hence by Lemma 3.3 there is non-linear $\theta \in \mathrm{Irr}(P)$ with $\theta(1)|p^{a-(d-c)}p^{d-c} = p^a$, a contradiction. Hence $\bar{b}$ may be taken to have defect group D/Q' as claimed.

Note that $\chi \in \mathrm{Irr}_{d'-a}(H/Q', \bar{b})$, where $p^{d'} = [D : Q']$. Since $[D : Q] = p^a$, it follows that irreducible characters of D/Q' have degree dividing p^a. But there is no non-linear irreducible character of D with degree p^a or less, so every irreducible character of D/Q' is linear, i.e., D/Q' is abelian.

Since Conjecture 3.1 holds for $\bar{b}$, we have

$$k(H/Q', \bar{b}) = k(N_H(D)/Q', \bar{b}) = k_{d'}(N_H(D)/Q', \bar{b}) = k_{d'}(H/Q', \bar{b})$$

by [22, 5.1]. We conclude that $k_{d'-a}(H/Q', \bar{b}) = 0$, a contradiction since $\chi \in \mathrm{Irr}_{d'-a}(H/Q', \bar{b})$. □

A careful analysis of the above proof gives the following strengthening of Theorem 3.4 when considering characters of height one:

Theorem 3.5. *Let B be a block of a finite group G with defect group D of order p^d, and suppose that Conjecture 3.1 holds for B. If $k_{d-1}(G,B) \neq 0$, then $k_{d-1}(D) \neq 0$.*

Proof. In the proof of Theorem 3.4, we see that Conjecture 3.1 is applied directly only to B and to the block $\overline{b}$ of $N_G(Q)/Q'$. The result then follows by the proof of Theorem 3.4 and the observation that Lemma 3.2 applies to $\overline{b}$. $\square$

The following is another consequence of Conjecture 3.1 (see [22]):

Conjecture 3.6 (Robinson)**.** *Let B be a p-block of G with defect group D and let $\chi \in \mathrm{Irr}(B)$. Then there is a radical p-subgroup Q with $C_D(Q) \leq Q \leq D$ and $\theta \in \mathrm{Irr}(Q)$ such that $|Q|/\theta(1) = p^{d(\chi)}$.*

If Conjecture 3.6 holds for B and $p^{\mathrm{mh}(B)} < \min\{[D:Q] : Q \leq D, Q \in \mathcal{R}_0(G)\}$, then by Lemma 3.3 $\mathrm{mh}(B) \geq \mathrm{mh}(D)$. Further, if Conjecture 3.1 holds for B and $p^{\mathrm{mh}(B)} < \min\{[D:Q] : Q \leq D, Q \in \mathcal{R}_0(G)\}$, then $\mathrm{mh}(B) = \mathrm{mh}(D)$. We remind the reader that every radical p-subgroup is an intersection of (possibly more than two) distinct defect groups.

Hence, by [2], it follows that if a block B has trivial intersection defect group D, then $\mathrm{mh}(B) = \mathrm{mh}(D)$.

4 Nonsolvable groups

In this section we prove Theorem D. We start with the general linear group.

Theorem 4.1. *Let $G = \mathrm{GL}(n,q)$ for some n and some prime power $q = p^e$. Let B be a p-block of G with defect group D. Then $\mathrm{mh}(B) = \mathrm{mh}(D)$.*

Proof. By [5], B is either a block of full defect or a block of defect zero. In the latter case, the result is obvious, so we consider the former case. We may assume that $n > 2$ (if $n = 2$, then $\mathrm{mh}(B) = \mathrm{mh}(D) = \infty$). By [11] and [10], $\mathrm{mh}(D) = e$. By [16] and [20], we also have that $\mathrm{mh}(B) = e$. The result follows. $\square$

Theorem 4.2. *Let $G = S_n$ for some n and let B be a p-block of G with defect group D, where $p \geq 5$. Then $\mathrm{mh}(B) = \mathrm{mh}(D) = 1$ or $\mathrm{mh}(B) = \mathrm{mh}(D) = \infty$. If $p < 5$, then $\mathrm{mh}(D) = 1$ or ∞.*

Proof. Let w be the weight of B, and let $w = a_0 + a_1 p + \cdots + a_r p^r$ be the p-adic decomposition of w. Then $D \cong (D_1)^{a_1} \times \cdots \times (D_t)^{a_t}$, where D_i is isomorphic to a Sylow p-subgroup of S_{p^i}, so D_i is of the form $((\cdots((C_p \wr C_p) \wr C_p) \wr \ldots \wr C_p$. If $r = 0$, then D is abelian. If $r \geq 1$, then by considering the appropriate inflations, $\mathrm{cd}(C_p \wr C_p) \subseteq \mathrm{cd}(D)$. But $p \in \mathrm{cd}(C_p \wr C_p)$, so $\mathrm{mh}(D) = 1$.

By [16, 4.1] the set of heights of irreducible characters in B contains the set $\{0, 1, 2, \ldots, (w - a_0 - a_1 - \cdots - a_r)/(p-1)\}$ whenever $p \geq 5$. If $r = 0$, then all irreducible characters have height zero. If $r \geq 1$, then $w - a_0 - a_1 - \cdots - a_r \geq p - 1$, so $\mathrm{mh}(B) = 1$.

□

As in [16], we expect this to hold for $p < 5$ also, but the calculations are too lengthy to include here.

We conclude with the sporadic groups

Theorem 4.3. *Let B be a block of a sporadic simple group. with defect group D. Then $\mathrm{mh}(B) = \mathrm{mh}(D)$. Further, if D is nonabelian then $\mathrm{mh}(B) = 1$ unless B is the principal 3-block of Co_3, in which case $\mathrm{mh}(B) = 2$.*

Proof. Using the GAP computer algebra package. □

We think that the case of the principal 3-block of the third Conway group is particularly meaningful. We expect that it should be possible to prove Conjecture A for a much wider class of simple, almost simple, or quasisimple groups. It would be very interesting to settle the case of p-solvable groups (one way or the other), but as yet we have been unable to do this. We present the results that we have obtained in this direction in the remainder of this paper.

5 Character degrees of Sylow p-subgroups and p-length

We digress from the main topic in this paper to prove Theorem E. We need the following lemma.

Lemma 5.1 (Isaacs [12])**.** *Let G be a finite solvable group and $O_p(G) = 1$, $P \in \mathrm{Syl}_p(G)$. Let V be a faithful completely irreducible G-module. If $P > 1$ is abelian, then there is a P-orbit on V of length p.*

Proof. Suppose that no orbit of size p exists and that $|G||V|$ is minimal with this property. By Hall-Higman, G has p-length one so we may assume that $G = PK$, where $K = O_{p'}(G)$. For the reader's convenience, we split the proof in a series of steps.

Step 1. If $M < K$ admits P, then P centralizes M.

Write $U = O_p(MP)$. It suffices to show that $U = P$. Let $V_0 = C_V(U)$, so that MP acts on V_0 and U is in the kernel of this action. We claim that U is the full kernel. Otherwise, since the kernel is normal in MP, it is not a p-group and $M_0 = C_M(V_0) > 1$. Note that $M_0 \lhd MP$ since V_0 is MP-invariant. Now let $V_1 = C_V(M_0) \supseteq V_0$.

By Fitting's lemma, $V = V_1 \times V_2$, where $V_2 = [V, M_0]$, and we have $V_2 > 1$ since $M_0 > 1$, and so M_0 acts nontrivially on V. Also, V_2 is MP-invariant, and thus U has nontrivial fixed points in V_2 and $V_2 \cap V_0 > 1$. This is a contradiction, however, since $V_2 \cap V_1 = 1$ and $V_1 \supseteq V_0$. This contradiction shows that U is the full kernel of the action of MP on V_0. Thus MP/U acts faithfully on V_0 and we have $O_p(MP/U) = 1$. But $|MP/U| < |G|$ and P/U does not have an orbit of size p on V_0. Thus P/U is trivial and $P = U$, as wanted.

Step 2. K is a q-group for some prime q, K/K' is elementary and P acts irreducibly on this group.

This argument is fairly standard. Since P does not centralize K, we can choose a prime q dividing $[K : C_K(P)]$. and a P invariant Sylow q-subgroup Q of K. Since P acts nontrivially on Q, we see from Step 1 that $Q = K$, and thus K is a q-group, as wanted.

If $[K, P] < K$, then by Step 1, we have $[K, P, P] = 1$ and thus $[K, P] = 1$, which is not the case since P acts nontrivially on K. Thus $[K, P] = K$ and it follows by Fitting's lemma applied to the action of P on K/K' that $C_{K/K'}(P) = 1$. But then by Step 1 again, no proper subgroup of the nontrivial abelian q-group K/K' admits P and it follows that K/K' is elementary and that the action of P on this group is irreducible.

Step 3. Final contradiction.

Since P acts faithfully and irreducibly on K/K' and P is abelian, we see that P is cyclic and we let a be a generator. Since P acts nontrivially on V, it follows by the Jordan form that we can find linearly independent vectors $v_0, v_1, \ldots, v_r$ in V , with $r > 1$ and such that $(v_i)^a = v_i v_{i+1}$ for $1 \leq r \leq r-1$ and $(v_r)^a = v_r$. It is easy now to see that v_{r-1} lies in an orbit of size p under $P = \langle a \rangle$. This is the desired contradiction. $\square$

We remark that Isaacs actually proves a stronger result in [12], but we just need this case. If G is a p-solvable group, then write $l_p(G)$ for the p-length of G. Now, we are ready to prove Theorem E, which we restate.

Theorem 5.2. *Let G be a finite solvable group, and $P \in \mathrm{Syl}_p(G)$, where p is odd. If $\mathrm{cd}(P)$ is a subset of $\{1, p^a, \ldots, p^{2a}\}$ containing p^a for some $a \geq 2$, then $l_p(G) = 1$.*

Proof. Let G be a counterexample with $|G|$ minimized. Then $G = O^{p'}(G)$. There is $M \lhd G$, where $l_p(G/M) = 2$ and $O_{p'}(G/M) = 1$. Since solvable groups with abelian Sylow p-subgroups have p-length one, $P/(P \cap M)$ is nonabelian. Since $\{1\} \neq \mathrm{cd}(P/(P \cap M)) \subseteq \mathrm{cd}(P)$, it follows by minimality that $M = 1$. Hence $O_{p'}(G) = 1$ and $l_p(G) = 2$.

Write $N = O_p(G)$ and $K = O_{p,p'}(G)$, and note that $O_{p,p',p}(G) = G$.

Now $\Phi(G) \leq F(G) = N$. Since $F(G)/\Phi(G) = F(G/\Phi(G))$, it follows that $\Phi(G) \neq F(G)$. Hence $l_p(G/\Phi(G)) = 2$, since otherwise $O_p(G/\Phi(G))$ would be a Sylow p-subgroup of $G/\Phi(G)$, and $O_p(G) \in \mathrm{Syl}_p(G)$. Hence $P/\Phi(G)$ is nonabelian, and so by minimality we have $\Phi(G) = 1$, and N is elementary abelian.

A theorem of Gaschütz (see [15, 1.12]) says that $F(G)/\Phi(G) = N$ is a completely reducible and faithful $G/F(G) = G/N$-module, and that $G/\Phi(G) = G$ splits over $F(G)/\Phi(G) = N$. So $G = N \rtimes H$ for some H. Let $Q \in \mathrm{Syl}_p(H)$ such that $P = N \rtimes Q$. Note that $\mathrm{Irr}(N)$ is a completely reducible and faithful H-module. If Q is abelian, then by Lemma 5.1 there is some $\theta \in \mathrm{Irr}(N)$ such that $[Q : C_Q(\theta)] = p$. Since G splits over N and θ is linear, θ extends to a linear character of $I_P(\theta)$, and so P possesses an irreducible character of degree p, a contraction. So Q is nonabelian. Since $\mathrm{cd}(Q) \subseteq \mathrm{cd}(P)$, it follows that Q possesses an irreducible character of degree at least p^a, and so $|Q| \geq p^{2a+1}$.

A theorem of Espuelas (see [15, 7.3]) says that, since $O_p(H) = 1$ and $\mathrm{Irr}(N)$ is a finite and faithful H-module, Q has a regular orbit on $\mathrm{Irr}(N)$. Hence P has an irreducible character of degree at least $|Q| \geq p^{2a+1}$, a contradiction, and we are done.

□

Note that that have only used the oddness hypothesis on p to apply Espuelas' theorem. We conjecture that the oddness hypothesis can be removed.

This result is reminiscent of the Hall-Higman theorem. The celebrated Hall-Higman theorem gives information on the p-length of a finite p-solvable group in terms of certain invariants (the derived length, the nilpotence class, etc.) of the Sylow p-subgroup. We think that it would be interesting to study how the set of character degrees of a Sylow p-subgroup affects the p-length of a p-solvable group.

For the purpose of discussion, we say that a set $\mathcal{S}$ of powers of p containing 1 *bounds the p-length* if whenever $\mathcal{S}$ is the set of character degrees of a Sylow p-subgroup of a p-solvable group G, the p-length of G is ≤ 1. In view of the formal similarity between Theorem A in [13] and Theorem

E it seems natural to ask whether, as was done in [13], a set $\mathcal{S}$ bounds the p-length if and only if $p \notin \mathcal{S}$. But we would not be surprised if one could use the p-groups in the proof of Theorem 8.4 of [14] to build a counterexample.

Question 5.3. *Which are the sets of powers of p that bound the p-length?*

6 The converse inequality for p-solvable groups

We return to our main problem. We have already seen at the end of Section 3 that if a block B has trivial intersection defect group D, then $\mathrm{mh}(B) = \mathrm{mh}(D)$ and that the same result holds for certain nonsolvable groups in Section 4 . In this section, we present several other particular cases where equality holds.

If $Q \lhd H$ and $\mu \in \mathrm{Irr}(Q)$, then write $\mathrm{mh}(H|\mu)$ for the minimal non-zero height amongst irreducible characters lying over μ. Define $\mathrm{m}(P|\mu)$ similarly for $Q \lhd P$ a p-group.

Theorem 6.1. *Let $G = E \ltimes X$, where $E \in \mathrm{Syl}_p(G)$ acts on an elementary abelian p-group V. Let $P = EV$. If for each $Q \in \mathrm{Syl}_p(GV) - \{P\}$ we have $[P : P \cap Q] > \mathrm{m}(E)$, then $\mathrm{mh}(B) = \mathrm{mh}(P)$ whenever B is a block of GV with defect group P.*

Proof. Write $m(E) = p^e$. By Theorem 2.5 $\mathrm{m}(E) \geq p^{\mathrm{mh}(B)} \geq \mathrm{m}(P)$. Let $\mu \in \mathrm{Irr}(V)$, and choose $\chi \in \mathrm{Irr}(B, \mu)$ with $\chi(1)_p$ minimised.

Suppose first that $[G : I_G(\mu)]_p > p^e$. Then $\chi(1)_p > p^e$, and characters in $\mathrm{Irr}(B, \mu)$ do not contribute to $\mathrm{mh}(B)$. Also $[P : I_P(\mu)] > p^e$, so characters in $\mathrm{Irr}(P, \mu)$ do not contribute to $\mathrm{m}(P)$.

Suppose that $[G : I_G(\mu)]_p = 1$. Let $Q \in \mathrm{Syl}_p(GV)$ with $I_P(\mu) \leq Q \leq I_{GV}(\mu)$. Then $I_P(\mu) \in P \cap Q$, so either $[P : I_P(\mu)] > p^e$, in which case irreducible characters covering μ do not contribute to $\mathrm{m}(P)$, or $I_P(\mu) = P$, in which case $\mathrm{m}(P|\mu) = p^e$. So, replacing μ by a G-conjugate if necessary, $p^{\mathrm{mh}(B)} \leq \mathrm{m}(E) = \mathrm{m}(P|\mu)$.

Now suppose that $[G : I_G(\mu)]_p = p^t$ where $0 < t < e$. Then $\chi(1)_p = p^t$. Note that if $Q \in \mathrm{Syl}_p(GV)$, then $|I_Q(\mu)| \leq p^t$. By replacing μ by a G-conjugate if necessary, we may assume $|I_P(\mu)| = p^t$, and $\mathrm{m}(P|\mu) = p^t$. So

$$\mathrm{mh}(B) \leq \mathrm{mh}(B|\mu) = \mathrm{mh}(P|\mu) \leq \mathrm{mh}(P) \leq \mathrm{mh}(B),$$

and we are done.

$\square$

We list some cases where the hypotheses of Theorem 6.1 are satisfied.

Lemma 6.2. *Let $G = E \ltimes X$, where E is a non-abelian p-group acting faithfully and irreducibly on an elementary abelian q-group X. Then for each $Q \in \mathrm{Syl}_p(GV) - \{P\}$ we have $[P : P \cap Q] > \mathrm{m}(E)$.*

Proof. Write $\mathrm{m}(E) = p^e$. Let $x \in X$, and write $H = E^x \cap E$. Suppose that $1 < [E : H] \leq p^e$. Then $H \lhd E^x, E$ (consider the permutation character for the action of E on the E-conjugates of H, which must have only trivial constituents), so $H \lhd \langle E^x, E\rangle$. Now $O_q(\langle E^x, E\rangle) = X \cap \langle E^x, E\rangle$, so $O_q(\langle E^x, E\rangle) = X$ by irreducibility. Hence $\langle E^x, E\rangle = G$, and $H \lhd G$. But $H \cap X = 1$, so $[X, H] = 1$, so $H = 1$ since E acts faithfully on X, a contradiction. $\square$

Lemma 6.3. *Let $p = 2$, and let $G = EX$, where E is a generalized quaternion group acting on a group X of odd order with $C_E(X) = 1$. Then for each $Q \in \mathrm{Syl}_2(GV) - \{P\}$ we have $[P : P \cap Q] > \mathrm{m}(E)$.*

Proof. Write $\mathrm{m}(E) = p^e$. Let $x \in X$, and suppose that $1 < [E : E^x \cap E] \leq p^e$. Then $E^x \cap E \lhd E, E^x$, and so $Z(E) \leq E \cap E^x$. Hence $H = Z(E^x) = Z(E)^x = H^x$. But $C_G(HX) = 1$, so $N_{HX}(H) = H$, and $x \notin N_G(H)$, a contradiction. $\square$

We move on to some results of a different type, showing that equality occurs when we impose some restrictions on the set of character degrees of the defect group.

Theorem 6.4. *Let B be a p-block of a finite solvable group G with defect group D, where p is odd. If $\mathrm{cd}(D)$ is a subset of $\{1, p^a, \ldots, p^{2a}\}$ containing p^a for some $a \geq 2$, then $\mathrm{mh}(B) = \mathrm{mh}(D) = a$.*

Proof. Let B be a counterexample with $[G : Z(G)]$ minimized. By reductions as in the proof of Theorem 2.5, we have $O_{p'}(G) \leq Z(G)$. Hence $C_G(O_p(G)) \leq O_p(G)Z(G)$, and $D \in \mathrm{Syl}_p(G)$. Since $a > 1$, by Theorem 5.2 $l_p(G) = 1$, so $D = O_p(G)$ and B is not a counterexample after all.

$\square$

Proposition 6.5. *Let B be a p-block of solvable group with defect group D such that $\mathrm{cd}(D) = \{1, p\}$, where p is odd. Then $\mathrm{mh}(B) = \mathrm{mh}(D) = 1$.*

Proof. Let B be a counterexample with $[G : Z(G)]$ minimized. Then, as in the proof of Theorem 2.5, we have $O_{p'}(G) \leq Z(G)$. Hence $C_G(O_p(G)) \leq O_p(G)Z(G)$, and $D \in \mathrm{Syl}_p(G)$. Since $|\mathrm{cd}(D)| = 2$, the derived length of D is at most two, and so $l_p(G) \leq 2$. Then $l_p(G) = 2$, since otherwise $D = O_p(G)$.

Write $N = O_p(G)$ and $K = O_{p,p'}(G)$. As in the proof of Theorem 5.2, we have $\Phi(G) < F(G) = N$ and $l_p(G/\Phi(G)) = 2$. Note that $D/\Phi(G)$ is nonabelian, so $\mathrm{cd}(D/\Phi(G)) = \{1, p\}$.

By [8] there is a block $\overline{B}$ of $G/\Phi(G)$ with defect group $D/\Phi(G)$ such that $\mathrm{cd}(\overline{B}) \subseteq \mathrm{cd}(B)$. Hence by minimality $\Phi(G) = 1$, and in particular N is elementary abelian.

By [15, 1.12] N is a faithful G/N-module, and so by [15, 7.3] it follows that D/N has a regular orbit on $\mathrm{Irr}(N)$. So D/N, and so D, possesses an irreducible character of degree at least $|D/N|$. Hence $|D/N| = p$. By [9], B possesses an irreducible character of positive height. Since K has a normal abelian Sylow p-subgroup and $[G : K] = p$, this irreducible character must have height one, and we are done. $\square$

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