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Extending Coarse-Grained Measures

by

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Summary. In [4] it is proved that a measure on a finite coarse-grained space extends, as a signed measure, over the entire power algebra. In [7] this result is reproved and further improved. Both the articles [4] and [7] use the proof techniques of linear spaces (i.e. they use multiplication by real scalars). In this note we show that all the results cited above can be relatively easily obtained by the Horn–Tarski extension technique in a purely combinatorial manner. We also characterize the pure measures and settle the dimension of the normalized-measure space. We then comment on a consequence of the results for circulant matrices. Finally, we take up the case of circle coarse-grained space and also establish a measure-extension result.

1. Introduction. The problem pursued in this paper is motivated by the measurement theory and the theory of quantum logics (see [4], [6] and [9] for more motivation details). The coarse-grained structures constitute an important example of so-called "concrete quantum logics". Let us recall the definition of the coarse-grained space and the relevant notions we shall use.

Let $n \geq 2$ and $l \geq 2$ be natural numbers. Let $\Omega = \{0, 1, \ldots, nl - 1\}$. Denote by $\Delta_{n,l}$ the smallest system of subsets of Ω that contains all sets of the type

$$I_h = \{h, h+1, \ldots, h+l-1\} \quad (h \in \Omega)$$

(the sum is understood modulo nl) and that is closed under the formation of complements in Ω and disjoint unions. The collection $\Delta_{n,l}$ is called the *coarse-grained additive class* generated by the system I_h, and the sets I_h are called the *generating sets* of $\Delta_{n,l}$.

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Let $m : \Delta_{n,l} \to \mathbb{R}$ be a mapping such that

- $m(\emptyset) = 0$,
- $m(A \cup B) = m(A) + m(B)$ for arbitrary disjoint sets A and B in $\Delta_{n,l}$.

The mapping m is said to be a *coarse-grained signed measure* on $\Delta_{n,l}$. If a coarse-grained signed measure attains only non-negative values, it is called a *coarse-grained measure*. We shall omit the phrase coarse-grained and simply speak of signed measures (resp. measures) on $\Delta_{n,l}$.

We want to present another proof technique for the extension results for measures (resp. signed measures) on $\Delta_{n,l}$ (see [4] and [7]) and add a few new results. Our approach uses the classical Horn–Tarski extension theorem. This we believe simplifies the arguments in places and allows one to have a better insight into the question considered.

2. Measures on a finite coarse-grained additive class (extensions). We use the notations introduced above.

THEOREM 2.1. *Let $n, l \in \mathbb{N}$, $n \geq 2$, $l \geq 2$ and let $\Delta_{n,l}$ be the coarse-grained additive system of subsets of $\Omega = \{0, 1, \ldots, nl-1\}$.*

(i) *Each signed measure on $\Delta_{n,l}$ can be extended as a signed measure over the Boolean algebra* $\exp \Omega$ *of all subsets of Ω.*

(ii) *If $n \geq 3$ or if $n = l = 2$, then each measure on $\Delta_{n,l}$ can be extended as a measure over* $\exp \Omega$*. Moreover, if the measure on $\Delta_{n,l}$ we start with is two-valued, the extension over* $\exp \Omega$ *can also be required to be two-valued.*

(iii) *Let $\mathcal{M}(\Delta_{n,l})$ denote the set of all normalized measures on $\Delta_{n,l}$ (a measure m is called normalized if $m(\Omega) = 1$). Then t is an extreme point of $\mathcal{M}(\Delta_{n,l})$ if and only if it is a two-valued measure. Moreover,* $\dim \mathcal{M}(\Delta_{n,l}) = l(n-1) + 1$.

(iv) *Suppose that $i_1, \ldots, i_{l-1} \in \Omega$ lie in different classes modulo l. Suppose that real numbers $r_1, \ldots, r_{l-1}$ are given. Then any measure m on $\Delta_{n,l}$ can be uniquely extended over* $\exp \Omega$ *as a signed measure t such that $t(i_j) = r_j$ $(j \leq l-1)$.*

The proof of Theorem 2.1(i) has been given in [4] where also (ii) appeared, though in a rather erroneous form, as observed in [7]. A complete proof of Theorem 2.1(i), (ii) has been published in [7]. Our proof essentially differs from the previous ones and is based on the following well known result due to Horn and Tarski (see [1] and [5]).

PROPOSITION 2.2. *Let $\mathcal{C}$ be a collection of subsets of a set Ω.*

(i) *A set function $m : \mathcal{C} \to \mathbb{R}$ can be extended as a signed measure over the power algebra* $\exp \Omega$ *if the following implication holds true* *(by χ_Y we denote the characteristic function of the set Y): If A_i $(i = 1, \ldots, p)$ and B_j $(j = 1, \ldots, q)$ are sets of $\mathcal{C}$, then*

$$\sum_{i=1}^{p} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j} \quad \textit{implies} \quad \sum_{i=1}^{p} m(A_i) = \sum_{j=1}^{q} m(B_j).$$

(ii) *A non-negative set function $m : \mathcal{C} \to \mathbb{R}$ can be extended as a measure over the power algebra $\exp \Omega$ if the following implication holds true: If A_i $(i = 1, \ldots, p)$ and B_j $(j = 1, \ldots, q)$ are sets of $\mathcal{C}$, then*

$$\sum_{i=1}^{p} \chi_{A_i} \leq \sum_{j=1}^{q} \chi_{B_j} \quad \textit{implies} \quad \sum_{i=1}^{p} m(A_i) \leq \sum_{j=1}^{q} m(B_j).$$

According to Proposition 2.2, to prove statement (i) in Theorem 2.1 it is sufficient to verify the validity of the implication in Proposition 2.2(i). This will be done in our next proposition. Prior to that, let us note that every measure on $\Delta_{n,l}$ is *uniquely determined* by its values on the generating sets I_h. Indeed, suppose that two measures m and m' coincide on all generators I_h. Since the family $\mathcal{F}$ of sets A in $\Delta_{n,l}$ for which $m(A) = m'(A)$ is closed under the formation of disjoint unions and complements and since $\mathcal{F}$ contains all generators I_h, we see that $\mathcal{F}$ necessarily coincides with the entire $\Delta_{n,l}$.

PROPOSITION 2.3. *Let I_h be the generating sets of the additive class $\Delta_{n,l}$ on $\Omega = \{0, 1, \ldots, nl-1\}$. Let $m : \Delta_{n,l} \to \mathbb{R}$ be a signed measure. If for some p and q and for some generating sets A_i and B_j we have $\sum_{i=1}^{p} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j}$, then $\sum_{i=1}^{p} m(A_i) = \sum_{j=1}^{q} m(B_j)$.*

Proof. Assume that $A_1, \ldots, A_p$, $B_1, \ldots, B_q$ are generating sets of $\Delta_{n,l}$. It follows that $A_i = I_r$ and $B_j = I_s$ for $i \leq p$, $j \leq q$ and some r and s in $\{0, 1, \ldots, nl-1\}$. Assume that $\sum_{i=1}^{p} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j}$. Since all A_i and B_j are of the same cardinality (equal to l), we see that $p = q$. We can assume $A_i \neq B_j$ for any i, j (if the same set appears in both families $\{A_i\}_{i \leq p}$ and $\{B_j\}_{j \leq q}$, we can cross it out). Consider the number of times each element of Ω appears in the two families. That is, given $\omega \in \Omega$, set

$$\mathcal{N}(\omega) = \sum_{i=1}^{p} \chi_{A_i}(\omega) = \sum_{j=1}^{p} \chi_{B_j}(\omega).$$

First observe that the function $\mathcal{N} : \Omega \to \mathbb{N}$ is constant. If not, take an $\omega \in \Omega$ such that $\mathcal{N}(\omega) < \mathcal{N}(\omega + 1)$. Such an element certainly exists as we easily see once we take into account that the sum in Ω is understood modulo nl. Supposing the last inequality holds true, there is an $i_0 \in \{1, \ldots, p\}$ such that $A_{i_0} = I_{\omega+1} = \{\omega + 1, \ldots, \omega + l\}$. Indeed, if not then any set containing $\omega + 1$ also contains ω. Then $\mathcal{N}(\omega) \geq \mathcal{N}(\omega + 1)$ contrary to assumption. The same argument applies to the family $\{B_j\}$. It means that $A_{i_0} = B_{j_0}$ for some indices i_0, j_0. This is impossible since we have crossed out all equal elements. Thus, $\mathcal{N}(\omega)$ is a constant function; denote its value by c.

To complete the proof, take the set A_1. For some $h \in \Omega$ we have $A_1 = I_h = \{h, h+1, \ldots, h+l-1\}$. There exists $j \in \{2, 3, \ldots, p\}$ such that $A_j = I_{h+l}$. If not, any set of the family $\{A_i\}_{i \le p}$ containing $h+l$ also contains $h+l-1$, which is impossible. Going on this way, we see that the family $\{A_i\}$ is a union of partitions of Ω (the number of these partitions is then c). The same is true for the family $\{B_j\}$. The proof is then complete since m was supposed to be additive on $\Delta_{n,l}$ and therefore $\sum_{i=1}^{p} m(A_i) = \sum_{j=1}^{p} m(B_j) = c \cdot m(\Omega)$. ∎

Note that in the previous proof we did not need to know what a general element of $\Delta_{n,l}$ looks like. In the proof of Theorem 2.1(ii) to follow we do need it. For that, recall the following result by Ovchinnikov [7] (this seems to be the only point where our procedure overlaps with his).

Proposition 2.4. *Let $\Delta_{n,l}$ be the coarse-grained additive class on $\Omega = \{0, 1, \ldots, nl-1\}$ generated by the sets I_h ($h \in \Omega$). If $n \ge 3$, then an l-element set $I = \{a_0, a_1, \ldots, a_{l-1}\} \subset \Omega$ belongs to $\Delta_{n,l}$ if and only if for each $t \in \{0, 1, \ldots, l-1\}$ there is exactly one element in I congruent to t modulo l.*

The previous proposition gives a complete description of the atoms in $\Delta_{n,l}$, and therefore in the case of $n \ge 3$ it gives a complete description of all elements of $\Delta_{n,l}$: The atoms are exactly the l-element subsets of Ω containing precisely one element of each residue class modulo l. Denoting by $\mathcal{A}$ the family of all atoms in $\Delta_{n,l}$, this implies that (upon denoting by $\mathcal{R}_i$ the elements congruent to i modulo l)

$$\mathcal{A} = \{\{a_0, a_1, \ldots, a_{l-1}\} : a_i \in \mathcal{R}_i,\ i = 0, 1, \ldots, l-1\}.$$

(Note that for $n = 2$ the situation is rather different. In this case $\Delta_{2,l} = \{\emptyset, I_0, I_1, \ldots, I_{nl-1}, \Omega\}$.)

Remark. Before taking up the proof of Theorem 2.1(ii), observe that the question of extending measures to measures is subtler than in the case of signed measures. The complication is that there are non-negative evaluations of the sets I_h $(h = 0, 1, \ldots, nl-1)$ which are additive on the sets I_h but which do not allow for non-negative extensions over all atoms of $\Delta_{n,l}$ (the circumstance overlooked in the erroneous Theorem 3 of [4]). This peculiarity may even occur for a two-valued evaluation. If e.g. we take $n = 3$ and $l = 3$ and consider the function $t : \Omega \to \mathbb{R}$ defined by setting $t(0) = 1$, $t(1) = -1$, $t(2) = 1$ and $t(i) = 0$ for all i, $3 \le i \le 8$, we get a signed measure on $\exp \Omega$ such that $t(I_h) \in \{0, 1\}$ $(h \in \{0, 1, \ldots, 8\})$. We easily see that t as an evaluation of I_h cannot be additively and non-negatively extended over all atoms of $\Delta_{3,3}$. Indeed, the value of the extension would have to be -1 on the set $\{1,5,6\}$.

We are now able to prove Proposition 2.2(ii) and thus provide the essential part of the proof of Theorem 2.1(ii) (the case $n=l=2$ is evident).

PROPOSITION 2.5. *Let $n \geq 3$. Let I_h ($h \in \Omega$) be the generating sets of the coarse-grained additive class $\Delta_{n,l}$ on $\Omega = \{0,1,\dots,nl-1\}$. Let $m : \Delta_{n,l} \to \mathbb{R}$ be a measure. If $\sum_{i=1}^{p} \chi_{A_i} \leq \sum_{j=1}^{q} \chi_{B_j}$, where A_i and B_j are some generating sets I_h ($h = 1,\dots,nl-1$), then $\sum_{i=1}^{p} m(A_i) \leq \sum_{j=1}^{q} m(B_j)$.*

Proof. Since card $A_i =$ card B_j for any i,j, we see that $p \leq q$. The case of $p=q$ is equivalent to $\sum_{i=1}^{p} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j}$ and this has been argued in Proposition 2.2(i).

Assume $p<q$. Since each of the sets A_i, B_j belongs to $\mathcal{A}$ and therefore contains exactly one element of each residue class modulo l, it is possible to add to $\{A_i\}_{i \leq p}$ some atoms $A_{p+1},\dots,A_q$ of $\Delta_{n,l}$ such that

$$\sum_{i=1}^{q} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j}.$$

Indeed, consider the collection E of all "exceeding" elements of the right-hand side of the inequality $\sum_{i=1}^{p} \chi_{A_i} < \sum_{j=1}^{q} \chi_{B_j}$, i.e. set $E = \{\omega \in \Omega : \sum_{i=1}^{p} \chi_{A_i}(\omega) < \sum_{j=1}^{q} \chi_{B_j}(\omega)\}$. The equalities

$$\sum_{\omega \in \mathcal{R}_i} \sum_{i=1}^{p} \chi_{A_i} = p, \qquad \sum_{\omega \in \mathcal{R}_i} \sum_{j=1}^{q} \chi_{B_j} = q$$

together with the inequality

$$\sum_{i=1}^{p} \chi_{A_i} < \sum_{j=1}^{q} \chi_{B_j}$$

ensure that it is possible to choose l elements in the set E, one for each residue class $\mathcal{R}_i$,

$$\omega_0 \in \mathcal{R}_0,\ \omega_1 \in \mathcal{R}_1,\ \dots,\ \omega_{l-1} \in \mathcal{R}_{l-1},$$

in such a way that, denoting by A_{p+1} the l-element set $\{\omega_0,\omega_1,\dots,\omega_{l-1}\}$, we obtain

$$\sum_{i=1}^{p+1} \chi_{A_i} \leq \sum_{j=1}^{q} \chi_{B_j} = q.$$

If $p+1=q$ we are done. Otherwise, let us again set

$$E = \left\{\omega \in \Omega : \sum_{i=1}^{p+1} \chi_{A_i}(\omega) < \sum_{j=1}^{q} \chi_{B_j}(\omega)\right\}$$

and repeat the procedure to obtain A_{p+2}, etc. After $q-p$ steps, we have produced the sets $A_{p+1},\dots A_q$ so that the desired equality

$$\sum_{i=1}^{q}\chi_{A_i}=\sum_{j=1}^{q}\chi_{B_j}$$

is valid. The non-negativity of m then yields

$$\sum_{i=1}^{p}m(A_i)\le\sum_{i=1}^{q}m(A_i).$$

An application of Proposition 2.2(i) gives the existence of a signed measure $\widetilde{m}$, extending m to $\exp\Omega$. This obviously implies the equality $\sum_{i=1}^{q}m(A_i)=\sum_{j=1}^{q}m(B_j)$ (in fact, for each set C in $\Delta_{n,l}$ we have $m(C)=\sum_{\omega\in C}\widetilde{m}(\{\omega\})$) and completes the proof. ∎

Remark. Note that the example we provided in the Remark above does not fulfil the condition in the statement of the last proposition. For instance, take $p=1$, $q=2$, $A_1=I_2$, $B_1=I_1$ and $B_2=I_4$.

For the proof of the rest of Theorem 2.1(ii), we need the following proposition.

Proposition 2.6. *Assume $n\ge 3$ or $n=l=2$. Let s be a normalized measure on $\Delta_{n,l}$ (i.e. a measure with $s(\Omega)=1$). Then the following statements are equivalent*:

(i) *s is two-valued*,
(ii) *s is concentrated at a point of Ω (it is a Dirac measure)*,
(iii) *s is an extreme point of the (compact convex) set of all normalized measures on $\Delta_{n,l}$.*

Proof. (i)⇒(ii). For $n=l=2$ the situation is obvious. Assume $n\ge 3$. Let $s:\Delta_{n,l}\to\{0,1\}$. Since $s(I_k)\in\{0,1\}$ for any $k\in\Omega$ and $\sum_{k\in\Omega}s(I_k)=l$, it follows that there are $\omega_1,\dots,\omega_l$ in Ω such that $s(I_{\omega_j})=1$ for any $j=1,\dots,l$. We therefore see that for any ω among the remaining $(n-1)l$ elements of Ω the value of s on I_ω is zero. We can then write $\Omega=\Omega_1\cup\Omega_2$, where $\mathrm{card}(\Omega_1)=l$ and $s(I_k)=1$ for $k\in\Omega_1$, while $\mathrm{card}(\Omega_2)=(n-1)l$ and $s(I_k)=0$ for $k\in\Omega_2$. Observe now that if we manage to show that Ω_1 consists of consecutive elements, we are done. Indeed, if this is the case, typically $\Omega_1=\{h+1,h+2,\dots,h+l\}=I_{h+1}$ for a certain h in Ω, then the measure s is concentrated at the element $h+l$ (the atoms I_k with $k\in\Omega_1$ are exactly those which contain $h+l$). Assume that Ω_1 does not consist of consecutive elements. Then we make use of the fact that $n>2$ and we find $h,k\in\Omega_1$ with $I_h\cap I_k=\emptyset$. However, this is impossible since then $s(I_h\cup I_k)=2$. This proves (i)⇒(ii).

(ii)⇒(iii). This implication is obvious.

(iii)⇒(i). The proof follows from a well known result in convex analysis: the restriction map $\mathbb{R}^{\exp \Omega} \to \mathbb{R}^{\Delta_{n,l}}$ can be viewed as a continuous linear operator and when specified to normalized measure spaces, every extreme point in the range allows for a preimage extreme point in the domain. Let us present a simple direct proof. Let s be a normalized measure on $\Delta_{n,l}$ which is not two-valued. Proposition 2.3 ensures the existence of an extension, $\widetilde{s}$, of s over the power algebra $\exp \Omega$. The state $\widetilde{s}$ can be written as a convex combination of Dirac measures $\delta_{x_0}, \delta_{x_1}, \ldots, \delta_{x_{nl-1}}$ on $\exp \Omega$,

$$\widetilde{s} = \sum_{i=0}^{nl-1} \alpha_i \delta_{x_i} \quad \text{with } \alpha_i \geq 0 \text{ and } \sum_{i=0}^{nl-1} \alpha_i = 1.$$

Without any loss of generality we can assume $\alpha_0 \in (0,1)$. If we denote by t the measure on $\exp \Omega$ defined by

$$t = \frac{1}{1-\alpha_0} \sum_{i=1}^{nl-1} \alpha_i \delta_{x_i}$$

then $\widetilde{s}$ becomes a convex combination of two measures δ_{x_0} and t,

$$\widetilde{s} = \alpha_0 \delta_{x_0} + (1-\alpha_0) t.$$

Observe that the restrictions s_0 and t_0 (of δ_{x_0} and t, respectively) to $\Delta_{n,l}$ do not agree. In fact, since s is not concentrated at x_0, the same is true for t, which is then different from s_0. ∎

From the previous result it follows that each two-valued measure on $\Delta_{n,l}$ can be extended as a two-valued measure over the entire $\exp \Omega$. Using this, we want to show that $\dim \mathcal{M}(\Delta_{n,l}) = l(n-1)+1$. Let s_i be the Dirac measure concentrated at $\{i\}$. We want to show that the measures $s_0, s_1, \ldots, s_{(n-1)l}$ form an affine basis of $\mathcal{M}(\Delta_{n,l})$. Let us first check that they are linearly independent. Assume that $\sum_{i=0}^{(n-1)l} \lambda_i s_i = 0$. Then

$$\sum_{i=0}^{(n-1)l} \lambda_i s_i(I_{(n-1)l+1}) = \lambda_0 s_0(I_{(n-1)l+1}) = \lambda_0$$

and therefore $\lambda_0 = 0$. Further,

$$\begin{aligned} \sum_{i=0}^{(n-1)l} \lambda_i s_i(I_{(n-1)l+2}) &= \lambda_0 s_0(I_{(n-1)l+2}) + \lambda_1 s_1(I_{(n-1)l+2}) \\ &= \lambda_1 s_1(I_{(n-1)l+2}) = \lambda_1 \end{aligned}$$

and therefore $\lambda_1 = 0$, etc. We inductively obtain $\lambda_0 = \lambda_1 = \cdots = \lambda_{(n-1)l} = 0$.

Second, we are going to show that any measure s can be expressed as a linear combination of s_i $(0 \leq i \leq (n-1)l)$. For that it is sufficient to check that given arbitrary values $(v_0, v_1, \ldots, v_{(n-1)l})$, there exist coefficients

$(\lambda_0, \lambda_1, \dots, \lambda_{(n-1)l})$ such that the measure $s = \sum_{i=0}^{(n-1)l} \lambda_i s_i$ attains the value v_i on the generating set I_i $(0 \le i \le (n-1)l)$. Since the values of s on the remaining sets I_i, $i > (n-1)l$, are already determined by the values of s on I_i $(0 \le i \le (n-1)l)$, this will complete the proof.

Fix $(v_0, v_1, \dots, v_{(n-1)l})$. Knowing that $s_i(I_{(n-1)l}) = 0$ for any $i < (n-1)l$ and wanting $s(I_{(n-1)l}) = v_{(n-1)l}$, we must have

$$\sum_{i=0}^{(n-1)l} \lambda_i s_i(I_{(n-1)l}) = \lambda_{(n-1)l} s_{(n-1)l}(I_{(n-1)l}) = \lambda_{(n-1)l}.$$

This yields $\lambda_{(n-1)l} = v_{(n-1)l}$. For $s(I_{(n-1)l-1}) = v_{(n-1)l-1}$ we must have

$$\begin{aligned}\sum_{i=0}^{(n-1)l} \lambda_i s_i(I_{(n-1)l-1}) &= \lambda_{(n-1)l-1} s_{(n-1)l-1}(I_{(n-1)l-1}) \\ &\quad + \lambda_{(n-1)l} s_{(n-1)l}(I_{(n-1)l-1}) \\ &= \lambda_{(n-1)l-1} + v_{(n-1)l-1}.\end{aligned}$$

This yields $\lambda_{(n-1)l-1} = v_{(n-1)l-1} - v_{(n-1)l}$. Going on this way, we will determine all the coefficients λ_i $(i \le (n-1)l)$ and complete the proof of Theorem 2.1(iii). Theorem 2.1(iv) easily follows from Theorem 2.1(iii).

Let us show by examples that the extension results given by Theorem 2.1 are in a sense best possible. Firstly, in Theorem 2.1 we have to assume $n > 2$ as the following simple example shows.

Example 2.7. Take $n = 2$ and $l = 3$. Then $\Delta_{2,3} = \{\emptyset, I_0, I_1, \dots, I_5, \Omega\}$. It is immediate to check that the (two-valued) measure on $\Delta_{2,3}$ such that

$$s(I_0) = s(I_2) = s(I_4) = 1, \quad s(I_1) = s(I_3) = s(I_5) = 0$$

cannot be extended over $\exp \Omega$ as a measure.

Secondly, it is worth observing that Theorem 2.1 cannot be generalized to arbitrary additive classes (see also [8]). In fact, an extension may not exist even if the original measure is two-valued.

Example 2.8. Let $\Omega = \{0, 1, 2, 3, 4, 5\}$ and consider the additive class Δ generated by the sets $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{3, 4, 5\}$, and $D = \{1, 3, 5\}$. If m is a signed measure on $\exp \Omega$, then

$$m(C) + m(A) + m(B^{\mathrm{c}}) + m(D^{\mathrm{c}}) = 2m(\Omega).$$

Analogously,

$$m(C^{\mathrm{c}}) + m(A^{\mathrm{c}}) + m(B) + m(D) = 2m(\Omega).$$

The (two-valued) measure t on Δ defined by setting

$$t(A) = 0, \quad t(B) = 1, \quad t(C) = 0, \quad t(D) = 1$$

cannot be extended as a signed measure on $\exp \Omega$. Indeed, if we compute the sums above, we obtain 0 for the first sum and 4 for the second.

3. A link of coarse-grained measures with circulant matrices. Let $n, l \in \mathbb{N}$ and $n \geq 2$, $l \geq 2$. Denote by $\mathcal{M}^{nl}$ the set of all $nl \times nl$ matrices. Let $M \in \mathcal{M}^{nl}$. We say that M is an *elementary circulant matrix* if M is a circulant matrix (see [2]) with the first row $(a_{1,1}, a_{1,2}, \ldots, a_{1,nl})$ such that $a_{1,j} = 1$ for all $j \leq l$, $a_{1,k} = 0$ otherwise. It is easy to see that if we write out the extension problem as a collection of equations for the values of the potential extension, we obtain a system of linear equations with an elementary circulant matrix. Our result can then be expressed in the following form:

THEOREM 3.1. *Let $M \in \mathcal{M}^{nl}$ and M be an elementary circulant matrix. Consider the equation $M\vec{x} = \vec{b}$. Let $\vec{b} = (b_1, \ldots, b_{nl})$.*

(i) *The system $M\vec{x} = \vec{b}$ has a solution if and only if there is a $c \in \mathbb{R}$ such that for each $h \in \{1, \ldots, l\}$ we have $\sum_{i=1}^{n} b_{h+il} = c$ (the sum is understood modulo nl).*

(ii) *Suppose that $b_i \geq 0$ for each i, $1 \leq i \leq nl$. Then the system $M\vec{x} = \vec{b}$ has a non-negative solution if and only if the following implication holds true (with $\vec{r}_i$ denoting the i-th row of M):*

If $\sum_{i=1}^{nl} c_i\vec{r}_i \geq \sum_{i=1}^{nl} d_i\vec{r}_i$ for some non-negative integers c_i, d_i $(1 \leq i \leq nl)$, then $\sum_{i=1}^{nl} c_i b_i \geq \sum_{i=1}^{nl} d_i b_i$.

4. Measures on a circle coarse-grained additive class (extensions). In this section we shall consider a continuous analogy of finite coarse-graining. This has already been initiated in [4], though the measure extension of finitely additive measures has not been pursued: the authors only analyzed a measure extension problem of analytic nature based on σ-additivity. We want to show that there is an extension theorem analogous to Theorem 2.1(i) valid in this "continuous" case, and that this result can also be derived from the Horn–Tarski theorem.

Let C be the unit circle in the plane parametrized by $[0, 2\pi)$. Fix an integer $n \geq 2$ and denote by Δ_n the smallest system of subsets of C that contains all the (generating) sets of the type $[\alpha, \alpha + 2\pi/n)$, $\alpha \in [0, 2\pi)$ (the sum is understood modulo 2π) and that is closed under the formation of complements in C and disjoint unions. Call Δ_n the coarse-grained additive system on C. With the intention to obtain an extension result for finitely additive measures defined on Δ_n, we shall verify the Horn–Tarski condition (Proposition 2.2(i)) for signed measures (we have not been able to verify the Horn–Tarski condition for measures, so this rather interesting question is left open).

PROPOSITION 4.1. *Let $A_1, \dots, A_p$ and $B_1, \dots, B_q$ be generating sets of the additive class Δ_n and let $m : \Delta_n \to \mathbb{R}$ be a signed measure. If for some p and q we have $\sum_{i=1}^{p} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j}$, then $\sum_{i=1}^{p} m(A_i) = \sum_{j=1}^{q} m(B_j)$.*

Proof. By assumption, for some sets of the type $A_i = [\alpha_i, \alpha_i + 2\pi/n)$ $(1 \le i \le p)$ and $B_j = [\beta_j, \beta_j + 2\pi/n)$ $(1 \le j \le q)$ we have the equality

$$\sum_{i=1}^{p} \chi_{A_i} = \sum_{j=1}^{q} \chi_{B_j}.$$

We can assume $A_i \neq B_j$ for each i, j. This gives

$$\{\alpha_1, \dots, \alpha_p\} \cap \{\beta_1, \dots, \beta_q\} = \emptyset.$$

Consider the number of times an element of C appears in the two families $\{A_i\}_{i \le p}$ and $\{B_j\}_{j \le q}$. That is, for each $x \in C$ set

$$\mathcal{N}(x) = \sum_{i=1}^{p} \chi_{A_i}(x) = \sum_{j=1}^{q} \chi_{B_j}(x).$$

We first want to show that $\mathcal{N} : C \to \mathbb{N}$ is constant. Suppose it is not. Then the set $\{x \in C : \mathcal{N}(x) > \lim_{y \to x^-} \mathcal{N}(y)\}$ is not empty. The points in this set are necessarily the left end points of A_i and B_j. Then the inclusion

$$\{x \in C : \mathcal{N}(x) > \lim_{y \to x^-} \mathcal{N}(y)\} \subseteq \{\alpha_1, \dots, \alpha_p\} \cap \{\beta_1, \dots, \beta_q\} = \emptyset$$

gives a contradiction.

We have shown that the function $\mathcal{N}$ is constant; denote its value by c. This means that $\{A_i\}_{i=1}^{p}$ and $\{B_j\}_{j=1}^{p}$ are c-fold coverings of C (this can be easily shown by proving that if $[\alpha, \alpha + 2\pi/n) = A_i$, then $[\alpha + 2\pi/n, \alpha + 4\pi/n)$ must be one of the A_j's $(j \neq i)$). We infer that

$$\sum_{i=1}^{p} m(A_i) = \sum_{j=1}^{q} m(B_j) = c \cdot m(C). \quad \blacksquare$$

We have proved the following theorem:

THEOREM 4.2. *Let C be the unit circle in the plane and let Δ_n be the coarse-grained additive system of subsets of C generated by all half-open intervals of the type $[\alpha, \alpha + 2\pi/n)$, $\alpha \in [0, 2\pi)$. Let $m : \Delta_n \to \mathbb{R}$ be a (finitely additive) measure on Δ_n. Then m can be extended over the power algebra $\exp C$ as a signed measure.*

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References

[1] K. P. S. Bhaskara Rao and M. Bhaskara Rao, *Theory of Charges*, Academic Press, 1983.
[2] P. J. Davis, *Circulant Matrices*, Wiley, New York, 1979.
[3] S. Gudder, *An extension of classical measure theory*, SIAM Rev. 26 (1984), 71–89.
[4] S. Gudder and J. P. Marchand, *A coarse-grained measure theory*, Bull. Polish Acad. Sci. Math. 28 (1980), 557–564.
[5] A. Horn and A. Tarski, *Measures in Boolean algebras*, Trans. Amer. Math. Soc. 64 (1948), 467–497.
[6] M. Navara and P. Pták, *Almost Boolean orthomodular posets*, J. Pure Appl. Algebra 60 (1989), 105–111.
[7] P. G. Ovchinnikov, *Measures on the Gudder–Marchand logics*, in: Constructive Theory of Functions and Functional Analysis 8, Kazan. Gos. Univ., 1992, 95–98.
[8] R. E. Prather, *Generating the k-subsets of an n-set*, Amer. Math. Monthly 87 (1980), 740–743.
[9] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, 1991.

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