Czechoslovak Mathematical Journal

Semra Doğruöz

Extending modules relative to a torsion theory

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 2, 381–393

Persistent URL: http://dml.cz/dmlcz/128264

Czechoslovak Mathematical Journal, 58 (133) (2008), 381–393

EXTENDING MODULES RELATIVE TO A TORSION THEORY

Semra Doğruöz, Aydin

(Received February 12, 2006)

Abstract. An R-module M is said to be an extending module if every closed submodule of M is a direct summand. In this paper we introduce and investigate the concept of a type 2 τ-extending module, where τ is a hereditary torsion theory on Mod-R. An R-module M is called type 2 τ-extending if every type 2 τ-closed submodule of M is a direct summand of M. If τ_I is the torsion theory on Mod-R corresponding to an idempotent ideal I of R and M is a type 2 τ_I-extending R-module, then the question of whether or not M/MI is an extending R/I-module is investigated. In particular, for the Goldie torsion theory τ_G we give an example of a module that is type 2 τ_G-extending but not extending.

Keywords: torsion theory, extending module, closed submodule

MSC 2000: 16S90

1. Introduction

Extending modules have been studied extensively in recent years, see [2], [4], [11] and [14]. In [5] and [6] the authors investigated extending modules relative to certain classes of modules. Our purpose is to define and study extending modules relative to a torsion theory τ on Mod-R. This brings out a new and more general concept of extending modules, and we present some of the fundamental properties of these modules. Throughout the paper R will denote an associative ring with identity, Mod-R will be the category of unitary right R-modules, and unless stated otherwise, all modules and module homomorphisms will belong to Mod-R.

If $\tau := (\mathscr{T}, \mathscr{F})$ is a torsion theory on Mod-R, then τ is uniquely determined by its associated torsion class $\mathscr{T}$ of τ-torsion modules. Modules in $\mathscr{T}$ will be called *τ-torsion* and modules in $\mathscr{F}$ are said to be *τ-torsion free.* If $\tau(M)$ denotes the sum of the τ-torsion submodules of M, then $\tau(M)$ is necessarily the unique largest τ-torsion submodule of M and $\tau(M/\tau(M)) = 0$ for an R-module M. $\tau(M)$ is referred to as the *τ-torsion submodule* of M and it follows that $\mathscr{T} := \{M \in \text{Mod-}R;\ \tau(M) = M\}$

and $\mathscr{F} := \{M \in \text{Mod-}R;\ \tau(M) = 0\}$. For every torsion theory τ, both the torsion class $\mathscr{T}$ and the torsion-free class $\mathscr{F}$ of R-modules contain the zero module and both are closed under isomorphisms; that is, if $N \in \mathscr{T}$ and $N' \cong N$, then $N' \in \mathscr{T}$, and similarly for $\mathscr{F}$. A *$\mathscr{T}$-submodule* (or *$\mathscr{F}$-submodule*) of M is a submodule N of M such that N belongs to $\mathscr{T}$ (or $\mathscr{F}$).

For a torsion theory $\tau := (\mathscr{T}, \mathscr{F})$, $\mathscr{T} \cap \mathscr{F} = 0$ and the torsion class $\mathscr{T}$ is closed under homomorphic images, direct sums and extensions; and $\mathscr{F}$ is closed under submodules, direct products and extensions. A class $\mathscr{C}$ of modules is said to be closed under extensions if whenever $0 \to M' \to M \to M'' \to 0$ is exact in Mod-R and $M', M'' \in \mathscr{C}$, then $M \in \mathscr{C}$.

The torsion theories on Mod-R can be partially ordered by using their torsion classes. If $\sigma := (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ and $\tau := (\mathscr{T}_\tau, \mathscr{F}_\tau)$ are torsion theories on Mod-R, then we write $\sigma \leqslant \tau$ whenever $\mathscr{T}_\sigma \subseteq \mathscr{T}_\tau$. Throughout, ξ will denote the torsion theory in which only the zero module is torsion and χ will denote the torsion theory in which every module is torsion. Clearly $\xi \leqslant \tau \leqslant \chi$ for every torsion theory τ. All torsion theories τ are assumed to be *hereditary*, that is, we assume that submodules of τ-torsion modules are τ-torsion, unless stated otherwise. If I is an idempotent ideal of R, then it is well known that I determines a hereditary torsion theory τ_I with torsion class $\{M;\ MI = 0\}$. We refer to τ_I as the torsion theory corresponding to I. If τ_G is the Goldie torsion theory [12], then τ_G is hereditary and the τ_G-torsion submodule $\tau_G(M)$ of an R-module M is just the second singular submodule of M. That is, $\tau_G(M)$ is the submodule $Z_2(M)$ of M such that $Z_2(M)/Z(M) = Z(M/Z(M))$, where for an R-module M, $Z(M)$ denotes the *singular submodule* of M [12]. Additional information on torsion theory can be found in [3], [10] and [12] and we refer to [1], [4] for general information on rings and modules.

A nonzero submodule N of an R-module M is said to be *essential* in M if N has nonzero intersection with each nonzero submodule of M, written $N \leqslant_e M$; then a *closure* of N (in M) is a submodule K of M such that K is maximal among the submodules of M such that N is essential in K. A submodule N of M is called *closed* (in M) if N has no proper essential extension in M, written $K \leqslant_c M$. A Zorn's Lemma argument shows that for each submodule N of M there is a closed submodule K of M such that N is essential in K. Given a submodule N of M, by a *complement* (in M) we mean a submodule L of M that is maximal among the submodules H of M such that $H \cap N = 0$. A submodule L of M is said to be a *complement* if there is a submodule N of M such that L is a complement of N. It is well known that a submodule K of M is a complement if K is closed in M.

Let $\tau := (\mathscr{T}, \mathscr{F})$ be a torsion theory and M an R-module. A submodule N of M is called *τ-essential* in M if N is essential in M and M/N is τ-torsion; this is denoted $N \leqslant_{\tau_e} M$. A submodule N of M is called *type* 1 *τ-closed* in M if N has no proper τ-essential extension in M; this is denoted $N \leqslant_{\tau_1 c} M$ [7]. In this work we call a submodule N of M *type 2 τ-closed* in M if M/N is τ-torsion and N is closed in M; this will be denoted by $N \leqslant_{\tau_2 c} M$. A module M is *type* 1 *τ-extending* if every type 1 τ-closed submodule is a direct summand [7]. We will call a module M *type* 2 *τ-extending* if every type 2 τ-closed submodule is a direct summand. A submodule N of a module M is called *τ-dense* in M if M/N is a τ-torsion module. A module M is *τ-complemented* if every submodule is τ-dense in a direct summand. Hence M is τ-complemented if and only if for any $N \leqslant M$ there exists a direct summand K containing N with K/N is τ-torsion. A submodule N of M is said to be *τ-cotorsion free* [12], if N has no proper τ-dense submodules.

2. Type 2 τ-extending modules

Recall that an R-module M is said to be type 2 *τ-extending* if every type 2 τ-closed submodule of M is a direct summand of M. Semisimple modules, uniform modules and injective modules provide examples of modules that are type 2 τ-extending.

In this section we will be mainly interested in studying basic properties of type 2 τ-extending modules. In this and later sections we shall also be interested in the following questions posed by P. F. Smith from Glasgow after he has read the first draft of this paper.

Question 1. Let τ_I be the torsion theory corresponding to an idempotent ideal I of R. Is it true that an R-module M is type 2 τ_I-extending if and only if M/MI is an extending R/I-module?

Question 2. When is a finite direct sum of type 2 τ-extending modules type 2 τ-extending?

Question 3. Is there a module that is type 2 τ_G-extending but not extending?

Before considering Question 1, we prove several fundamental properties of type 2 τ-extending modules. The following lemma gives some immediate consequences of the definitions.

Lemma 2.1. *The following hold for an R-module M.*

(1) *If N is a type 2 τ-closed submodule of M, then N is closed in M.*
(2) *If N is a type 2 τ-closed submodule of K and K is a type 2 τ-closed submodule of M, then N is type 2 τ-closed in M.*
(3) *If M is extending, then M is type 2 τ-extending.*
(4) *If M is τ-torsion and type 2 τ-extending, then M is extending.*

(5) *If M is τ-torsion free, then M is type 2 τ-extending.*

(6) *Every direct summand of a type 2 τ-extending module is type 2 τ-extending.*

Proof. (1) Clear from the definitions.

(2) If N and K are as described in the lemma, then K/N and M/K are τ-torsion. Since the torsion class of τ is closed under extensions, it follows that M/N is τ-torsion. On the other hand, N is closed in K and K closed in M implies that N is closed in M (see [4, Page 6, Property 4]). Hence N is type 2 τ-closed in M.

(3) Let M be an extending module and let N be a type 2 τ-closed submodule of M. By (1), N is closed, so N is a direct summand of M. Hence, M is type 2 τ-extending.

(4) Let M be a τ-torsion module and suppose that M is type 2 τ-extending. If N is a closed submodule of M, since M/N is τ-torsion, N is type 2 τ-closed in M. By assumption N is a direct summand of M, so we see that M is extending.

(5) Let N be a type 2 τ-closed submodule of a τ-torsion free R-module M. Since N is closed in M, there exists a submodule K of M such that N is maximal with respect to $N \cap K = 0$. It follows that K is isomorphic to a submodule of the τ-torsion module M/N, so K is τ-torsion. Since M is τ-torsion free, $K = 0$. Therefore $M = N$, and so M is type 2 τ-extending.

(6) Let an R-module $M = M_1 \oplus M_2$ be the direct sum of the submodules M_1 and M_2, and suppose that M is type 2 τ-extending. If N is a type 2 τ-closed submodule of M_1, then M_1/N is τ-torsion and N is closed in M_1. If $N' = N \oplus M_2$, then $M/N' = (M_1 \oplus M_2)/(N \oplus M_2) \cong M_1/N$, so M/N' is τ-torsion. We claim that N' is closed in M. If N' is essential in a submodule T of M, then N is an essential submodule of $M_1 \cap T \subseteq M_1$. So it follows that $N = M_1 \cap T$. Since $M_2 \subseteq T$ and $M = M_1 \oplus M_2$, by modularity, $T = M_2 + (T \cap M_1) = M_2 + N = N'$. Thus N' is closed in M. But we have just seen that M/N' is τ-torsion, so N' is type 2 τ-closed in M. By hypothesis, $M = N' \oplus K$ for some submodule K of M, so $M = N' \oplus K = N \oplus M_2 \oplus K$. By modularity we have $M_1 = N \oplus (M_1 \cap (M_2 \oplus K))$ which shows that M_1 is type 2 τ-extending. □

Example 2.2. Every R-module is type 2 ξ-extending, where $\xi := (0, \text{Mod-}R)$ is the torsion theory in which only the zero module is considered to be torsion.

Proof. This follows easily since N is a type 2 ξ-closed submodule of M, thus M/N ξ-torsion, and so $N = M$. □

Example 2.3. An R-module M is type 2 χ-extending if and only if it is extending, where χ is the torsion theory in which every module is considered to be torsion.

P r o o f. The sufficiency is clear by Lemma 2.1 part (3). So assume that M is type 2 χ-extending and let N be a closed submodule of M. Now every module is χ-torsion, so in particular, M/N is χ-torsion. Hence N is a type 2 χ-closed submodule of M. Therefore N is a direct summand of M, and so M is extending. □

The following example shows that there are torsion theories τ and modules M that are type 2 τ-extending but not extending. It also shows that there are modules which have closed submodules which are not type 2 τ-closed. Because of this example, we see that in (4) of Lemma 2.1, the assumption that M is τ-torsion is not superfluous.

Example 2.4. Consider the ring $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$, where F is a field. If τ_I is the torsion theory on Mod-R corresponding to the idempotent ideal $I = \begin{bmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, that is $\mathscr{T}_I := \{N \in \text{Mod-}R : NI = 0\}$, then the following hold for the right R-module $M := R_R$.

(1) M is a type 2 τ_I-extending module, but is not extending.

(2) M has direct summands that are not type 2 τ_I-closed.

(3) The τ_I-torsion submodule $\tau_I(M)$ of M is not a direct summand.

P r o o f. Note that $\tau_I(M) = \begin{bmatrix} 0 & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$, so M is not τ_I-torsion.

(1) Consider the submodules $K = \begin{bmatrix} 0 & F & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $K \cap L = 0$ and K is a maximal submodule of I with respect to the property of having zero intersection with L. Hence K is a complement in I of L, and therefore in M since I is a direct summand of M. But K is not a direct summand of M, so M is not an extending R-module. However M is type 2 τ_I-extending. To see this, let N be a type 2 τ_I-closed submodule of M. Then M/N is τ_I-torsion and N is a closed submodule of M. If N is τ_I-torsion submodule of M, then M will be τ_I-torsion which is not the case. Hence N is not a τ_I-torsion submodule and, in this case, it is easy to check that all possibilities for N are $N = I$ or $N = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $N = \begin{bmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}$ or $N = M$. Hence in all cases N is a direct summand of M, and so M is a type 2 τ_I-extending module.

(2) Let $V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}$. Then V is a direct summand of M. Since M/V is not τ_I-torsion, V is not type 2 τ_I-closed.

(3) It is easy to see that $\tau_I(M)$ is an essential submodule of M, and therefore it is not a direct summand of M. □

In the preceding example, we found a torsion theory τ_I and an R-module M such that M is type 2 τ_I-extending. In the following example, we provide an example of a torsion theory τ_J such that the same R-module M is not type 2 τ_J-extending. These examples show that whether or not a module is type 2 τ-extending depends on the particular torsion theory τ under consideration.

Example 2.5. Let R and M be as in Example 2.4 and let J be the idempotent ideal $J = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix}$ of R. If τ_J is the torsion theory on Mod-R corresponding to J, that is $\mathscr{T}_J := \{N \in \text{Mod-}R\colon\ NJ = 0\}$, then M is neither extending nor type 2 τ_J-extending.

P r o o f. To show that M is not type 2 τ_J-extending, let N be a type 2 τ_J-closed submodule of M. Then M/N is τ_J-torsion and N is a closed submodule of M. Hence $(M/N)J = 0$, and so $MJ \leqslant N$. Now

$$MJ = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix} \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix} = \begin{bmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & F \end{bmatrix} = J \leqslant N.$$

Since $(M/J)J = 0$, M/J is τ_J-torsion. It is easy to check that J is closed in M, so J is a type 2 τ_J-closed in M. But J is not a direct summand of M, and so M is not type 2 τ_J-extending. □

The following two examples provide a negative answer to Question 1. These examples demonstrate that for an idempotent ideal I of R it is possible for an R-module M to be such that M/MI is an extending R/I-module but M is not type 2 τ_I-extending, and that it is possible for an R-module M to be type 2 τ_I-extending even though M/MI is not an extending R/I-module.

Example 2.6. If R, M, J and τ_J are as in Example 2.5, then M/MJ is an extending R/J-module, but M is not type 2 τ_J-extending.

Proof. We have just seen in Example 2.5 that M is not type 2 τ_J-extending. So it remains to show that M/MJ is an extending R/J-module. Now the ring

$$R/J := \left\{ \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix} + J \colon \ a,b,c,d,e \in F \right\}$$

is isomorphic to the ring of upper triangular matrices over F. By [4, 13.5, 13.6] every right (R/J)-module is extending since R/J is an Artinian serial ring that is also a (left and right) hereditary right SI ring [4] with $J(R/J)^2 = 0$. Hence M/MJ is an extending (R/J)-module. □

Example 2.7. There exist a torsion theory τ_K corresponding to an idempotent ideal K of a ring R and a right R-module M such that M is type 2 τ_K-extending but M/MK is not R/K-extending.

Proof. Let $\mathbb{Z}$ denote the ring of integers and $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{bmatrix}$, and consider the idempotent ideal K of R, $K = \begin{bmatrix} 0 & 0 & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{bmatrix}$, and the right R-module $M = \begin{bmatrix} 0 & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. Let $\tau_K := (\mathscr{T}_K, \mathscr{F}_K)$ denote the torsion theory on Mod-R corresponding to the idempotent ideal K, that is $\mathscr{T}_K := \{N \in \text{Mod-}R \colon \ NK = 0\}$. For a submodule N of M, M/N is τ_K-torsion if and only if $(M/N)K = 0$ if and only if $MK \leqslant N$. Let N be a type 2 τ_K-closed submodule of M. Thus N is closed in M and $MK \leqslant N$. So it is easy to check that such a closed submodule N of M is M itself. Hence M is type 2 τ_K-extending. Now consider the R/K-submodules N/MK and L/MK of the module M/MK, $N/MK = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2n & 0 \\ 0 & n & 0 \end{bmatrix} + MK; \ n \in \mathbb{Z} \right\}$, $L/MK = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{bmatrix} + MK; \ a,b \in \mathbb{Z} \right\}$. It is a routine to check that the submodule N/MK is maximal with respect to the property $(N/MK) \cap (L/MK) = 0$. So N/MK is closed in M/MK. But it can not be a direct summand of M/MK as R/K-module. Hence M/MK is not an extending R/K-module. □

The following results deal with characterizations of type 2 τ-extending modules.

Proposition 2.8. *If M is an R-module, then for a τ-dense submodule N of M there is a type 2 τ-closed submodule K of M such that N is essential in K.*

Proof. Let N be a τ-dense submodule of M. By Zorn's Lemma, we may find a closed submodule K of M such that N is essential in K. Since M/K is a homomorphic image of M/N, M/K is τ-torsion. Thus K is a type 2 τ-closed submodule of M such that N is essential in K. □

Lemma 2.9. *The following are equivalent for an R-module M.*

(1) *M is type 2 τ-extending.*

(2) *For each τ-dense submodule N of M, there is a direct summand A of M such that N is essential in A.*

Proof. (1) $\Rightarrow$ (2). Let N be a τ-dense submodule of M. By Proposition 2.8, we can find a type 2 τ-closed submodule K of M such that N is essential in K. By (1) K is a direct summand of M, and so (2) holds.

(2) $\Rightarrow$ (1). Let N be a type 2 τ-closed submodule of M. Then N is τ-dense and closed in M. By (2) there exists a direct summand A of M such that N is essential in A. Since N is closed in M, $N = A$. Hence (1) holds. □

Proposition 2.10. *The following hold for a τ-torsion free R-module M.*

(1) *M has no proper type 2 τ-closed submodules.*

(2) *Every τ-dense submodule N of M is essential in M.*

Proof. (1) Let N be a type 2 τ-closed submodule of M. Then M/N is τ-torsion and N is closed in M. By hypothesis $M = N \oplus N'$ for some submodule N' of M, and so N' is clearly τ-torsion. Since $\tau(M) = 0$, $N' = 0$, and so $M = N$.

(2) (See also [13, Lemma 1.7].) Let N be a τ-dense submodule of M. By Proposition 2.8, there is a type 2 τ-closed submodule K of M such that N is essential in K. By (1) $K = M$. So N is essential in M. □

Proposition 2.11. *If N is type 2 τ-closed submodule of an R-module M, then there is a submodule K of M such that N is maximal with respect to the property $N \cap K = 0$. In this case $N \oplus K$ is τ-essential in M. Conversely, if K is a τ-torsion submodule of M such that a submodule N of M is maximal with respect to the property $N \cap K = 0$, then N is a type 2 τ-closed submodule of M.*

Proof. If N is a type 2 τ-closed submodule of M, then N is τ-dense and closed in M. Then N is a complement in M (see [4, p. 6]). Let K be a submodule of M such that N is a maximal submodule of M such that $N \cap K = 0$.

Next we show that $N \oplus K$ is τ-essential in M. Assume that $N \oplus K$ is not essential in M. Then there is a nonzero submodule N_1 of M such that $(N \oplus K) \cap N_1 = 0$.

This gives $(N_1 \oplus N) \cap K = 0$ which is a contradiction. Hence $N \oplus K$ is essential in M. Finally $M/(N \oplus K)$ is a homomorphic image of M/N, so $M/(N \oplus K)$ is τ-torsion. Therefore $N \oplus K$ is τ-essential in M.

Conversely, suppose that there exists a submodule K of M with the property that K is τ-torsion and that N is maximal with respect to the property $N \cap K = 0$. We claim that N is type 2 τ-closed in M. It is clear that N is closed in M, so we need to show that N is τ-dense in M. Since K and $M/(N \oplus K)$ are τ-torsion, by exact sequence

$$0 \to K \cong (K \oplus N)/N \to M/N \to M/(K \oplus N) \to 0,$$

shows that M/N is τ-torsion. Hence N is type 2 τ-closed in M. □

It was our hope that an investigation of the decomposition of type 2 τ-extending modules would not only provide information about such decompositions but that it would indirectly shed light on Question 2: When is a finite direct sum of type 2 τ-extending modules type 2 τ-extending? Unfortunately this does not seem to be the case, so this question is yet to be resolved. In this vein we discuss the following example in the hope to shed light for further study toward the question.

Example 2.12. There exists a ring R, a torsion theory τ and type 2 τ-extending modules M_1, M_2 such that $M = M_1 \oplus M_2$ is a τ-torsion module but need not be type 2 τ-extending.

Proof. Let p be a prime integer and consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}_p$ and $M_2 = \mathbb{Z}_{p^3}$ and $M = M_1 \oplus M_2$. It is well known that M is not extending (see namely [9]). Let $\tau_p = \tau := (\mathscr{T}_p, \mathscr{F}_p)$ denote the torsion theory on Mod-$\mathbb{Z}$ where

$$\mathscr{T}_p := \{K \in \text{Mod-}\mathbb{Z};\ \text{for each } k \in K \text{ there exists a positive integer } t \text{ depending on } k \text{ with } kp^t = 0\}.$$

Since M_1 and M_2 are uniform, they are extending, in particular they are type 2 τ_p-extending modules. Clearly M is also a τ_p-torsion module and since it is not extending, it is not type 2 τ_p-extending. In order to see this directly, let $N = (\bar{1}, \bar{p})\mathbb{Z}$. Then it is easy to check that N is type 2 τ_p-closed submodule of M but not a direct summand. Hence M is not a type 2 τ_p-extending $\mathbb{Z}$-module. □

Let U and M both be R-modules. We say that U is M-*injective* if, for every submodule N of M, every homomorphism $\varphi\colon N \to U$ can be extended to a homomorphism $\psi\colon M \to U$ such that $\psi(x) = \varphi(x)$, for all $x \in N$. A class of R-modules $\{M_i\colon i \in I\}$, where I is an index set, is called *relatively injective* if M_i is M_j-injective for every pair of distinct $i, j \in I$.

We mention Theorem 2.13 relating to Question 2.

Theorem 2.13. *Let τ be a hereditary torsion theory and let M be an R-module which is a direct sum $M = M_1 \oplus M_2$ of two relatively injective τ-torsion submodules M_1 and M_2. Then M is type 2 τ-extending if and only if both M_1 and M_2 are type 2 τ-extending.*

Proof. The necessity is clear by Lemma 2.1 part (6). For the sufficiency let M_1 and M_2 be relatively injective τ-torsion type 2 τ-extending submodules. It is easy to see that M_1 and M_2 are both extending submodules. By [9, Theorem 8] M is extending. Therefore by Lemma 2.1, M is type 2 τ-extending. □

3. Type 2 τ_G-extending modules

In this section we investigate extending modules relative to Goldie torsion theory τ_G on Mod-R. Recall that the singular submodule of an R-module M is given by $Z(M) := \{x \in M;\ xE = 0, E$ an essential right ideal of $R\}$ and that M is τ_G-torsion free if and only if $Z(M) = 0$.

We show that if τ is a torsion theory such that $\tau_G \leqslant \tau$, then a singular module M is type 2 τ-extending if and only if it is type 2 τ_G-extending. We also investigate Question 3: Is there an example of a type 2 τ_G-extending module which is not extending? We begin with the following proposition.

Proposition 3.1. *If M is type 2 τ_G-extending module, then $\tau_G(M)$ is a direct summand of M.*

Proof. Let M be a type 2 τ_G-extending module. If M is τ_G-torsion, then by Lemma 2.1 part (4), M is extending. So suppose that M is not τ_G-torsion. If K is a complement of $\tau_G(M)$ in M, then $K \oplus \tau_G(M)$ is an essential submodule of M, and by [8, Proposition 3.26] $M/(K \oplus \tau_G(M))$ is τ_G-torsion. Since $\tau_G(M) \cong (K \oplus \tau_G(M))/K$ is τ_G-torsion, the short exact sequence

$$0 \to (K \oplus \tau_G(M))/K \to M/K \to M/(K \oplus \tau_G(M)) \to 0$$

shows that M/K is τ_G-torsion. Hence K is type 2 τ_G-closed in M. By assumption K is a direct summand of M, so $M = K \oplus K'$ for some submodule K' of M. Thus K is τ_G-torsion free (i.e., non-singular) and $\tau_G(M)$ is contained in K'. Moreover, $M/K \cong K'$, so K' is τ_G-torsion. But $\tau_G(M)$ is the largest τ_G-torsion submodule of M, so $\tau_G(M) = K'$. Therefore, we have $M = \tau_G(M) \oplus K$. □

Proposition 3.2. *Let τ and ϱ be torsion theories such that $\tau \leqslant \varrho$. If an R-module M is type 2 ϱ-extending, then M is type 2 τ-extending.*

Proof. Assume that M is a type 2 ϱ-extending R-module and that N is a type 2 τ-closed submodule of M. Then M/N is τ-torsion and N is closed in M, so since $\tau \leqslant \varrho$, M/N is ϱ-torsion and N is closed in M. Thus N is type 2 ϱ-closed in M. By assumption M is type 2 ϱ-extending, so N is a direct summand of M. Therefore M is type 2 τ-extending. □

The following example shows that converse of Proposition 3.2 does not hold.

Example 3.3. If R, I, τ_I and M are as in Example 2.4, then $\tau_I \leqslant \chi$ and M is type 2 τ_I-extending but not type 2 χ-extending.

Proof. We saw in Example 2.4 that M is a type 2 τ_I-extending module that is not extending. But for the torsion theory χ, an R-module is type 2 χ-extending if and only if it is extending (see Example 2.3). Thus, M is type 2 τ_I-extending but not type 2 χ-extending. □

Corollary 3.4. *Let τ be a torsion theory such that $\tau \leqslant \tau_G$. The following hold for an R-module M.*

(1) *If M is non-singular, then M is type 2 τ_G-extending.*
(2) *If M is type 2 τ_G-extending, then M is type 2 τ-extending.*
(3) *If M is non-singular, then M is type 2 τ-extending.*

Proof. (1) If M is a non-singular R-module, then $Z(M) = 0$, and so $\tau_G(M) = 0$. Thus M is τ_G-torsion free. By Lemma 2.1 part (5), M is type 2 τ_G-extending.

(2) By Proposition 3.2, this is clear.

(3) By part (1) and (2), this is also clear. □

The converses of Corollary 3.4 part (1) and (3) are not true in general as the following example shows.

Example 3.5. For $\mathbb{Z}$ the ring of integers, $\mathbb{Z}_{p^2}$ is a type 2 τ_G-extending $\mathbb{Z}$-module but it is not a non-singular $\mathbb{Z}$-module.

Proof. Since $\mathbb{Z}_{p^2}$ is a uniform $\mathbb{Z}$-module, we know that for the Goldie torsion theory τ_G, every uniform module is type 2 τ_G-extending. Thus $\mathbb{Z}_{p^2}$ is type 2 τ_G-extending and also type 2 τ-extending since $\mathbb{Z}_{p^2}$ is uniform. It is easy to see that $\mathbb{Z}_{p^2}$ is not a non-singular $\mathbb{Z}$-module. □

What can we say about the converse of 3.4 part (2)? Is there a torsion theory τ such that $\tau \leqslant \tau_G$ and a module which is type 2 τ-extending but not type 2 τ_G-extending? We leave this question open.

Further we ask whether there exists or not a hereditary torsion theory τ and a module M with $\tau \leqslant \tau_G$ such that M is type 2 τ-extending but not type 2 τ_G-extending.

We also have the following characterization.

Theorem 3.6. *Suppose that τ is a torsion theory such that $\tau_G \leqslant \tau$. Then a singular R-module M is type 2 τ-extending if and only if it is type 2 τ_G-extending.*

Proof. Since $\tau_G \leqslant \tau$, by Proposition 3.2, if M is type 2 τ-extending, then M is type 2 τ_G-extending.

Conversely, suppose that M is type 2 τ_G-extending. Let N be a type 2 τ-closed submodule of M. Then N is closed in M and N is τ-dense in M. Since M is a singular module, $Z(M) = M$, and so $\tau_G(M) = M$. Thus M is τ_G-torsion, and so M/N is τ_G-torsion as a homomorphic image of M. Thus N is a type 2 τ_G-closed submodule of M. By hypothesis N is a direct summand of M. Therefore M is type 2 τ-extending. □

We conclude our discussion of type 2 τ_G-extending modules with the following example of a module which is type 2 τ_G-extending but not extending.

Example 3.7. Consider the ring $R = \left\{ \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix} \right\}$, where $\mathbb{Z}$ is the ring of integers. If M is the R-module R_R, then M is a nonsingular type 2 τ_G-extending, but not extending R-module.

Proof. For $1 \leqslant i,j \leqslant 2$, let e_{ij} denote the matrix unit with 1 in the (i,j)th position and the other entries 0. Then M is not an extending module since $N = (e_{12} + e_{22}2)R$ is a closed submodule of M that is not a direct summand (see also [2, Example 6.2]). It is easy to check that M is non-singular, so M is a τ_G-torsion free. Lemma 2.1 shows that M is type 2 τ_G-extending. □

Acknowledgement. The author would like to thank referee for his/her valuable suggestions and comments. This research was supported in part by TUBITAK-NATO while the author was visiting the Department of Mathematics of the University of Nebraska-Lincoln during the academic year 2004/2005.

References

[1] *F. W. Anderson and K. R. Fuller*: Rings and categories of modules. Springer-Verlag, New York, 1974.

[2] *A. W. Chatters and C. R. Hajarnavis*: Rings in which every complement right ideal is a direct summand. Quart. J. Math. Oxford. *28* (1977), 61–80.

[3] *P. E. Bland*: Topics in torsion theory. Math. Research, Berlin, Wiley-VCH Verlag, p. 103, 1998.

[4] *N. Viet Dung, D. Van Huynh, P. F. Smith and R. Wisbauer*: Extending modules. Longman, Harlow, 1994.
[5] *S. Doğruöz and P. F. Smith*: Modules which are extending relative to module classes. Communications in Algebra *26* (1998), 1699–1721.
[6] *S. Doğruöz and P. F. Smith*: Modules which are weak extending Relative to Module Classes. Acta Math. Hungarica *87* (2000), 1–10.
[7] *S. Doğruöz*: Classes of extending modules associated with a torsion theory. East-west J. Math. (2007), to appear.
[8] *K. R. Goodearl and R. B. Warfield*: An introduction to noncommutative Noetherian rings. London Math. Society Student Texts *16* (1989).
[9] *A. Harmanci and P. F. Smith*: Finite direct sums of CS-modules. Houston J. Math. *19* (1993), 523–532.
[10] *S. G. Jonathan*: Torsion theories. Longman Scientific and Technical, 1986.
[11] *M. A. Kamal and B. J. Muller*: Extending modules over commutative domains. Osaka J. Math. *25* (1988), 531–538.
[12] *B. Stenström*: Rings of Quotients. Springer-Verlag: Berlin, 1975.
[13] *P. F. Smith, Ana M. de Viola-Prioli and Jorge E. Viola-Prioli*: Modules complemented with respect to a torsion theory. Communications in Algebra *25* (1997), 1307–1326.
[14] *L. Zhongkui*: On X-Extending and X-Continuous modules. Communications in Algebra *29* (2001), 2407–2418.

Author's address: Semra Doğruöz, Department of Mathematics, Adnan Menderes University, 09100 Aydin, Türkiye, e-mail: `sdogruoz@adu.edu.tr`.