Extending Parikh q-matrices

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ABSTRACT

The notion of extending Parikh q-matrix with respect to a word instead of an ordered alphabet is introduced. Some basic properties of this extending Parikh q-matrices have been investigated. Also it has been shown that the extending Parikh q-matrix mapping can be obtained as a composition of a Parikh q-matrix mapping and a word substitution morphism.

Keywords

Parikh q-matrix, Extending Parikh q-matrix, scattered subword, alternating Parikh q-matrix, q-counting subwords.

1. INTRODUCTION

The Parikh mapping, an important tool in the theory of formal languages, introduced by R. J. Parikh in [9] - gives the number of occurrences of letters in the word as a numerical vector. The main result of this mapping is that the image obtained by the Parikh mapping of a context free language is always a semilinear set. Over unary alphabet this mapping determines the word uniquely. But once the size of the alphabet increases it fails to determine the word uniquely, in other words the Parikh mapping is not injective. Some extensions of the Parikh vector mapping to matrices have been introduced in [6, 8, 12]. These matrices provide more information of the given word in terms of its scattered subwords. However none of these mappings determine the word uniquely. Various theoretical properties of a Parikh matrix, such as the notion of alternate Parikh matrix of a word, the relation between Parikh matrix of a word and its inverse, palindromic amiability of words have been studied in [2, 3, 8]. Since the Parikh matrix mapping is not injective, the M-ambiguity, N-ambiguity, γ-property of words have been extensively investigated in [1, 2, 3, 7, 8, 10, 11, 13].

Omer Egecioglu [5] further extended the notion of Parikh q-matrix mapping to Parikh q-matrix encoding which is indeed injective. However this encoding is not a morphism.

In [12], Serbanuta extended the Parikh matrix mapping to extending Parikh matrix mapping induced by a word instead of being defined with respect to an ordered alphabet $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$. It has similar properties as of classical Parikh matrix. This extending Parikh matrix coincides with Parikh matrix if it is defined with respect to the word $a_1a_2...a_k$.

In [4] it was shown some algebraic properties of Parikh q-matrices which was introduced in [6] and the notion of alternate q-counting scattered subword of a word to compute the inverse of a Parikh q-matrix was introduced. The Parikh q-matrix mapping [6] of a word w over an ordered alphabet Σ_k, q-counts certain scattered subwords of w of the form $a_{i,j}$, $1 \leq i \leq j \leq k$ with respect to the ordered alphabet $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$, where $a_i \neq a_j$ for all $i \neq j$. Thus Parikh q-matrix of a word w does not q-count scattered subwords of w with repeated letters and also of scattered subwords not necessarily in the order of Σ_k. To facilitate this, in this paper, the concept of q-counting of scattered subword of w with respect to a word u (which may have repeated letters) is discussed. Substituting the value $q = 1$, one can obtain the extending Parikh matrix mapping defined in [12].

The paper is structured as follows: Section 2 provides some basic notations and definitions. In section 3, the concept of extending Parikh q-matrix mapping with respect to a word is defined. Also the notion of extending q-counting scattered subword is introduced and the entries of extending Parikh q-matrix of a word induced by a word in terms of extending q-counting scattered subwords has been characterized. In section 4, the alternate extending Parikh q-matrix mapping has been defined and a relation between the extending Parikh q-matrix of a word and its alternate has been established. In section 5, it is shown that the extending Parikh q-matrix mapping can be obtained as a composition of a Parikh q-matrix mapping and a word substitution morphism. We end the paper by giving few concluding remarks.

2. PRELIMINARIES

This section presents some basic notations and definitions. The set of all non-negative integers (all integers) is denoted by $\mathbb{N}$ ($\mathbb{Z}$ respectively). $\mathbb{N}[q]$ ($\mathbb{Z}[q]$) denotes the collection of polynomials in the variable q with coefficients from $\mathbb{N}$ ($\mathbb{Z}$ respectively).
Let $\Sigma = \{a_1, a_2, ..., a_k\}$ be an alphabet. The a_i's are called the letters of the alphabet Σ. The set of all words (non-empty words) over Σ is denoted by Σ^* (Σ^+ respectively) and the empty word is denoted by λ. An ordered alphabet Σ is an alphabet Σ with a linear order " $<$ " on it. For example, $\Sigma = \{a_1, a_2, ..., a_k\}$ with the order relation $a_1 < a_2 < \cdots < a_k$, is an ordered alphabet. We denote an ordered alphabet $\{a_1 < a_2 < \cdots < a_k\}$ by Σ_k. For a word $w \in \Sigma^*$, $|w|$ denotes the length of the word w.
A word $v \in \Sigma^*$ is said to be a factor of w if there exist words u and $x \in \Sigma^*$ such that $w = uvx$. v is said to be prefix (suffix respectively) if $u = \lambda$ ($x = \lambda$ resp.).

A word $u \in \Sigma^*$ is called a (scattered) subword of w if there exist words $x_1, x_2, ..., x_n$ and $y_0, y_1, y_2, ..., y_n$ (some of them possibly empty) over Σ such that $u = x_1x_2...x_n$ and $w = y_0x_1y_1x_2y_2...x_ny_n$.
The mirror image of a word $w \in \Sigma^*$, denoted by $mi(w)$, is defined as:

(1) $mi(\lambda) = \lambda$,
(2) $mi(x_1x_2\cdots x_n) = x_n\cdots x_2x_1$, where $x_i \in \Sigma, 1 \leq i \leq n$.

A monoid $M_k[q]$ whose elements are the set of all $k \times k$ upper triangular matrices with polynomial entries from $\mathbb{N}[q]$ with respect to the usual matrix multiplication of matrices and has the identity I_k was defined in [6]. The notion of Parikh q-matrix mapping is defined as follows.

DEFINITION 1. *([6]) Let $\Sigma_k = \{a_1 < a_2 < ... < a_k\}$, be an ordered alphabet where $k \geq 1$. The Parikh q-matrix mapping, denoted by ψ_q, is the morphism:*

$$\psi_q : \Sigma_k^* \to M_k[q]$$

defined as follows:

$$\psi_q(a_l) = (m_{ij})_{1 \leqslant i,j \leqslant k}$$

where,

- $m_{ll} = q$
- $m_{ii} = 1$, *for* $1 \leq i \leq k, i \neq l$
- $m_{l(l+1)} = 1$, *if* $l < k$
- *All other entries are zero.*

Observe that the above mapping can be extended from Σ_k to Σ_k^* such that

- $\psi_q(\lambda) = I_k$
- $\psi_q(w_1w_2\cdots w_n) = \psi_q(w_1)\psi_q(w_2)\cdots\psi_q(w_n)$, for $w_i \in \Sigma_k$.

EXAMPLE 1. *Consider $\Sigma_4 = \{a < b < c < d\}$ and $w = aab$. Then*

$$\begin{aligned}\psi_q(aab) &= \psi_q(a).\psi_q(a).\psi_q(b) \\ &= \begin{pmatrix}q&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}q&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&q&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \\ &= \begin{pmatrix}q^2&q+q^2&0&0\\0&q&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\end{aligned}$$

3. EXTENDING PARIKH Q-MATRIX MAPPING

Let us define the q-analogue of extending Parikh q-matrix mapping as follows.

DEFINITION 2. *Let Σ_k be an ordered alphabet and $u = b_1b_2\cdots b_{|u|}$, $b_i \in \Sigma_k$ for all $1 \leq i \leq |u|$, be a word over Σ_k. The Parikh q-matrix mapping induced by the word $u \in \Sigma_k^*$, denoted by ψ_q^u, is the morphism*

$$\psi_q^u : (\Sigma_k^*, \cdot, \lambda) \to (M_{|u|+1}[q], \cdot, I_{|u|+1})$$

defined by: if $a \in \Sigma_k$ and $\psi_q^u(a) = (m_{ij})_{1\leq i,j\leq |u|+1}$, then

- $m_{ij} = 1$, *if $i = j$ and zero otherwise.*
- *If $\delta_{b_l,a} = 1$, $1 \leq l \leq |u|$ then update the entries $m_{ll} = q$ and $m_{l(l+1)} = 1$.*

It is clear that if $a \in \Sigma_k$ and $|u|_a = 0$, then $\psi_q^u(a) = I_{|u|+1}$. One can easily verify that the Parikh q-matrix mapping is a special case of this extending Parikh q-matrix mapping when $u = a_1a_2\cdots a_k$. Let us see an example.

EXAMPLE 2. *Consider $\Sigma_2 = \{a < b\}$ and $u = baa$. Then*

$$\psi_q^u(a) = \begin{pmatrix}1&0&0&0\\0&q&1&0\\0&0&q&1\\0&0&0&1\end{pmatrix} \text{ and } \psi_q^u(b) = \begin{pmatrix}q&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and for a word $w = aab$,

$$\begin{aligned}\psi_q^u(aab) &= \psi_q^u(a).\psi_q^u(a).\psi_q^u(b) \\ &= \begin{pmatrix}1&0&0&0\\0&q&1&0\\0&0&q&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&q&1&0\\0&0&q&1\\0&0&0&1\end{pmatrix}\begin{pmatrix}q&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \\ &= \begin{pmatrix}q&1&0&0\\0&q^2&2q&1\\0&0&q^2&q+1\\0&0&0&1\end{pmatrix}\end{aligned}$$

Given a word $u = b_1b_2\cdots b_{|u|}$, we use the notation $u_{i,j}$ for the word $b_ib_{i+1}\cdots b_j$, where $1 \leq i \leq j \leq |u|$. Let $u_{i,j}$ be a scattered subword of a word w. Then the q-counting of a scattered subword $u_{i,j}$ of a word w with respect to a word $u = b_1b_2\cdots b_{|u|}$, $b_i \in \Sigma_k$ represented by $S^u_{w,u_{i,j}}(q)$ is a polynomial in q and is defined as follows.

DEFINITION 3. **(Extending q-counting scattered subwords)** *Let $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$, be an ordered alphabet where $k \geq 1$, $w \in \Sigma_k^*$ and $u_{i,j}$, where $1 \leq i \leq j \leq |u|$, be a scattered subword of w. Then, the q-counting of a scattered subword $u_{i,j}$ of a word w with respect to a word u, denoted by $S^u_{w,u_{i,j}}(q)$ is defined by*

$$S^u_{w,u_{i,j}}(q) = \sum_{w=v_ib_iv_{i+1}\cdots v_jb_jv_{j+1}} q^{\sum_{t=i}^{j+1}|v_t|_{b_t}}$$

EXAMPLE 3. *Consider the alphabet $\Sigma_3 = \{a < b < c\}$, $u = baa$ and let $w = ab^3ab$. Then, to compute the q-counting of ba of w with respect to the word $u = baa$, we consider the possible factorizations:*

$$(a)b(b^2)a(b),\ (ab)b(b)a(b) \text{ and } (ab^2)b(\lambda)a(b)$$

Thus we have

$$S^u_{ab^3ab,ba}(q) = 1 + q + q^2. \tag{1}$$

To compute the q-counting of aa of w with respect to the word $u = baa$ the only possible factorization is

$$(\lambda)a(b^3)a(b)$$

Therefore

$$S^u_{ab^3ab,aa}(q) = q^{|\lambda|_a + |b^3|_a + |b|_c} \tag{2}$$

Now substituting the value $q = 1$ in the above two equations (1) and (2) we get $S^u_{ab^3ab,ba}(1) = 3 = |ab^3ab|_{ba}$ and $S^u_{ab^3ab,aa}(1) = 1 = |ab^3ab|_{aa}$.

The entries of extending Parikh matrix of a word w with respect to a word u was given in [12] as follows.

THEOREM 4. *([12]) Consider* $u = b_1 b_2 \cdots b_{|u|} \in \Sigma_k^*$, $b_i \in \Sigma_k$, *for all* $1 \le i \le |u|$ *and* $w \in \Sigma_k^*$. *Then the extending Parikh matrix* $\psi_{\Sigma_k,u}(w) = (m_{ij}(q))_{1\le i,j\le |u|+1}$ *has the following properties:*

(1) $m_{ij} = 0$, *for all* $1 \le j < i \le |u|+1$,

(2) $m_{ii} = 1$, *for all* $1 \le i \le |u|+1$,

(3) $m_{i(j+1)} = |w|_{u_{i,j}}$, *for all* $1 \le i \le j \le |u|$.

Similarly the entries of extending Parikh q-matrix mapping of a word w induced by u can be expressed in terms of its extending q-counting scattered subword as follows.

THEOREM 5. *Let* $\Sigma_k, k \ge 1$ *be an ordered alphabet and assume that* $w \in \Sigma_k^*$ *and* $u = b_1 b_2 \cdots b_{|u|}$. *The matrix* $\psi_q^u(w) = (m_{ij}(q))_{1\le i,j\le |u|+1}$ *has the following properties.*

(1) $m_{ij} = 0$, *for all* $1 \le j < i \le |u|+1$,

(2) $m_{ii} = q^{|w|_{b_i}}$, *for all* $1 \le i \le |u|+1$,

(3) $m_{i(j+1)} = S^u_{w,u_{i,j}}(q)$, *for all* $1 \le i \le j \le |u|$.

PROOF. The proof is done by induction on $|w| = n$. If $|w| = 1$, then $w = a$, for some $a \in \Sigma_k$. And from the definition of extending Parikh q-matrix mapping the result holds. Suppose the result holds for every word of length less than or equal to n. Let w be of length $n+1$ and $w = w'a$, where $|w'| = n$ and $a \in \Sigma_k$. Then we have

$$\psi_q^u(w) = \psi_q^u(w')\psi_q^u(a).$$

Let,

$$\psi_q^u(w') = (m'_{ij}(q))_{1\le i,j\le |u|+1},$$

$$\psi_q^u(a) = (n_{ij}(q))_{1\le i,j\le |u|+1},$$

and

$$\psi_q^u(w) = (m_{ij}(q))_{1\le i,j\le |u|+1}.$$

Then we have

$$\begin{aligned} m_{i(j+1)} &= \sum_{l=1}^{|u|+1} m'_{il} n_{l(j+1)} \\ &= m'_{ij} n_{j(j+1)} + m'_{i(j+1)} n_{(j+1)(j+1)} \\ &= m'_{ij}.\delta_{b_j,a} + m'_{i(j+1)} \end{aligned}$$

By induction hypothesis we have,

$$m'_{i(j+1)} = S^u_{w,u_{i,j)}}(q)$$

Therefore,

$$m_{i(j+1)} = S^u_{w,u_{i,j}}(q) + \delta_{b_j,a}.S^u_{w,u_{i,(j-1)}}(q)$$

which is nothing but $S^u_{w,u_{i,j}}(q)$, for all $1 \le i \le j \le |u|$.
Thus $m_{i(j+1)} = S^u_{w,u_{i,j}}(q)$, for all $1 \le i \le j \le |u|$. □

EXAMPLE 4. *Consider the alphabet* $\Sigma_3 = \{a < b < c\}$, $u = baa$ *and let* $w = ab^3ab$. *Then, the Parikh q-matrix of w with respect to the word* $u = baa$ *is*

$$\begin{aligned} \psi_q^u(ab^3ab) &= \begin{pmatrix} q^{|w|_b} & S^u_{w,b}(q) & S^u_{w,ba}(q) & S^u_{w,baa}(q) \\ 0 & q^{|w|_a} & S^u_{w,a}(q) & S^u_{w,aa}(q) \\ 0 & 0 & q^{|w|_a} & S^u_{w,a}(q) \\ 0 & 0 & 0 & q^{|w|_c} \end{pmatrix} \\ &= \begin{pmatrix} q^4 & (q+q^2+2q^3) & (1+q+q^2) & 0 \\ 0 & q^2 & 2q & 1 \\ 0 & 0 & q^2 & 1+q \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

4. ALTERNATE EXTENDING PARIKH Q-MATRIX MAPPING

In [8] the notion of alternating Parikh matrix was defined and the authors have shown that the inverse of the Parikh matrix of a word w can be computed directly from the alternate Parikh matrix of the mirror image of the word w. Later in [6] the authors extended this property for Parikh q-matrix. They define the notion of alternate Parikh q-matrix and produce similar result with some modification. Now we recall the notion of alternate Parikh q-matrix and the result concerning the Parikh q-matrix and its alternate Parikh q-matrix below.

DEFINITION 6. *([6]) Let* $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ *be an alphabet and* $w \in \Sigma_k^*$. *The alternate Parikh q-matrix mapping* $\overline{\psi}_q$ *is a morphism from* Σ_k^* *to a collection of* k*-dimensional upper triangular matrices over* $\mathbb{Z}[q]$

$$\overline{\psi}_q : \Sigma_k^* \to M_k[q]$$

defined as follows:

$$\overline{\psi}_q(a_l) = (m_{ij})_{1\le i,j\le k}$$

such that

- $m_{ll} = 1$
- $m_{ii} = q, i \ne l$
- $m_{l(l+1)} = -1$ *if* $l < k$
- *all other entries are zero.*

Consider the alphabet $\Sigma_3 = \{a < b < c\}$. Then,

$$\overline{\psi}_q(a) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}, \overline{\psi}_q(b) = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & q \end{pmatrix},$$
$$\overline{\psi}_q(c) = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\overline{\psi}_q$ is a morphism,

$$\begin{aligned} \overline{\psi}_q(babba) &= \overline{\psi}_q(b)\overline{\psi}_q(a)\overline{\psi}_q(b)\overline{\psi}_q(b)\overline{\psi}_q(a) \\ &= \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & q \end{pmatrix} \\ &\quad \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix} \\ &= \begin{pmatrix} q^3 & -(q^2+q^3) & q^2+q^3 \\ 0 & q^2 & -(q^2+q^3+q^4) \\ 0 & 0 & q^5 \end{pmatrix} \end{aligned}$$

Now we extend the concept of alternate Parikh q-matrix with respect to a word u as follows.

DEFINITION 7. *Let $\Sigma_k, k \geq 1$ be an ordered alphabet and assume that $w \in \Sigma_k^*$ and $u = b_1 b_2 \cdots b_{|u|}$ with $b_i \neq b_{i+1}$ for all $1 \leq i < |u|$. Then the alternate Parikh q-matrix mapping with respect to the word u is a morphism*

$$\overline{\psi_q^u} : \Sigma_k^* \longleftrightarrow M_{|u|+1}[q]$$

defined by the following conditions:
Let $1 \leq i_1 < i_2 < \cdots i_t \leq |u|$ be such that $\delta_{b_{i_s},a} = 1$, for all s such that $1 \leq s \leq t$. If $\overline{\psi_q^u}(a) = (m'_{ij}(q))_{1 \leq i,j \leq |u|+1}$, then

$$m'_{ij} = \begin{cases} 0 & j < i, \\ q & j = i \neq i_s\,, 1 \leq s \leq t, \\ 1 & j = i = i_s\,, 1 \leq s \leq t, \\ -1 & j = i_s + 1\,, i = i_s\,, 1 \leq s \leq t, \\ 0 & otherwise. \end{cases}$$

EXAMPLE 5. *Consider the alphabet $\Sigma_3 = \{a < b < c\}$, $u = baa$ and let $w = ab^3ab$. Then*

$$\overline{\psi_q^u}(a) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and

$$\overline{\psi_q^u}(b) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

and for the word w,

$$\overline{\psi_q^u}(ab^3ab) = \overline{\psi_q^u}(a)\overline{\psi_q^u}(b)\overline{\psi_q^u}(b)\overline{\psi_q^u}(b)\overline{\psi_q^u}(a)\overline{\psi_q^u}(b)$$

$$= \begin{pmatrix} q^2 & -(2q^2+q^3+q^4) & (q^2+q^3+q^4) & 0 \\ 0 & q^4 & -2q^4 & q^4 \\ 0 & 0 & q^4 & -(q^4+q^5) \\ 0 & 0 & 0 & q^6 \end{pmatrix}$$

DEFINITION 8. *Let $\Sigma_{k+1} = \{a_1 < a_2 < \cdots < a_k < a_{k+1}\}$ be an alphabet and $u, w \in \{a_1 < a_2 < \cdots < a_k\}^*$, $u = b_1 b_2 \cdots b_{|u|}$. The alternating q-counting scattered subword $b_{i,j}$ of a word w with respect to the word u denoted by $S'^u_{w,b_{i,j}}(q)$ is defined by*

$$S'^u_{w,b_{i,j}}(q) = (-1)^{i+j+1} \sum_{w = v_i b_i \cdots v_j b_j v_{j+1}} q^{\sum_{t=i}^{j+1} (|v_t| - |v_t|_{b_t})},$$
$$1 \leq i \leq j \leq |u|.$$

THEOREM 9. *Let $\Sigma_{k+1} = \{a_1 < a_2 < \cdots < a_k < a_{k+1}\}$ be an alphabet and $u, w \in \{a_1 < a_2 < \cdots < a_k\}^*$, $u = b_1 b_2 \cdots b_{|u|}$. The extending alternate Parikh q-matrix $\overline{\psi_q^u}(w) = (m_{ij})_{1 \leq i,j \leq |u|+1}$ with respect to the word u has the following properties:*

(1) $m_{ij} = 0, 1 \leq j < i \leq |u| + 1$
(2) $m_{ii} = q^{|w| - |w|_{u_i}}$, $1 \leq i \leq |u| + 1$
(3) $m_{i(l+1)} = S'^u_{w,u_{i,l}}(q)$, *where* $1 \leq i \leq l \leq |u|$.

PROOF. We prove by induction on $|w| = n$. If $n = 0$, that is $w = \lambda$, the proof is obvious, since $\overline{\psi_q^u}(w) = I_{|u|+1}$. For $n = 1$, that is $w = a \in \Sigma_k$ it follows from the definition.
Let us assume that the properties are true for all words of length atmost n and let w be of length $n + 1$. Hence $w = w'a$, where $|w'| = n$ and $a \in \Sigma_k$. We have

$$\overline{\psi_q^u}(w) = \overline{\psi_q^u}(w').\overline{\psi_q^u}(a).$$

Let

$$\overline{\psi_q^u}(w) = (m'_{ij})_{1 \leq i,j \leq |u|+1}\,, \overline{\psi_q^u}(a) = (n_{ij})_{1 \leq i,j \leq |u|+1}$$

and

$$\overline{\psi_q^u}(w) = (m_{ij})_{1 \leq i,j \leq |u|+1}$$

Since the product of two upper triangular matrices is an upper triangular matrix we have $m_{ij} = 0$, for $i > j$.
Now we have $m_{ij} = \sum_{p=1}^{|u|+1} m'_{ip} n_{pj}$.
When $i = j$, $m_{ii} = \sum_{p=1}^{|u|+1} m'_{ip} n_{pi} = m'_{ii} n_{ii}$.
Let $1 \leq i_1 < i_2 < \cdots i_t \leq |u|$ be such that $\delta_{b_{i_s},a} = 1$, for $1 \leq s \leq t$.
If $i \neq i_s, 1 \leq s \leq t$, then $m'_{ii} = q^{|w'| - |w'|_{u_i}}$, by induction hypothesis and $n_{ii} = q$. Therefore

$$m_{ii} = q^{|w'| - |w'|_{u_i} + 1} = q^{|w'a| - |w'|_{u_i}} = q^{|w| - |w'a|_{u_i}} = q^{|w| - |w|_{u_i}}$$

If $i = i_s, 1 \leq s \leq t$, then $m'_{ii} = q^{|w'| - |w'|_{u_i}}$, by induction hypothesis and $n_{ii} = 1$. Therefore

$$m_{ii} = q^{|w'| - |w'|_{u_i}} = q^{|w'a| - |w'a|_{u_i}} = q^{|w| - |w|_{u_i}}.$$

To prove $m_{i(l+1)} = S'^u_{w,u_{i,l}}(q)$, where $1 \leq i \leq l \leq |u|$, we see that

$$m_{i(l+1)} = \sum_{p=1}^{|u|+1} m'_{ip} n_{p(l+1)} = m'_{il} n_{l(l+1)} + m'_{i(l+1)} n_{(l+1)(l+1)}$$

(1) If $b_l \neq a$, $n_{l(l+1)} = 0$ and we have

$$m_{i(l+1)} = m'_{i(l+1)} n_{(l+1)(l+1)}.$$

There may arise two cases:
(a) If $b_{l+1} = a$, then $n_{(l+1)(l+1)} = 1$ and

$$m_{i(l+1)} = m'_{i(l+1)} = S'^u_{w',u_{i,l}}(q) = S'^u_{w,u_{i,l}}(q)$$

(b) If $b_{l+1} \neq a$, then $n_{(l+1)(l+1)} = q$ and

$$m_{i(l+1)} = q.m'_{i(l+1)} = qS'^u_{w',u_{i,l}}(q) = S'^u_{w,u_{i,l}}(q)$$

(2) If $b_l = a$, then $n_{l(l+1)} = -1$ and $n_{(l+1)(l+1)} = q$ and we have

$$m_{i(l+1)} = -m'_{il} + q.m'_{i(l+1)} \tag{3}$$

By induction hypothesis we have

$$m'_{il} = S'^u_{w',u_{i,l-1}}(q) \text{ and } m'_{i(l+1)} = S'^u_{w',u_{i,l}}(q)$$

We see that there are two ways to choose the scattered subword $u_{i,l} = b_{i,l}$ from $w = w'b_l$. First one is to choose the subword $b_{i,l-1}$ from w' and the last letter b_l and the second one is choose the subword $b_{i,l}$ from w' itself. In the first case,

$$
\begin{aligned}
-m'_{il} &= -S'^{u}_{w',b_{i,l-1}}(q) \\
&= (-1).(-1)^{i+l-1+1} \sum_{w'=v_i b_i \cdots v_{l-1} b_{l-1} v_l} q^{\sum_{t=i}^{l}(|v_t|-|v_t|_{b_t})} \\
&= (-1)^{i+l+1} \sum_{w=v_i b_i \cdots v_{l-1} b_{l-1} v_l b_l \lambda} q^{\sum_{t=i}^{l+1}(|v_t|-|v_t|_{b_t})}
\end{aligned}
$$

and in the second case,

$$
\begin{aligned}
qm'_{i(l+1)} &= q.S'^{u}_{w',b_{i,l}}(q) \\
&= (-1)^{i+l+1} q \sum_{w'=v_i b_i \cdots v_l b_l v_{l+1}} q^{\sum_{t=i}^{j+1}(|u_t|-|u_t|_{b_t})} \\
&= (-1)^{i+l+1} \sum_{w'=v_i b_i \cdots v_l b_l v_{l+1}} q^{\sum_{t=i}^{l+1}(|v_t|-|v_t|_{b_t})+1} \\
&= (-1)^{i+l+1} \sum_{w=v'_i b_i \cdots v'_l b_l v'_{l+1}} q^{\sum_{t=i}^{l+1}(|v'_t|-|v'_t|_{b_t})}
\end{aligned}
$$

where $v'_i = v_i$ for all i except $i = l+1$ and $v'_{l+1} = v_{l+1} b_l$.
From (3) we have

$$
\begin{aligned}
m_{i(l+1)} &= -m'_{il} + q.m'_{i(l+1)} \\
&= -S'^{u}_{w',b_{i,l-1}}(q) + q.S'^{u}_{w',b_{i,l}}(q) \\
&= S'^{u}_{w,b_{i,l}}(q)
\end{aligned}
$$

and the proof follows from induction.

□

The next result shows the relation between the extending Parikh q-matrix mapping of a given word and its alternate.

THEOREM 10. *Let $\Sigma_k, k \geq 1$ be an ordered alphabet and assume that $w \in \Sigma_k^*$ and $u = b_1 b_2 \cdots b_{|u|}$ with $b_i \neq b_{i+1}$ for all $1 \leq i < |u|$. Then*

$$\psi_q^u(w).\overline{\psi_q^u}(mi(w)) = q^{|w|} I_{|u|+1}.$$

PROOF. We prove it by induction on the length of the word w. If $|w| = 1$, then $w = a$, for some $a \in \Sigma_k$. We also have $mi(w) = w$. We will show that

$$\psi_q^u(a).\overline{\psi_q^u}(a) = qI_{|u|+1}.$$

Let $1 \leq i_1 < i_2 < \cdots i_t \leq |u|$ be such that $\delta_{b_{i_s},a} = 1$, for all s such that $1 \leq s \leq t$. Then if $\psi_q^u(a) = (m_{ij}(q))_{1 \leq i,j \leq |u|+1}$ and $\overline{\psi_q^u}(a) = (m'_{ij}(q))_{1 \leq i,j \leq |u|+1}$, we have

$$
m_{ij} = \begin{cases}
0 & : j < i, \\
1 & : j = i \neq i_s \text{ , } 1 \leq s \leq t, \\
q & : j = i = i_s \text{ , } 1 \leq s \leq t, \\
1 & : j = i_s + 1 \text{ , } i = i_s \text{ , } 1 \leq s \leq t, \\
0 & : otherwise.
\end{cases}
$$

and

$$
m'_{ij} = \begin{cases}
0 & : j < i, \\
q & : j = i \neq i_s \text{ , } 1 \leq s \leq t, \\
1 & : j = i = i_s \text{ , } 1 \leq s \leq t, \\
-1 & : j = i_s + 1 \text{ , } i = i_s \text{ , } 1 \leq s \leq t, \\
0 & : otherwise.
\end{cases}
$$

Let $\psi_q^u(a).\overline{\psi_q^u}(a) = (n_{ij}(q))_{1 \leq i,j \leq |u|+1}$. Then we have,

$$n_{ij} = \sum_{l=1}^{|u|+1} m_{il} m'_{lj}.$$

When $j = i$, $n_{ii} = m_{ii}.m'_{ii} = q$, for $1 \leq i \leq |u|+1$.
Since the product of two upper triangular matrices is upper triangular,

$$n_{ij} = 0, \text{ for } i > j.$$

When $j > i+1$, $n_{ij} = m_{ii} m'_{ij} + m_{i(i+1)} m'_{(i+1)j} = 0$, since $m'_{ij} = 0$ and $m'_{(i+1)j} = 0$. When $j = i+1$,

$$n_{i(i+1)} = m_{ii} m'_{i(i+1)} + m_{i(i+1)} m'_{(i+1)(i+1)}.$$

Now we have two cases either $m_{ii} = q$ or $m_{ii} = 1$ depending on $\delta_{b_i,a} = 1$.

(1) If $m_{ii} = q$, then $m_{i(i+1)} = 1$ and since u does not have consecutive equal letters, $m_{(i+1)(i+1)} = 1$, therefore $m'_{(i+1)(i+1)} = q$ and $m'_{i(i+1)} = -1$. Thus

$$n_{i(i+1)} = q.(-1) + 1.q = 0.$$

(2) If $m_{ii} = 1$, then $m_{i(i+1)} = 0$ and $m'_{i(i+1)} = 0$. Therefore $n_{i(i+1)} = 0$.

Thus we have shown that

$$
n_{ij} = \begin{cases}
q & \text{if } i = j, \\
0 & otherwise.
\end{cases}
$$

Hence

$$\psi_q^u(a).\overline{\psi_q^u}(a) = qI_{|u|+1}.$$

If $|w| > 1$, then $w = w'a$ with $|w'| = |w| - 1$ and

$$
\begin{aligned}
\psi_q^u(w).\overline{\psi_q^u}(mi(w)) &= \psi_q^u(w'a).\overline{\psi_q^u}(a.mi(w')) \\
&= \psi_q^u(w').qI_{|u|+1}.\overline{\psi_q^u}(mi(w')) \\
&= q.\psi_q^u(w').\overline{\psi_q^u}(mi(w')) \\
&= q.q^{|w'|} I_{|u|+1}, \text{ by induction hypothesis.} \\
&= q^{|w|} I_{|u|+1}
\end{aligned}
$$

and the proof follows. □

If u has consecutive equal letters, then the above result does not hold. For example, consider $\Sigma_2 = \{a < b\}$, $u = baa$ and $w = a$. Then we have $|u| + 1 = 4$ and

$$
\psi_q^{baa}(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & q & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \overline{\psi_q^{baa}}(a) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & q \end{pmatrix}
$$

$$\psi_q^{baa}(a).\overline{\psi_q^{aa}}(a) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 1-q & -1 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \neq q.I_4$$

However, if u does not have consecutive equal letters, then the result holds.

5. REDUCTION TO PARIKH Q-MATRIX

In this section we show that the Parikh q-matrix mapping induced by a word is a composition of Parikh q-matrix mapping and a word substitution morphism. Using this it can be easily proved that the value of each minor of an arbitrary Parikh q-matrix is a non negative integer for non negative q.

DEFINITION 11. *([12]) Let $\Sigma_k, k \geq 1$ be an ordered alphabet and $u = b_1 b_2 \cdots b_{|u|}$ with $b_i \in \Sigma_k$ for all $1 \leq i \leq |u|$. Also let $\Sigma'_{|u|} = \{c_1 < c_2 < \cdots < c_{|u|}\}$ be an ordered alphabet. For each $a \in \Sigma_k$ the corresponding word for a induced by u in $\Sigma'_{|u|}$ is the word $s(a) = c_{i_1} c_{i_2} \cdots c_{i_t}$, where $t = |u|_a$ and $1 \leq c_{i_1} < c_{i_2} < \cdots < c_{i_t} \leq |u|$ such that $b_{i_s} = a$ for all s such that $1 \leq s \leq t$. The $\Sigma_k, u, \Sigma'_{|u|}$ substitution morphism is the monoid morphism*

$$\sigma_{\Sigma_k, u, \Sigma'_{|u|}} : (\Sigma_k^*, \cdot, \lambda) \to (\Sigma'^*_{|u|}, \cdot, \lambda)$$

defined by

$$\text{for all } a \in \Sigma_k \text{ , } \sigma_{\Sigma_k, u, \Sigma'_{|u|}}(a) = mi(s(a)),$$

where $s(a)$ is the corresponding word for a induced by u in $\Sigma'_{|u|}$.

For example,

EXAMPLE 6. *Let $\Sigma_3 = \{a < b < c\}$ and $u = abaa$ and $\Sigma_4 = \{d < e < f < g\}$. Then $s(a) = dfg$, $s(b) = e$, $s(c) = \lambda$ and therefore $\sigma(a) = gfd$, $\sigma(b) = e$ and $\sigma(c) = \lambda$. And for a word $w = abacab \in \Sigma_3^*$,*

$$\begin{aligned} \sigma(w) &= \sigma(a)\sigma(b)\sigma(a)\sigma(c)\sigma(a)\sigma(b) \\ &= gfd.e.gfd.\lambda.gfd.e = gfdegfdgfde \end{aligned}$$

Now we have the theorem as follows.

THEOREM 12. *Let $\Sigma_k, k \geq 1$ be an ordered alphabet and $u = b_1 b_2 \cdots b_{|u|}$ with $b_i \in \Sigma_k$ for all $1 \leq i \leq |u|$. Also let $\Sigma'_{|u|} = \{c_1 < c_2 < \cdots < c_{|u|}\}$ be an ordered alphabet. Then*

$$\psi_q^u = \psi_q^{\Sigma'} \circ \sigma_{\Sigma_k, u, \Sigma'_{|u|}}$$

where $\psi_q^{\Sigma'_{|u|}}$ is the Parikh q-matrix mapping over the ordered alphabet $\Sigma'_{|u|}$.

PROOF. It is enough to show for letters in Σ_k. Let $a \in \Sigma_k$. We have to show that

$$\psi_q^u(a) = \psi_q^{\Sigma'_{|u|}}(\sigma(a)).$$

Let $1 \leq i_1 < i_2 < \cdots i_t \leq |u|$ be such that $\delta_{b_{i_s}, a} = 1$, for $1 \leq s \leq t$ and $\psi_q^u(a) = (m_{ij}(q))_{1 \leq i,j \leq |u|+1}$. Then we have

$$m_{ij} = \begin{cases} 0 & : j < i, \\ 1 & : j = i \neq i_s \text{ , } 1 \leq s \leq t, \\ q & : j = i = i_s \text{ , } 1 \leq s \leq t, \\ 1 & : j = i_s + 1 \text{ , } i = i_s \text{ , } 1 \leq s \leq t, \\ 0 & : otherwise. \end{cases}$$

Now $\sigma(a) = c_{i_t} c_{i_{t-1}} \cdots c_{i_2} c_{i_1}$. Let $\psi_q^{\Sigma'_{|u|}}(\sigma(a)) = (n_{ij}(q))_{1 \leq i,j \leq |u|+1}$, then using Theorem 5 we have

$$n_{ij} = \begin{cases} 0 & : j < i, \\ 1 & : j = i \neq i_s \text{ , } 1 \leq s \leq t, \\ q & : j = i = i_s \text{ , } 1 \leq s \leq t, \\ 1 & : j = i_s + 1 \text{ , } i = i_s \text{ , } 1 \leq s \leq t, \\ 0 & : \text{otherwise, since there is no scattered} \\ & \quad \text{subword } c_{i,j} \text{ in } \sigma(a) \text{ where } j > i. \end{cases}$$

Therefore $m_{ij} = n_{ij}$, for all i, j. i.e.

$$\psi_q^u(a) = \psi_q^{\Sigma'_{|u|}}(\sigma(a)).$$

Hence the theorem follows. □

We have shown in [4] that the minor of an arbitrary Parikh q-matrix is a polynomial in q with non negative co-efficients. From Theorem 12 it follows that the algebraic properties of Parikh q-matrix hold for extending Parikh q-matrices with respect to a word u. Therefore we have the following corollary. The value of each minor of an arbitrary extending Parikh q-matrix with respect to a word is non negative for non negative q.

6. CONCLUSION

In this paper the notion of Parikh q-matrix with respect to a word has been introduced. Some basic properties of this extending matrix have been investigated. Also the alternate extending Parikh q-matrix of a word is defined to find the inverse of extending Parikh q-matrix of the word. It has been shown that this extending Parikh q-matrix mapping can be obtained from Parikh q-matrix mapping and a word substitution morphism.

We know that Parikh q-matrix mapping is not injective, a similar analogue can be drawn for this extending Parikh q-matrix mapping. i.e. one can study the ambiguity of this extending Parikh matrix mapping And the properties of these kind of ambiguous words with respect to a given word in future. It will be quite interesting to study commutativity of this mapping, in other words, given a word u, when the Parikh q-matrices of two words with respect to u commute. Also one can study the behavior of this mapping under morphisms like Istrail, Fibonacci and Tribonacci etc.

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