Extending perfect matchings to Hamiltonian cycles in line graphs

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Abstract

A graph admitting a perfect matching has the Perfect-Matching-Hamiltonian property (for short the PMH-property) if each of its perfect matchings can be extended to a Hamiltonian cycle. In this paper we establish some sufficient conditions for a graph G in order to guarantee that its line graph $L(G)$ has the PMH-property. In particular, we prove that this happens when G is (i) a Hamiltonian graph with maximum degree at most 3, (ii) a complete graph, or (iii) an arbitrarily traceable graph. Further related questions and open problems are proposed along the paper.

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1 Introduction

The main property studied in this paper is related to two of the most studied concepts in graph theory: perfect matchings and Hamiltonian cycles. Let us recall that a *perfect matching* of a graph G is a set of independent edges of G that covers all the vertices in G, and a *Hamiltonian cycle* is a cycle passing through all vertices of G. If such a cycle exists then G is said to be *Hamiltonian*.

The *complete graph* on n vertices, denoted by K_n, is the graph in which every two vertices are adjacent. For any graph G, K_G denotes the complete graph on the same vertex set $V(G)$ of G. Let G be of even order. A perfect matching of K_G is said to be a *pairing* of G. In [2], the authors say that a graph G has the *Pairing-Hamiltonian property* (for short the PH-property) if every pairing M of G can be extended to a Hamiltonian cycle H of K_G in which $E(H)-M \subseteq E(G)$, where $E(H)$ is the set of edges of H. Amongst other results, the authors show that the only cubic graphs having the PH-property are K_4, the complete bipartite graph $K_{3,3}$ and the 3-cube. Adopting a similar terminology, we say that a graph G admitting a perfect matching has the *Perfect-Matching-Hamiltonian property* (for short the PMH-property) if every perfect matching of G can be extended to a Hamiltonian cycle of G. We only consider graphs admitting a perfect matching to avoid trivial cases. This has already been studied in literature, and graphs having this property are also known as F-Hamiltonian, where F is a perfect matching (see [9, 23]). Henceforth, if a graph has the Perfect-Matching-Hamiltonian property, we say that it is a PMH-graph or simply that it is PMH. Note that since every perfect matching of G is a pairing of G, clearly, a graph having the PH-property is also a PMH-graph.

In the 1970s, Las Vergnas [13] (see Theorem 1) and Häggkvist [9] (see Theorem 2) gave two sufficient Ore-type conditions for a graph to be PMH.

Theorem 1. *[13] Let G be a bipartite graph, with partite sets U and V, such that $|U| = |V| = \frac{n}{2} \geqslant 2$. If for each pair of non-adjacent vertices $u \in U$ and $v \in V$ we have $deg(u) + deg(v) \geqslant \frac{n}{2} + 2$, then G is PMH.*

Theorem 2. *[9] Let G be a graph, such that the order of G is even and at least 4. If for each pair of non-adjacent vertices u and v we have $deg(u) + deg(v) \geqslant n + 1$, then G is PMH.*

Later on, in 1993, Ruskey and Savage [17] asked whether every matching in the n-dimensional hypercube Q_n, for $n \geqslant 2$, extends to a Hamiltonian cycle of Q_n. This was in fact shown to be true for $n = 2, 3, 4$ (see [8]) and for $n = 5$ (see [22]). Moreover, Fink [8] also showed that Q_n has the PH-property. This clearly implies that Q_n is a PMH-graph, and thus answers a conjecture made by Kreweras (see [12]). Finally, Amar, Flandrin and Gancarzewicz in [3] gave a degree sum condition for three independent vertices under which every matching of a graph lies in a Hamiltonian cycle. More results on PMH-graphs can be found in the paper by Yang [23].

The class of line graphs of connected graphs is a compelling class of graphs for which a great deal is known regarding Hamiltonicity and the existence of perfect matchings. Indeed, it is well-known that if G is connected and has an even number of edges, then its line graph admits a perfect matching (see Section 2 for more details), and so, in the sequel we shall tacitly assume that G is connected and of even size. Furthermore, Hamiltonicity of a line graph $L(G)$ is another extensively studied property: a necessary and sufficient condition for Hamiltonicity in $L(G)$ is proved in [10], while Thomassen conjectured in [21] that every 4-connected line graph is Hamiltonian.

Along these lines, we here deal with the line graph of a graph G and search for sufficient conditions on G which result in $L(G)$ being PMH. We will prove that $L(G)$ is PMH in all of the following cases:

- G is Hamiltonian with maximum degree $\Delta(G)$ at most 3 (Theorem 5),
- G is a complete graph (Theorem 14), and
- G is arbitrarily traceable from some vertex (Theorem 17).

In Section 3.2, we shall also discuss the line graph of complete bipartite graphs. Further related results and open problems regarding graphs which are hypohamiltonian, Eulerian or with large maximum degree are discussed along the paper.

1.1 Definitions and Notation

All graphs considered in this paper are finite, simple (without loops or multiple edges) and connected. Most of our terminology is standard, and we refer the reader to [4] for further definitions and notation not explicitly stated.

Unless otherwise stated, we let the order of G be n and denote the set of vertices of G by $\{v_1, v_2, \ldots, v_n\}$. For a graph G and $N \subseteq E(G)$, $G - N$ represents the resulting graph after deleting the edges in N from G.

A *walk* (of length k) in a graph G is a sequence $u_1, \ldots, u_{k+1}$ of vertices of G with corresponding edge set $\{u_iu_{i+1} : i \in [k]\}$. If $u_1 = u_{k+1}$, the walk is said to be *closed* and is denoted by $(u_1, \ldots, u_{k+1} = u_1)$. A *path* on t vertices, denoted by P_t, is a walk of length $t - 1$ in which all the vertices and edges are distinct. We may also refer to P_t as a *t-path*. A *cycle* of length k is a closed walk of length k in which all the vertices are distinct, except for the first and last. For simplicity, we denote a cycle of length k by $(u_1, \ldots, u_k)$, instead of $(u_1, \ldots, u_{k+1} = u_1)$.

A *tour* of G is a closed walk having no repeated edges. A graph G is *Eulerian* if there is a tour that traverses all the edges of G, called an *Euler tour*. A *dominating tour* of G is a tour in which every edge of G is incident with at least one vertex of the tour. In particular, a dominating tour which is 2-regular is referred to as a *dominating cycle*. In general, if a walk does not pass through some vertex v, we say that v is *untouched* or *uncovered*.

A *clique* in a graph G is a complete subgraph of G, and so K_n may sometimes be referred to as an *n-clique*.

2 Line graphs of graphs with small maximum degree

The *line graph* $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G, and two vertices of $L(G)$ are adjacent if the corresponding edges in G are incident to a common vertex. For some edge $e \in E(G)$, we refer to the corresponding vertex in $L(G)$ as e, for simplicity, unless otherwise stated. A *clique partition* of a graph G is a collection of cliques of G in which each edge of G occurs exactly once. For any $v \in V(G)$, let Q_v be the set of all the edges incident to v. Clearly, Q_v induces a clique in $L(G)$ and $\mathcal{Q} = \{Q_v : v \in V(G) \text{ with degree at least } 2\}$ is a clique partition of $L(G)$. We say that $\mathcal{Q}$ is the *canonical clique partition* of $L(G)$. In the sequel, we shall refer to Q_{v_i} simply as Q_i and in order to avoid trivial cases, from now on we always assume that G is a connected graph of order larger than 2. In what follows, we shall also say that a clique $Q' \in \mathcal{Q}$ is intersected by a set of edges N of $L(G)$, and by this we mean that $E(Q') \cap N \neq \varnothing$.

For a graph F, an *F-decomposition* of G is a collection of subgraphs of G whose edges form a partition of $E(G)$ such that each subgraph in the collection is isomorphic to F. In general, it is not hard to show that every connected graph G with $|E(G)|$ even has a P_3-decomposition. This is equivalent to saying that $L(G)$ has a perfect matching (see also Corollary 3 in [20]): indeed there is a natural bijection between the paths in a P_3-decomposition of G and the edges of the corresponding perfect matching M of $L(G)$, with the two edges in a P_3 corresponding to the two end-vertices of the respective edge in M. Since we are interested in line graphs which are PMH, a necessary condition is that $L(G)$ is Hamiltonian. Harary and Nash-Williams in [10] showed that $L(G)$ is Hamiltonian if and only if G admits a dominating tour. In particular, this implies that if G is Hamiltonian or Eulerian, then, $L(G)$ is also Hamiltonian, but the converse is not necessarily true (see also [5, 10, 19]).

The following technical lemma is the main tool we use to prove Theorem 5 as well as a series of related results contained in this section. It describes a necessary and sufficient condition to extend a given perfect matching to a Hamiltonian cycle in subcubic graphs.

Lemma 3. *Let G be a connected graph such that $\Delta(G) \leqslant 3$. A perfect matching M of $L(G)$ can be extended to a Hamiltonian cycle if and only if there exists a dominating cycle D of G such that the vertices in G untouched by D correspond to a subset of cliques in $\mathcal{Q}$ not intersected by M, where $\mathcal{Q}$ is the canonical clique partition of $L(G)$.*

Proof. Let M be a perfect matching of $L(G)$ which can be extended to a Hamiltonian cycle H_L of $L(G)$. For some orientation of H_L, let $Q_1, Q_2, \ldots, Q_s$ be the order in which $E(H_L)$ intersects at least one edge of the cliques in $\mathcal{Q}$, where $s \in [n]$. Since $\Delta(G) \leqslant 3$, $\mathcal{Q}$ consists of 2-cliques and 3-cliques, implying that the sequence $Q_1, Q_2, \ldots, Q_s$ does not have repetitions. We claim that $D = (v_1, v_2, \ldots, v_s)$ is a dominating cycle of G. Clearly, D is a cycle, since consecutive cliques in the sequence $Q_1, Q_2, \ldots, Q_s$ imply the existence of an edge between the corresponding two vertices in D. We then consider two cases. If every clique in $\mathcal{Q}$ is intersected by $E(H_L)$, then $(v_1, v_2, \ldots, v_s)$ is a Hamiltonian cycle, since $s = n$. Therefore, consider the case when $\mathcal{Q}$ contains a clique, say Q, not intersected by $E(H_L)$. The edges of the other cliques in $\mathcal{Q}$ which are incident to a vertex in Q must be intersected by $E(H_L)$, as otherwise the latter is not a Hamiltonian cycle of $L(G)$. Let these cliques be denoted by $Q_{j_1}, \dots, Q_{j_k}$, for $k = 2$ or 3 and $j_1, \dots, j_k \in [s]$. Let the corresponding vertices of Q and $Q_{j_1}, \dots, Q_{j_k}$, in G, be v and $v_{j_1}, \dots, v_{j_k}$, respectively. Also, since $v \neq v_t$ for all v_t in D, and M is a perfect matching of $L(G)$, the vertices $v_{j_1}, \dots, v_{j_k}$ are in the cycle D (not necessarily adjacent amongst themselves) and so the edges in G having v as an end-vertex have at least one end-vertex in D. Thus, since v was arbitrary, D is dominating. Moreover, every vertex in G untouched by D corresponds to a clique in $\mathcal{Q}$ not intersected by $E(H_L)$, which is a subset of the cliques in $\mathcal{Q}$ not intersected by M.

Conversely, let M be a perfect matching of $L(G)$ and let $D = (v_1, v_2, \dots, v_s)$ be a dominating cycle in G, for some $s \leqslant n$, such that the untouched vertices correspond to a subset of the cliques in $\mathcal{Q}$ not intersected by M. Note that there exists a one-to-one mapping between the untouched vertices in G and the unintersected cliques in $\mathcal{Q}$, which is not necessarily onto. We traverse the cliques in $\mathcal{Q}$ as follows. Let Q be a clique in $\mathcal{Q}$, with corresponding vertex $v \in V(G)$. We consider three cases.

Case 1: $E(Q) \cap M \neq \varnothing$.
By our assumption, $v = v_i$ for some $i \in [s]$, and we traverse Q $(= Q_i)$ using the unique path joining $V(Q_{i-1}) \cap V(Q_i)$ and $V(Q_i) \cap V(Q_{i+1})$ which contains $E(Q) \cap M$.

Case 2: $E(Q) \cap M = \varnothing$ and $v \in D$.
In this case, $v = v_j$ for some $j \in [s]$, and we traverse Q $(= Q_j)$ using the edge with end-vertices $V(Q_{j-1}) \cap V(Q_j)$ and $V(Q_j) \cap V(Q_{j+1})$.

Case 3: $E(Q) \cap M = \varnothing$ and $v \notin D$.
Since M is a perfect matching, all the cliques in $\mathcal{Q}$ sharing a vertex with Q (which must be triangles in this case) are intersected by M. These 3-cliques are traversed as in Case 1, and in this way the edges of Q are not intersected.

We traverse all the cliques in $\mathcal{Q}$ in the above way and let the resulting sequence of edges be H_L. We claim that H_L induces a Hamiltonian cycle of $L(G)$ containing M. By Case 1, H_L contains M and so every vertex of $L(G)$ is covered by H_L. Also, the sequence of cliques intersected by $E(H_L)$, i.e. $Q_1, Q_2, \dots, Q_s$, corresponds to the sequence of vertices in D, and so, since D is connected and 2-regular, H_L is a connected cycle, proving our claim. □

Remark 4. Note that Lemma 3 is not true in general for $\Delta(G) > 3$. An easy example is shown in Figure 1: indeed, an arbitrary perfect matching of $L(G)$ can be extended to a Hamiltonian cycle, i.e. $L(G)$ is PMH, but there is no dominating cycle in G.

By using Lemma 3, we can furnish a first sufficient condition on G assuring that its line graph is PMH.

Theorem 5. *Let G be a Hamiltonian graph such that $\Delta(G) \leqslant 3$. Then, $L(G)$ is PMH.*

Proof. Let H be a Hamiltonian cycle of G. Given any perfect matching M of $L(G)$, since the set of vertices untouched by H in G is empty, it is trivially a subset of the cliques in $\mathcal{Q}$ not intersected by M. Consequently, by Lemma 3, M can be extended to a Hamiltonian cycle of $L(G)$. Since M was arbitrary, G is PMH. □

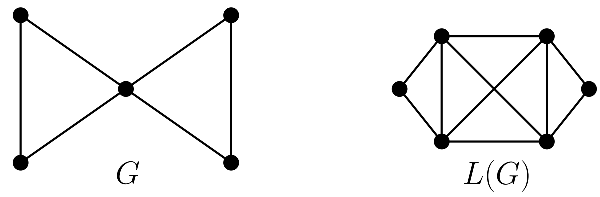

Figure 1: A graph with maximum degree 4 whose line graph is PMH.

In particular, Theorem 5 applies for all Hamiltonian cubic graphs. However, in the cubic case we can say more. In 1964, Kotzig [11] proved that the existence of a Hamiltonian cycle in a cubic graph is both a necessary and sufficient condition for a partition of $L(G)$ in two Hamiltonian cycles. We show the following.

Corollary 6. *Let G be a Hamiltonian cubic graph and M a perfect matching of $L(G)$. Then, $L(G)$ can be partitioned in two Hamiltonian cycles, one of which contains M.*

Proof. If we extend M to a Hamiltonian cycle of $L(G)$ using the method described in Lemma 3, we obtain a Hamiltonian cycle H_1 whose edge set intersects each triangle in $\mathcal{Q}$, since G is Hamiltonian. Moreover, since $E(H_1)$ intersects $Q \in \mathcal{Q}$ in one or two edges, the edges of $L(G) - E(H_1)$ intersect Q in two edges or one, respectively. Therefore, the edges in $L(G) - E(H_1)$ induce a Hamiltonian cycle H_2 of $L(G)$ whose edges intersect the triangles in $\mathcal{Q}$ in the same order as the edges in H_1. □

When considering Theorem 5, one could wonder if the two conditions on the maximum degree and the Hamiltonicity of G could be improved in some way. First of all, we remark that our result is best possible in terms of the maximum degree of G: indeed, if G is a Hamiltonian graph such that $\Delta(G) = 4$, then, $L(G)$ is not necessarily PMH. For instance, consider the Hamiltonian graph in Figure 2 having maximum degree 4, and let M be the perfect matching of $L(G)$ shown in the figure.

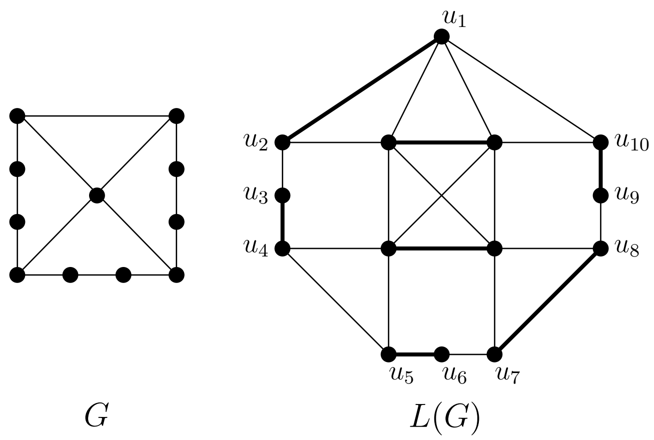

Figure 2: A Hamiltonian graph with maximum degree 4 whose line graph is not PMH.

Suppose M can be extended to a Hamiltonian cycle. Then, it should include all edges incident to its vertices of degree 2, and so it should contain the paths $u_1, u_2, \ldots, u_4$ and

$u_5, u_6, \ldots, u_{10}$. However, these two paths cannot be extended to a Hamiltonian cycle of $L(G)$ containing M, contradicting our assumption.

On the other hand, Hamiltonicity of G in Theorem 5 is not a necessary condition, since there exist non-Hamiltonian cubic graphs whose line graph is PMH. In particular, in Proposition 7 we prove that hypohamiltonian cubic graphs are examples of such graphs. Let us recall that a graph G is *hypohamiltonian* if G is not Hamiltonian, but for every $v \in V(G)$, $G - v$ has a Hamiltonian cycle.

Proposition 7. *Let G be a hypohamiltonian graph such that $\Delta(G) \leqslant 3$. Then, $L(G)$ is PMH.*

Proof. Let M be a perfect matching of $L(G)$. Since $|\mathcal{Q}| = |V(G)|$ is strictly larger than $|M| = \frac{|V(L(G))|}{2} \leqslant \frac{\frac{3}{2}|V(G)|}{2}$, there surely exists some clique $Q \in \mathcal{Q}$ which is not intersected by M. Let v be the corresponding vertex in G. Since G is hypohamiltonian, there exists a dominating cycle in G which passes through all the vertices of G except v, and so by Lemma 3, $L(G)$ is PMH, since M was arbitrary. □

Finally, another possible improvement of Theorem 5 could be a weaker assumption on the length of the longest cycle of G (i.e. the circumference of G, denoted by $circ(G)$). However, in Proposition 10 we exhibit cubic graphs having circumference just one less than the order of G whose line graphs are not PMH.

We will make use of the following standard operations on cubic graphs known as *Y-reduction* (shrinking a triangle to a vertex) and of its inverse, *Y-extension* (expanding a vertex to a triangle), illustrated in Figure 3.

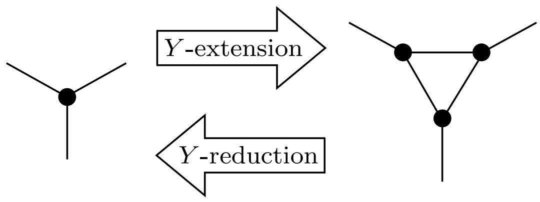

Figure 3: Y-operations

For the proof of Proposition 10, we also need to show that each edge of $L(G)$, where G is cubic and Hamiltonian, belongs to a perfect matching. This kind of property is extensively studied in many papers and a graph G is said to be *1-extendable* if every edge in G belongs to a perfect matching of G. Theorem 2.1 in [16] states that every claw-free 3-connected graph is 1-extendable. By recalling that every line graph is a claw-free graph, we have, in particular, that $L(G)$ is 1-extendable if G is cubic and 3-edge-connected. The generalisation to an arbitrary Hamiltonian cubic graph G is not hard to achieve by using such a result, but here we prefer to present a direct short proof which is valid for any bridgeless cubic graph and which makes use of the following tool from the proof of Proposition 2 in [14].

Remark 8. [14] Let G_1 be a cubic graph of even size and M a perfect matching of $L(G_1)$, with canonical clique partition $\mathcal{Q}$. The graph G_2 obtained by removing all the edges in M from $L(G_1)$ and then applying Y-reductions to all the triangles in $\mathcal{Q}$ not intersected by M, is isomorphic to G_1.

Remark 8 follows by considering the natural bijection ϕ between $V(G_1)$ and $\mathcal{Q}$, and the function ψ_M between $\mathcal{Q}$ and $V(G_2)$, where $\psi_M(Q)$, for $Q \in \mathcal{Q}$, is defined as follows. If $E(Q) \cap M = \varnothing$, Q is mapped to the vertex in G_2 obtained after applying a Y-reduction to Q. Otherwise, if $E(Q) \cap M \neq \varnothing$, Q is mapped to the vertex in G_2 corresponding to the vertex in Q unmatched by $E(Q) \cap M$. It is not hard to prove that $\psi_M \circ \phi$ is an isomorphism between G_1 and G_2.

Lemma 9. *Let G be a bridgeless cubic graph of even size. Then, every edge of $L(G)$ belongs to a perfect matching.*

Proof. Let $e \in E(L(G))$ and let M be a perfect matching of $L(G)$. Assume $e \notin M$, otherwise the statement holds. The graph $L(G) - M$ is cubic and by Remark 8 can be obtained by applying suitable Y-extensions to G. Since G is bridgeless, and the resulting graph after applying Y-extensions to a bridgeless graph is again bridgeless, we have that $L(G)-M$ is bridgeless as well. Moreover, in [18], Schönberger proved that every bridgeless cubic graph is 1-extendable: hence, there exists a perfect matching of $L(G) - M$ which contains e. Such a perfect matching is trivially also a perfect matching of $L(G)$ containing e. □

The following proposition shows that the Hamiltonicity condition in Theorem 5 cannot be relaxed to any other condition regarding the length of the longest cycle in G. Indeed, starting from an appropriate cubic graph and performing suitable Y-extensions, we obtain a graph of circumference one less than its order whose line graph is not PMH.

Proposition 10. *Let G be a hypohamiltonian cubic graph of odd size. Let G' be a graph obtained by performing a Y-extension to all vertices of G except one. Then, $circ(G') = |V(G')| - 1$ and $L(G')$ is not PMH.*

Proof. Let v be the vertex of G to which we do not apply a Y-extension, and let the resulting graph be G', with the vertex of G' corresponding to v denoted by v'. Since G is hypohamiltonian, G admits a cycle C of length $|V(G)| - 1$ which passes through all the vertices of G except v. Consequently, G' admits a cycle C' which passes through all the vertices of G' except v' and whose edges intersect the Y-extended triangles in the same order that C passes through all the corresponding vertices in G, resulting in the three vertices of each Y-extended triangle being consecutive in C'. Since G' is not Hamiltonian, $circ(G') = |V(G')| - 1$. We proceed by supposing that $L(G')$ is PMH, for contradiction. Denote by $Q_{v'}$ the triangle in the canonical clique partition of $L(G')$ which corresponds to the vertex v'. By construction of G', we have $|E(G')| = |E(G)| + 3(|V(G)| - 1)$. Since both $|V(G)| - 1$ and $|E(G)|$ are odd, $|E(G')|$ is even, i.e. $L(G')$ has even order. Moreover, since G is hypohamiltonian, G is bridgeless. Consequently, G' is bridgeless as well, since it is obtained by applying Y-extensions to G, and so, by Lemma 9, there exists a perfect matching M of $L(G')$ which intersects a chosen edge of $Q_{v'}$. Lemma 3 assures that there exists a dominating cycle D in G' such that the set of its uncovered vertices does not contain v'. Furthermore, the edge set of every dominating cycle of G', in particular $E(D)$, intersects at least one edge of all the Y-extended triangles. Consequently, the dominating cycle D induces a cycle in G which passes through v and also through every other vertex of G, making G Hamiltonian, a contradiction. □

As already remarked, the graph in Figure 2 is Hamiltonian, but not every perfect matching in its line graph can be extended to a Hamiltonian cycle. Such an example is not regular, and we are not able to find a regular one. A most natural question to ask is whether the Hamiltonicity and regularity of a graph are together sufficient conditions to guarantee the PMH-property of its line graph. Thus, we suggest the following problem.

Problem 11. Let G be an r-regular Hamiltonian graph of even size, for $r \geqslant 4$. Does $L(G)$ have the PMH-property?

To conclude this section, let us note that not all 4-regular (and so not all Eulerian) graphs of even size have a PMH line graph. A non-Hamiltonian example is given in Figure 4. It is not hard to check that every perfect matching of $L(G)$ which contains the edges e_1e_2 and e_3e_4 cannot be extended to a Hamiltonian cycle of $L(G)$.

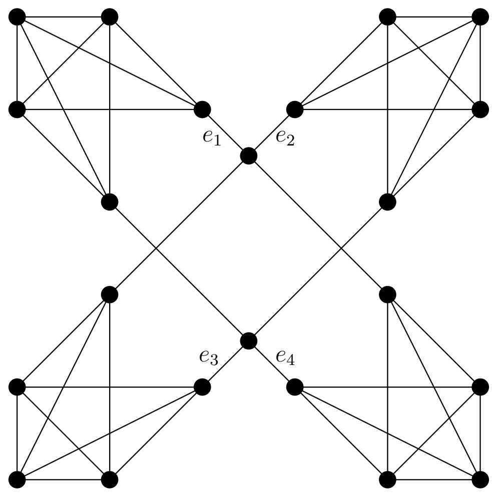

Figure 4: A non-Hamiltonian 4-regular graph whose line graph does not have the PMH-property.

Since the graphs in Figure 2 and Figure 4 are both not simultaneously Eulerian and Hamiltonian, we pose a further problem.

Problem 12. Let G be a graph of even size which is both Eulerian and Hamiltonian. Does $L(G)$ have the PMH-property?

3 Other classes of graphs whose line graphs are PMH

The complete graph K_n, for even n, and the complete bipartite graph $K_{m,m}$, for $m \geqslant 2$, are clearly PMH. To stay in line with the contents of this paper, we now see whether their line graphs are also PMH. To this purpose, given an edge-colouring (not necessarily proper) of a Hamiltonian graph, a Hamiltonian cycle in which no two consecutive edges have the same colour will be referred to as a *properly coloured Hamiltonian cycle.*

3.1 Complete graphs

First of all, we note that the line graph of a complete graph K_n has a perfect matching if and only if the number of edges in K_n is even. Hence, in the sequel we consider only complete graphs with $n \equiv 0, 1 \mod 4$.

We denote the vertices of K_n by $\{v_i : i \in [n]\}$ and the edges of K_n by $\{e_{i,j} = v_i v_j : i \neq j\}$. Moreover, $V(L(K_n))$ is denoted by $\{v_{i,j} : i \neq j\}$ where the vertex $v_{i,j}$ corresponds to the edge $e_{i,j}$ of K_n. Finally, we denote the edges of $L(K_n)$ by $\{e^i_{j,k} = v_{i,j}v_{i,k} : j \neq k\}$. Note that the upper index in the notation $e^i_{j,k}$ immediately indicates that the considered edge belongs to the clique Q_i in the canonical clique partition of $L(K_n)$, while the order of lower indices is irrelevant.

The proof of our main theorem in this section, Theorem 14, makes use of a special case of a result by Daykin [6] from 1976 which asserts the existence of a properly coloured Hamiltonian cycle if the edges of K_n are coloured according to the following constraints.

Theorem 13. *[6] If the edges of the complete graph K_n, for $n \geqslant 6$, are coloured in such a way that no three edges of the same colour are incident to any given vertex, then there exists a properly coloured Hamiltonian cycle.*

In the following proof, the process of traversing one path after another will be called *concatenation of paths.* If two paths P^1 and P^2 have end-vertices x, y and y, z, respectively, we write P^1P^2 to denote the path starting at x and ending at z obtained by traversing P^1 and then P^2.

Theorem 14. *For $n \equiv 0, 1 \mod 4$, $L(K_n)$ is PMH.*

Proof. Since K_4 is Hamiltonian and cubic, by Theorem 5, the result holds for $n = 4$. Therefore, we can assume $n > 4$.

Let M be a perfect matching of $L(K_n)$. We colour the $\frac{1}{4}n(n-1)$ edges of M with $\frac{1}{4}n(n-1)$ different colours. For all $e^i_{j,k} \in M$, we colour the edges $e_{i,j}$ and $e_{i,k}$ in K_n with the same colour given to the edge $e^i_{j,k}$ in $L(K_n)$. This gives a P_3-decomposition of K_n in which each P_3 is monochromatic and the colours of all the 3-paths are pairwise distinct.

If $n = 5$, the total number of Hamiltonian cycles in K_5 is $\frac{4!}{2} = 12$. Each of the five monochromatic 3-paths in K_5 is on exactly two distinct Hamiltonian cycles. Therefore, the number of Hamiltonian cycles containing a monochromatic P_3 is at most 10, hence K_5 contains at least two (complementary) properly coloured Hamiltonian cycles. Without loss of generality, let one of them be H, say $H = (v_1, v_2, \dots, v_5)$.

For $n \geqslant 8$, by Theorem 13, there exists a properly coloured Hamiltonian cycle H in K_n and again, without loss of generality, we can assume $H = (v_1, v_2, \ldots, v_n)$.

Now, for all $n \geqslant 5$ and $n \equiv 0, 1 \mod 4$, we will use the properly coloured Hamiltonian cycle H in K_n to obtain a Hamiltonian cycle H_L in $L(K_n)$ containing the perfect matching M. We construct the Hamiltonian cycle H_L in such a way that it enters and exits each clique in the canonical clique partition $\mathcal{Q}$ of $L(K_n)$ exactly once. More precisely, we construct a suitable path P^i in each clique Q_i and we obtain H_L as a concatenation of such paths following the order determined by H. Consider the $(n-1)$-clique Q_i and its two vertices $v_{i-1,i}$ and $v_{i,i+1}$. The corresponding edges $e_{i-1,i}$ and $e_{i,i+1}$, in K_n, are not of the same colour since they are consecutive in H, and so the edge $e^i_{i-1,i+1} \notin M$. We assign a linear order $<_i$ to the set of edges $M \cap E(Q_i)$, with $(M \cap E(Q_i), <_i) = \mu_i$, such that:

(i) if $M \cap E(Q_i)$ contains an edge incident to $v_{i-1,i}$, such an edge is the first edge of μ_i, and

(ii) if $M \cap E(Q_i)$ contains an edge incident to $v_{i,i+1}$, such an edge is the last edge of μ_i,

Note that $<_i$ exists since $e^i_{i-1,i+1} \notin M$. Next, we construct an M-alternating path in Q_i, which we denote by P^i, starting at $v_{i-1,i}$ and ending at $v_{i,i+1}$ as follows: P^i alternates between an edge of μ_i and an edge which is simultaneously adjacent to two consecutive edges in μ_i, except possibly the first and/or last edge in P^i. Note that the choice of edges not belonging to $M \cap E(Q_i)$ as given above is always possible since Q_i is a clique. Consequently, $M \cap E(Q_i) \subset E(P_i)$.

Now we define H_L to be $P^1 P^2 \ldots P^n$. Note that H_L is a cycle since the paths P^i are all internally and pairwise disjoint, and the beginning of P^1 coincides with the end of P^n. Moreover, H_L is Hamiltonian because $M \subset E(H_L)$ and so each vertex of the line graph belongs to H_L. □

3.2 Complete bipartite graphs

In 1976, Chen and Daykin considered an analogous version of Theorem 13 for the complete bipartite graph $K_{m,m}$ (see [7]). A particular case of Theorem 1′ in [7] can be stated as follows.

Theorem 15. *[7] Consider an edge-colouring of the complete bipartite graph $K_{m,m}$ such that no vertex is incident to more than k edges of the same colour. If $m \geqslant 25k$, then there exists a properly coloured Hamiltonian cycle.*

By considering the case $k = 2$ in the previous theorem, i.e. $m \geqslant 50$, and by using an argument very similar to the one used for complete graphs in Section 3.1, one could obtain that $L(K_{m,m})$ is PMH for every even $m \geqslant 50$. However, in a forthcoming paper, three of the authors give a more complete result and extend this by using a different and more technical approach, which goes beyond the scope of this paper. They prove the following theorem.

Theorem 16. *[1] Let m_1 be an even integer and let $m_2 \geqslant 1$. Then, $L(K_{m_1,m_2})$ does not have the PH-property if and only if $m_1 = 2$ and m_2 is odd.*

3.3 Arbitrarily traceable graphs

A graph G is said to be *arbitrarily traceable* (or equivalently *randomly Eulerian*) from a vertex $v \in V(G)$ if every walk starting from v and not containing any repeated edges can be completed to an Eulerian tour. This notion was firstly introduced by Ore in [15], who proved that an Eulerian graph G is arbitrarily traceable from v if and only if every cycle in G touches v. Here we show that every perfect matching M of the line graph of an arbitrarily traceable graph can be extended to a Hamiltonian cycle.

Note that the technique used in this proof is in some way different from what was used in the case of complete graphs in Section 3.1. Again, a perfect matching M of $L(G)$ corresponds to a P_3-decomposition of G, but this time we construct an Euler tour of the original graph (instead of a Hamiltonian cycle) such that two edges in the same 3-path are consecutive in the Euler tour (as opposed to what was done in Section 3.1 where we forbade two edges in the same 3-path to be consecutive in the Hamiltonian cycle considered in K_n).

Theorem 17. *Let G be a graph of even size. If G is arbitrarily traceable from some vertex, then its line graph is PMH.*

Proof. Let M be a perfect matching of $L(G)$. Consider the P_3-decomposition of G induced by M. Since G is arbitrarily traceable from some vertex, there exists an Euler tour in which every pair of edges in the same 3-path are consecutive. The sequence of edges in this Euler tour corresponds to a sequence of vertices in $L(G)$ which gives a Hamiltonian cyle H of $L(G)$, and since the two edges of each 3-path in the P_3-decomposition are consecutive in the Euler tour, H contains all the edges of M, as required. □

4 Concluding remark

Along the paper, we have proposed several sufficient conditions of different types for a graph in order to guarantee the PMH-property in its line graph. The wide variety of such conditions, ranging between sparse and dense graphs, do not allow us to easily identify non-trivial necessary conditions to this problem. This could be seemingly hard, but we still consider it an intriguing problem to be addressed in the future.

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