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EXTENDING THE FORMULA TO CALCULATE THE SPECTRAL RADIUS OF AN OPERATOR

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Abstract. In a Banach space, Gelfand's formula is used to find the spectral radius of a continuous linear operator. In this paper, we show another way to find the spectral radius of a bounded linear operator in a complete topological linear space. We also show that Gelfand's formula holds in a more general setting if we generalize the definition of the norm for a bounded linear operator.

1. Introduction and basic definitions

In all that follows E stands for a linear vector space over the field $\mathbf{C}$ of complex numbers. $E[t]$ will denote a complete locally convex topological vector space, with a Hausdorff topology t, and $T : E \to E$ will be a linear map. Finally, $\vartheta(t)$ will be the filter of all balanced, convex and closed t-neighborhoods of zero (in E).

Definition 1. The linear operator $T : E[t] \to E[t]$ is said to be a bounded operator, if there is a neighborhood $U \in \vartheta(t)$ such that $T(U)$ is a bounded set.

If in the definition above $T(U)$ is a relatively compact set, then T is said to be a compact operator. Any compact operator is a bounded operator, and any bounded operator is continuous (with the t-topology) (see [5]).

We recall that, given any topological linear space $E[\omega]$ and $S : E[\omega] \to E[\omega]$ a linear operator, the resolvent of S is the set

$$\rho_\omega(S) = \Big\{\xi \in \mathbf{C} \,\Big|\, \xi I - S : E[\omega] \to E[\omega] \text{ is bijective and has a continuous inverse}\Big\}.$$

The spectrum of S is defined by $\sigma_\omega(S) = \mathbf{C} \setminus \rho_\omega(S)$ (the set-theoretic complement in $\mathbf{C}$ of the resolvent set), and the spectral radius by

$$sr_\omega(S) = \sup\Big\{|\lambda| \,\Big|\, \lambda \in \sigma_\omega(S)\Big\}.$$

Definition 2. A net $\{x_\alpha\}_J \subset E$ is said to be t-ultimately bounded (t-ub) if, given any $V \in \vartheta(t)$, there is a positive real number r and an index $\alpha_0 \in J$, both depending on V, such that $x_\alpha \in rV \ \forall\, \alpha \geq \alpha_0$.

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Let us denote by Γ the set of all t-ub nets in E.

Remark 1. Any bounded, convergent or Cauchy net is a t-ub net. For more details about t-ub nets we refer the reader to [1].

Definition 3. Let $\xi \in \mathbf{C}$. We will say that $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$ ($T^n = T \circ T \circ ... \circ T$, n times) if, given both $V \in \vartheta(t)$ and $\{x_\alpha\}_J \in \Gamma$, there exist $\alpha_0 \in J$ and $n_0 \in \mathbf{N}$ such that $\frac{1}{\xi^n}T^n(x_\alpha) \in V$ $\forall \alpha \geq \alpha_0$ and $\forall n \geq n_0$.

Definition 4. $\gamma_t(T) = \inf\left\{|\xi| \;\middle|\; \frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0\right\}$.

Remark 2. It is shown by Vera [4] that for a bounded operator T, we have:

(i) $\gamma_t(T) < \infty$, and for any $\xi \in \mathbf{C}$ such that $\gamma_t(T) < |\xi|$, $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$.
(ii) $sr_t(T) \leq \gamma_t(T)$, where $\mathrm{sr}_t(\mathrm{T})$ is the spectral radius of T.
(iii) When $E[t]$ is a Banach space, $\gamma_t(T) = r_t(T)$.

In [2] it was proved, based in the above result, that $\gamma_t(T) = sr_t(T)$ when T is a compact operator. In this paper we extend that result to any bounded operator.

2. Main results

From now on let $T : E[t] \to E[t]$ be a bounded operator and let $U \in \vartheta(t)$ be such that $T(U)$ is a bounded set.

Let P_U be the functional of Minkowski (see [3]) generated by U, which is a seminorm on E. Let $E[P_U]$ denote the vector space E with the topology given by the seminorm P_U.

Remark 3. The topology on E given by the seminorm P_U is coarser than the topology t ($P_U \leq t$).

Proposition 1. $T : E[P_U] \to E[P_U]$ *is a bounded operator (hence a continuous one).*

Proof. Since $T(U)$ is a bounded set and $P_U \leq t$, $T(U)$ is also a P_U-bounded set in $E[P_U]$. □

Definition 5. $\gamma_{P_U}(T) = \inf\left\{|\xi| \;\middle|\; \frac{1}{\xi^n}T^n \xrightarrow{\Gamma_{P_U}} 0\right\}$.

Here Γ_{P_U} convergence means that, given any net $\{x_\alpha\}_J \subset E$ such that for all α, $P_U(x_\alpha) \leq r$ for some positive real number r (P_U-bounded net), then $P_U(\frac{1}{\xi^n}T^n x_\alpha) \to 0$ as a net in $\mathbf{R}$ whose index set is $\mathbf{N} \times \mathbf{J}$.

Proposition 2. $\gamma_{P_U}(T) = \gamma_t(T)$.

Proof. Let $\xi \in \mathbf{C}$ be such that $\gamma_{P_U}(T) < |\xi|$, and let $V \in \vartheta(t)$ and $\{x_\alpha\}_J \in \Gamma$ be given. Since $\frac{1}{\xi}T(U)$ is a bounded set, there is a positive real number r_1 such that $\frac{1}{r_1\xi}T(U) \subset V$. In [1] is shown that $\{x_\alpha\}_J \in \Gamma \Rightarrow \{r_1 x_\alpha\}_J \in \Gamma$. This implies that there exist both $\alpha_0 \in J$ and $r_2 > 0$ such that $r_1 x_\alpha \in r_2 U \;\forall \alpha \geq \alpha_0$, i.e., $P_U(r_1 x_\alpha) \leq r_2$, that is, the net $\{x_\alpha\}_{\alpha \geq \alpha_0}$ is a P_U-bounded net; therefore, $\exists\, \alpha_1 \in J$ ($\alpha_1 \geq \alpha_0$) and $n_1 \in \mathbf{N}$ such that $P_U(\frac{1}{\xi^n}T^n(x_\alpha)) < 1 \;\forall \alpha \geq \alpha_1$, $n \geq n_1$, that is, $\frac{1}{\xi^n}T^n(x_\alpha) \in U$ for those indices. Hence

$$\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha = \frac{1}{r_1\xi}T(\frac{1}{\xi^n}T^n r_1 x_\alpha) \in \frac{1}{r_1\xi}T(U) \subset V \;\; \forall \alpha \geq \alpha_1 , n \geq n_1 ,$$

that is, $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$, and therefore, $\gamma_t(T) \le |\xi|$. This implies that $\gamma_{P_U}(T) \le \gamma_t(T)$.

On the other hand, let $\gamma_t(T) < |\xi|$ and $\{x_\alpha\}_J$, a P_U-bounded net; that is, $x_\alpha \in rU$ for all α and some $r > 0$. Then $\{\frac{1}{\xi}Tx_\alpha\}_J \subset \frac{r}{\xi}T(U)$, where $\frac{r}{\xi}T(U)$ is a t-bounded set; therefore, $\{\frac{1}{\xi}Tx_\alpha\}_J \in \Gamma$. Since $\frac{1}{\xi^n}T^n \xrightarrow{\Gamma_t} 0$, given $\epsilon > 0$, $\exists \alpha_0 \in J$ and $n_0 \in \mathbf{N}$ such that $\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha = \frac{1}{\xi^n}T^n(\frac{1}{\xi}Tx_\alpha) \in \epsilon U$ $\forall \alpha \ge \alpha_0$, $n \ge n_0$; that is, $P_U(\frac{1}{\xi^{n+1}}T^{n+1}x_\alpha) \le \epsilon$ for those indices. This says that $\frac{1}{\xi^n}T^n x_\alpha$ is P_U-convergent to 0; therefore, $\gamma_{P_U}(T) \le |\xi|$. This implies that $\gamma_t(T) \le \gamma_{P_U}(T)$. □

Definition 6.

$$L(E) = \Big\{S : E[t] \to E[t] \;\Big|\; S \text{ is a linear and continuous operator}\Big\},$$

$$L_U(E) = \Big\{S \in L(E) \;\Big|\; S(U) \text{ is a bounded set}\Big\},$$

$L_U(E)$ is a vector subspace of the complex vector space $L(E)$.

Remark 4. For the bounded operator T that we have been working on we have $T,\ T^n,\ \lambda T,\ \lambda T^n \in L_U(E)$ for all $n \in \mathbf{N}$ and all $\lambda \in \mathbf{C}$.

Moreover, for any $S \in L(E)$, $S \circ T, T \circ S \in L_U(E)$.

Definition 7. For any operator $S \in L_U(E)$, we define, taking into account that $S(U)$ is a bounded set, the following real number:

$$||S||_U = \sup\{P_U(Sx) \mid x \in U\}.$$

It easy to check that $||S^n||_U \le ||S||_U^n$ $\forall S \in L_U(E)$ and $\forall n \in \mathbf{N}$.

Theorem 1. *If* $S_n \xrightarrow{\Gamma_t} S$ *in L(E), then* $||S_n \circ T - S \circ T||_U \to 0$.

Proof. Let us prove it by way of contradiction.

Let $\epsilon > 0$ be such that there exist natural numbers $n_1 < n_2 < n_3 < ...$ such that $\epsilon < ||S_{n_k} \circ T - S \circ T||_U$; hence, for each of those n_k there is $x_{n_k} \in U$ such that $P_U[(S_{n_k} \circ T - S \circ T)x_{n_k})] > \epsilon$. Since $\{Tx_{n_k}\} \subset T(U)$, it is a bounded sequence; hence, for $V = \epsilon U \in \vartheta(t)$ there is an index $m_0 \in \mathbf{N}$ such that $(S_n - S)(Tx_{n_k}) \in V$ for all $n, n_k \ge m_0$; this implies that $P_U[(S_{n_k} \circ T - S \circ T)x_{n_k})] \le \epsilon$, which yields a contradiction. □

Proposition 3. $\rho_t(T) \subset \rho_{P_U}(T)$.

Proof. Let us suppose first that $\gamma_t(T) < 1$. Let $\xi \in \rho_t(T)$ be such that $|\xi| > \gamma_t(T)$. Then $S = \sum_{k=0}^{\infty} \frac{1}{\xi^{k+1}}T^k$ is a continuous operator and $S = (\xi I - T)^{-1}$. Set $S_n = \sum_{k=0}^{n} \frac{1}{\xi^{k+1}}T^k$. Then $S_n \xrightarrow{\Gamma_t} S$, and from Theorem 1 it follows that $||S_n \circ \frac{1}{\xi}T - S \circ \frac{1}{\xi}T||_U \to 0$. On the other hand, $S_n \circ \frac{1}{\xi}T = S_{n+1} - \frac{1}{\xi}I$ and $S \circ \frac{1}{\xi}T = S - \frac{1}{\xi}I$; hence $||S_{n+1} - S||_U \to 0$. Thereby, given $\{x_m\}_{\mathbf{N}} \subset E$ such that $P_U(x_m) \to 0$, then $P_U(Sx_m) \le P_U[(S - S_n)x_m] + P_U(S_n x_m) \to 0$. This proves that $S : E[P_U] \to E[P_U]$ is a continuous operator; hence $\xi \in \rho_{P_U}(T)$.

Now let $\xi \in \rho_t(T)$ be such that $|\xi| \le \gamma_t(T)$. Then $|\frac{1}{\xi}| > 1 > \gamma_t(T)$, which means that $\frac{1}{\xi}I - T : E[P_U] \to E[P_U]$ is a continuous operator. Since $\xi I - T =$

$(\xi - \frac{1}{\xi})I - (T - \frac{1}{\xi}I)$, we have that

$$(\xi I - T)^{-1} = (\xi - \frac{1}{\xi})^{-1} I \circ [(T - \frac{1}{\xi}I)^{-1} - (\frac{1}{\xi})^{-1} I] \circ (T - \frac{1}{\xi}I)^{-1};$$

since the right hand side is the composition of three continuous operators from $E[P_U]$ to $E[P_U]$ we have that $\xi \in \rho_{P_U}(T)$.

Finally, let T be such that $\gamma_t(T) < r < \infty$. Then $T_1 = \frac{1}{r}T \in L_U(E)$ is such that $\gamma_t(T_1) < 1$. Hence $\frac{1}{r}\rho_t(T) = \rho_t(T_1) \subset \rho_{P_U}(T_1) = \frac{1}{r}\rho_{P_U}(T)$, and therefore $\rho_t(T) \subset \rho_{P_U}(T)$. □

Definition 8. $N = \left\{ x \in E \,\middle|\, P_U(x) = 0 \right\}$.

Remark 5. Since $\left\{ x \in E \,\middle|\, P_U(x) \leq 1 \right\} \subset U$, $N \subset U$.

Theorem 2. *N is a closed linear subspace of E, and T(x) = 0 for all* $x \in N$.

Proof. The first claim follows from the fact that

$$P_U(\xi x + y) \leq |\xi| \; P_U(x) + P_U(y).$$

For the second claim let's take $x \in N$; then $mx \in N$ for $m = 1, 2, \ldots$. Let V be any balanced, convex and closed t-neighborhood of 0. Since $\{m\,T(x)\}_{m=1,2,3,\ldots} \subset T(N) \subset T(U)$ and the latter set is bounded, there exists $r \in \mathbf{R}^+$ such that $\{m\,T(x)\} \subset rV \;\Rightarrow\; T(x) \in \frac{r}{m}V \subset V$ when $m > r$. Since V was an arbitrary neighborhood of zero and $E[t]$ is Hausdorff, then $T(x) = 0$. □

Definition 9. Let E/N be the quotient linear space and let $\hat{P}_U$ be the norm on it defined by $\hat{P}_U(x + N) = P_U(x)$ (see [3]).

Remark 6. $(E/N)[\hat{P}_U]$ will denote the vector space E/N with the topology given by the norm $\hat{P}_U$.

Definition 10. Let $\hat{T} : (E/N) \to (E/N)$ be defined by $\hat{T}(x + N) = T(x) + N$.

Remark 7. It is easy to show that $\hat{T}$ is a well defined linear map.

Proposition 4. $\hat{T} : (E/N)[\hat{P}_U] \to (E/N)[\hat{P}_U]$ *is a linear and bounded operator (hence* $\hat{T}$ *is continuous).*

Proof. U/N is the unit ball in $(E/N)[\hat{P}_U]$ and $\hat{T}(U/N) = (T(U) + N)/N$. The latter set is $\hat{P}_U$-bounded because the canonical projection $E[P_U] \to (E/N)[\hat{P}_U]$ is a continuous map. □

Remark 8. Since $(E/N)[\hat{P}_U]$ is a norm space we can define, as usual, the norm of $\hat{T}$, and this will be denoted by $||\hat{T}||_{\hat{P}_U}$.

Proposition 5. $\gamma_{\hat{P}_U}(\hat{T}) = \gamma_{P_U}(T)$.

Proof. Set $|\xi| > \gamma_{P_U}(T)$. Let $\{x_\alpha + N\}_J$ be a $\hat{P}_U$-bounded net in E/N; then $\{x_\alpha\}_J$ is a P_U-bounded net in E; hence, given $\epsilon > 0$, there are indices $\alpha_0 \in J$ and $n_0 \in \mathbf{N}$ such that $\frac{1}{\xi^n}T^n x_\alpha \in \epsilon U \;\; \forall\, \alpha \geq \alpha_0$ and $n \geq n_0$. Thus

$$\frac{1}{\xi^n}\hat{T}^n(x_\alpha + N) = \frac{1}{\xi^n}T^n x_\alpha + N \in \;\; \epsilon \; (U/N), \quad \alpha \geq \alpha_0 \,, \; n \geq n_0$$

This implies that $\gamma_{\hat{P}_U}(T) \leq |\xi|$. Hence $\gamma_{\hat{P}_U}(T) \leq \gamma_{P_U}(T)$.

Set $|\xi| > \gamma_{\hat{P}_U}(T)$. Let $\{x_\alpha\}_J$ be a P_U-bounded net in E. Then $\{x_\alpha + N\}_J$ is a $\hat{P}_U$-bounded net in E/N; hence, given $\epsilon > 0$, there are indices $\alpha_0 \in J$ and $n_0 \in \mathbf{N}$ such that $\frac{1}{\xi^n}\hat{T}^n(x_\alpha + N) \in \epsilon\,(U/N)\ \forall\, \alpha \geq \alpha_0$, $n \geq n_0$. This implies that for those indices $\frac{1}{\xi^n}T^n x_\alpha = \epsilon u_\alpha + z_\alpha$, $u_\alpha \in U$, $z_\alpha \in N$; hence $P_U(\frac{1}{\xi^n}T^n x_\alpha) \leq P_U(\epsilon u_\alpha) + P_U(z_\alpha) \leq \epsilon + 0 = \epsilon$, and thus $|\xi| > \gamma_{P_U}(T)$. This implies that $\gamma_{\hat{P}_U}(T) \geq \gamma_{P_U}(T)$. □

Proposition 6. $\rho_{P_U}(T) = \rho_{\hat{P}_U}(\hat{T})$.

Proof. $\xi \in \rho_{P_U}(T) \Rightarrow \xi I - T : E[P_U] \to E[P_U]$ is bijective and has a continuous inverse.

Let us show that $A : (E/N)[\hat{P}_U] \to (E/N)[\hat{P}_U]$ defined by $A(x + N) = (\xi I - T)^{-1}(x) + N$, which is a linear and continuous map, is the inverse function of $\xi\hat{I} - \hat{T}$. For this, $A(\xi\hat{I} - \hat{T})(x + N) = A(\widehat{\xi I - T})(x + N) = A((\xi I - T)(x) + N) = (\xi I - T)^{-1}(\xi I - T)(x) + N = x + N$. In a similar way it can be proved that $(\xi\hat{I} - \hat{T}) \circ A = I$. This implies that $\xi \in \rho_{\hat{P}_U}(\hat{T})$.

It is just routine to prove the set contention in the other way around. □

Definition 11. $\widetilde{(E/N)}[\tilde{P}_U]$ will denote the completion (as a normed space) of $(E/N)[\hat{P}_U]$, and $\tilde{T}$ will denote the natural extension of $\hat{T}$.

Remark 9. $\widetilde{(E/N)}[\tilde{P}_U]$ is a Banach space. Besides, since $\hat{T}$ is a bounded operator, $\tilde{T} : \widetilde{(E/N)}[\tilde{P}_U] \to \widetilde{(E/N)}[\tilde{P}_U]$ is a bounded operator (see [3]).

Remark 10. Since $\widetilde{(E/N)}[\tilde{P}_U]$ is a Banach space we can define, as usual, the norm of $\tilde{T}$; this will be denoted by $||\tilde{T}||_{\tilde{P}_U}$.

Proposition 7. $\gamma_{\hat{P_U}}(\hat{T}) = \gamma_{\tilde{P_U}}(\tilde{T})$.

Proof. Since $\tilde{T}$ is an extension of $\hat{T}$, the proof follows immediately from the definitions of $\gamma_{\hat{P_U}}(\hat{T})$ and $\gamma_{\tilde{P_U}}(\tilde{T})$. □

Proposition 8. $\rho_{\hat{P_U}}(\hat{T}) = \rho_{\tilde{P_U}}(\tilde{T})$.

Proof. If $\xi \in \rho(\hat{T})$, then $\xi I - \hat{T} : (E/N)[\hat{P_U}] \to (E/N)[\hat{P_U}]$ is bijective and has a continuous inverse, so that both $\xi I - \hat{T}$ and $(\xi I - \hat{T})^{-1}$ have a continuous extension to $\widetilde{(E/N)}$, which are precisely $\xi I - \tilde{T}$ and $(\xi I - \tilde{T})^{-1}$ respectively. This implies that $\xi \in \rho(\tilde{T})$.

On the other hand, if $\xi \in \rho(\tilde{T})$, then $\xi I - \tilde{T} : \widetilde{(E/N)}[\tilde{P_U}] \to \widetilde{(E/N)}[\tilde{P_U}]$ is bijective and has a continuous inverse; hence the restrictions of those functions to $(E/N)[\hat{P_U}]$ are precisely $\xi I - \hat{T}$ and its inverse function, which are continuous functions for being the restrictions of continuous ones. Then $\xi \in \rho(\hat{T})$. □

Theorem 3. $\gamma_t(T) = sr_t(T)$.

Proof. By Remark 2 (ii) it suffices to show that $sr_t(T) \geq \gamma_t(T)$. Also, from Remark 2 (iii) we get

$$sr_{\tilde{P_U}}(\tilde{T}) = \gamma_{\tilde{P_U}}(\tilde{T}) \tag{1}$$

because $\widetilde{(E/N)}[\tilde{P_U}]$ is a Banach space. From Propositions 2, 5 and 7 we obtain

$$\gamma_t(T) = \gamma_{\tilde{P_U}}(\tilde{T}) \tag{2}$$

From Propositions 3, 6 and 8 we obtain

$$\rho_t(T) \subset \rho_{\tilde{P_U}}(\tilde{T});$$

this implies that

$$sr_{\tilde{P_U}}(\tilde{T}) \leq sr_t(T). \tag{3}$$

From (1), (2) and (3) we finally get

$$\gamma_t(T) \leq sr_t(T).$$

□

3. A generalization of Gelfand's formula

In this part we prove that Gelfand's formula (see [3]) applies for a bounded operator defined on a topological vector space. Following the notation from the sections above, we will show that we can use $||T||_U$ in Gelfand's formula to calculate the spectral radius of T.

Proposition 9. *For any* $T \in L_U(E)$, $||T||_U = ||\hat{T}||_{\hat{P}_U}$.

Proof. Set $r > ||T||_U$. Then $T(U) \subset rU$; hence $P_U(Tx) \leq r$ for all $x \in U$. This implies that $||\hat{T}||_{\hat{P}_U} \leq r$, and therefore $||\hat{T}||_{\hat{P}_U} \leq ||T||_U$.

Set $r < ||T||_U$. Then there exists $x \in U$ such that

$$r < P_U(Tx) = \hat{P}_U(\hat{T}(x+N)) \leq ||\hat{T}||_{\hat{P}_U}.$$

This implies that $||\hat{T}||_{\hat{P}_U} = ||T||_U$. □

Corollary 1. $||\tilde{T}||_{\tilde{P}_U} = ||\hat{T}||_{\hat{P}_U} = ||T||_U$.

Theorem 4. $sr_t(T) = \lim_{n\to\infty} ||T^n||_U^{\frac{1}{n}}$ *for any* $T \in L_U(E)$.

Proof. We recall first that $T \in L_U(E) \Rightarrow T^n \in L_U(E)$. From (1), (2), and Theorem 3 we obtain

$$sr_t(T) = sr_{\tilde{P_U}}(\tilde{T}). \tag{4}$$

Because $\widetilde{(E/N)}[\tilde{P_U}]$ is a Banach space, Gelfand's formula holds:

$$sr_{\tilde{P_U}}(\tilde{T}) = \lim_{n\to\infty} ||\tilde{T}^n||^{\frac{1}{n}}. \tag{5}$$

Finally, using (4) and (5) and the corollary above, we obtain

$$sr_t(T) = \lim_{n\to\infty} ||T^n||_U^{\frac{1}{n}}.$$

□

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