Extending the 'Oriented Smoothness Constraint' into the Temporal Domain and the Estimation of Derivatives of Optical Flow

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Abstract

Recent experimental results by *Schnörr 89* with an approach based on a simplified 'oriented smoothness constraint' show considerable improvement at expected discontinuities of the optical flow field. It thus appears justified to study whether the local gray value variation can be exploited in the temporal as well as in the spatial domain in order to achieve further improvements at discontinuities in the optical flow field associated with the image areas of moving objects in image sequences. An extension of the oriented smoothness constraint into the temporal domain is presented. In this context, a local estimation approach for the spatio-temporal partial derivatives of optical flow has been developed. This, in turn, is used to compare two approaches for the definition of optical flow.

1. Introduction

The notion of *optical flow* has been introduced in the course of studies of human perception. The 2-D optical flow can be an important clue both for the 3-D relative motion between camera and scene as well as for the relative depths of points in the scene. Since the concept of optical flow has originally been introduced only qualitatively, various possibilities exist for attempts to define it in such a manner that it becomes amenable to quantitative estimation.

Optical flow $\mathbf{u} = (u, v)^T$ is taken to describe the apparent shift of gray value structures $g(x,y,t) = g(\mathbf{x},t)$ in an image plane of a camera which moves relative to the depicted scene. Here, the 2-D vector $\mathbf{x} = (x, y)^T$ denotes a location in the image plane and t denotes time. One usually defines optical flow by the requirement that the gray value structure $g(\mathbf{x}+\mathbf{u}\delta t, t+\delta t)$ observed at time $t+\delta t$ at the location $\mathbf{x}+\mathbf{u}\delta t$ is the same as $g(\mathbf{x},t)$. This results in the so-called 'Brightness Change Constraint Equation (BCCE)' which expresses a single constraint between the components u and v of $\mathbf{u}$:

$$g_x u + g_y v + g_t = 0 \tag{1.1}$$

The derivatives of $g(\mathbf{x},t)$ with respect to x, y, and t are denoted by the corresponding subscripts. Since eq. (1.1) does not allow the estimation of both components of $\mathbf{u}$, *Horn and*

Schunck 81 formulated a minimization approach in order to estimate u and v as a function of $\mathbf{x}$:

$$\iint d\mathbf{x} \left\{ \left[g_x u + g_y v + g_t \right]^2 + \lambda^2 \left[u_x^2 + u_y^2 + v_x^2 + v_y^2 \right] \right\} => minimum \tag{1.2}$$

In order to reduce the smoothing effect of the second term - multiplied by λ^2 - in eq. (1.2) across potential discontinuities in the optical flow field, Nagel suggested to let the contribution of this smoothness term be controlled by the local gray value variation. This resulted in the following modification of eq. (1.2) :

$$\iint d\mathbf{x} \left\{ \left[g_x u + g_y v + g_t \right]^2 + \lambda^2 trace \left[(\nabla \mathbf{u})^T W (\nabla \mathbf{u}) \right] \right\} => minimum \tag{1.3}$$

where the weight matrix W is given by (*Nagel 87*) :

$$W = \frac{1}{g_x^2 + g_y^2 + 2\gamma} \begin{pmatrix} g_y^2 + \gamma & -g_x g_y \\ -g_x g_y & g_x^2 + \gamma \end{pmatrix} \tag{1.4}$$

Snyder 89 recently proved that - assuming general constraints for such an expression - the weight matrix given by eq. (1.4) is the only reasonable choice. In addition, *Schnörr 89* has just proved that the problem formulation according to eq. (1.3) with W according to (1.4) has a unique solution which depends continuously on the input data. *Schnörr 89* compared results obtained based on this oriented smoothness constraint with results based on eq. (1.2). This comparison supports the expectation that the oriented smoothness constraint contributes to a much better demarkation of the optical flow field arount moving object images than the isotropic smoothness constraint introduced by *Horn and Schunck 81*.

These encouraging results lead to the following consideration: just as the spatial gray value gradient may be used to constrain the strength and orientation of a spatial smoothness requirement for the optical flow estimates, strong temporal changes in the components of optical flow - i. e. potential discontinuities - could be estimated more reliably by the introduction of a temporal smoothness constraint which in turn would have to be controlled by the spatio-temporal gray value variation. This conjecture will be formulated quantitatively in the next section.

2. Extension of the oriented smoothness constraint into the temporal domain

The observed gray value function g(x,t) can be interpreted as a density in the three-dimensional space (x,t). The partial derivatives with respect to the variables (x,t) or (x,y,t) may be considered to form a three-component gradient vector $\nabla_{xt} = (\partial / \partial x , \partial / \partial y , \partial / \partial t)^T$. If we extend the definition of the optical flow vector **u** formally to $\acute{u} = (u, v, 1)^T$ by a third component with the constant value 1, we may write the BCCE (1.1) in the form

$$(\nabla_{xt} g)^T \acute{u} = 0 \tag{2.1}$$

This equation can be interpreted as the requirement that the optical flow vector $\acute{u}(x,t)$ is constrained to a plane defined by the normal vector $(\nabla_{xt}\, g)$ at (x,t). This leaves one degree of freedom to $\acute{u}(x,t)$, namely its orientation within this plane. We may interprete the optical flow vector $\hat{u}(x,t)$ as the tangent to a flow line through the point (x,t). If the gray value function g(x,t) does not vary with time, $\partial g / \partial t = 0$ and all tangent planes are parallel to the t-axis. In this case one would like to have flow lines which are parallel to the t-axis, too. If the gray value structure is translated with constant velocity in the image plane, the normal vector $(\nabla_{xt} g)$ will be slanted with respect to the t-axis, but one would like to retain flow lines which are still parallel to each other within tangent planes to g(x,t). This aim can be achieved by postulating that the direction of a flow line, i. e. the orientation of an optical flow vector, remains constant for infinitesimal displacements within the tangent plane to g(x,t). This is equivalent to the postulate that changes in $\hat{u}$ occur at most in the direction of the gradient $(\nabla_{xt} g)$. This in turn implies that changes in the components of $\hat{u}$, i. e. the vectors $(\nabla_{xt} u)$ and $(\nabla_{xt} v)$, are collinear with $(\nabla_{xt} g)$. These latter requirements can be expressed by demanding that the vector products between $(\nabla_{xt} g)$ and both $(\nabla_{xt} u)$ as well as $(\nabla_{xt} v)$ vanish :

$$[(\nabla_{xt} g) \times (\nabla_{xt} u)] = \begin{pmatrix} 0 & -\frac{\partial g}{\partial t} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial t} & 0 & -\frac{\partial g}{\partial x} \\ -\frac{\partial g}{\partial y} & \frac{\partial g}{\partial x} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} \end{pmatrix} = 0 \tag{2.2a}$$

and analogously for $(\nabla_{xt} v)$. Since such a requirement will in general be too strong for measurements of g(x,t) corrupted by noise, eq. (2.2a) will be replaced by the less stringent requirement that

$$\|[(\nabla_{xt} g) \times (\nabla_{xt} u)]\|^2 = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial t} \end{pmatrix} \begin{pmatrix} g_y^2 + g_t^2 & -g_x g_y & -g_x g_t \\ -g_x g_y & g_x^2 + g_t^2 & -g_y g_t \\ -g_x g_t & -g_y g_t & g_x^2 + g_y^2 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} \end{pmatrix} => minimum \tag{2.2b}$$

It is seen that the upper left 2x2 submatrix on the right hand side of eq. (2.2b) corresponds to the oriented smoothness expression for ∇u, provided g_t^2 is set equal to zero, γI with I as the 2x2 identy matrix is added and the sum is normalized by the trace of the resulting weight matrix. The generalized oriented smoothness constraint for the first component of **u** can thus be written as

$$\lambda^2 \frac{1}{2(g_x^2 + g_y^2 + g_t^2) + 3\gamma} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial t} \end{pmatrix} \begin{pmatrix} g_y^2 + g_t^2 + \gamma & -g_x g_y & -g_x g_t \\ -g_x g_y & g_x^2 + g_t^2 + \gamma & -g_y g_t \\ -g_x g_t & -g_y g_t & g_x^2 + g_y^2 + \gamma \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial t} \end{pmatrix} \tag{2.3}$$

The expression (2.3) and a corresponding expression for ∇v can thus be considered to represent a generalization of the two-dimensional oriented smoothness constraint from eqs. (1.3) and (1.4) to the three-dimensional (x,y,t)-space. The same weight matrix as in expression (2.3) has been used by *Krämer 89* in an attempt to estimate structure and motion directly from monocular image sequences.

During the development of earlier forms of the oriented smoothness constraint (see *Nagel 87*), analytical investigations of u and v as a function of the local spatio-temporal gray value variation turned out to be very useful. It thus is expected that similar investigations will advance the understanding of the implications of the generalized oriented smoothness constraint proposed in this section. As a preparation for such studies, u and v as well as their partial derivatives with respect to x,y, and t should be obtained as explicit functions of the local spatio-temporal gray value variation. This is the topic of subsequent sections.

3. Explicating local spatio-temporal gray value variations

In order to explicate the dependency of partial derivatives of g on x and t, we choose the point of interest as the origin of a local coordinate system and consider the Taylor expansion of the gray value gradient components up to first order terms. Moreover, in order to restrict this approximation to an explicitly characterized local spatio-temporal environment around the origin, this Taylor expansion is weighted by a trivariate Gaussian

$$g_x(\mathbf{x},t) = \left(g_x(0,0) + g_{xx}(0,0)\,x + g_{xy}(0,0)\,y + g_{xt}(0,0)\,t \right) \frac{c}{\left(\sigma_{\mathbf{x}}^2\, 2\pi\right)^{3/2}}\, e^{-\frac{x^2+y^2+c^2t^2}{2\sigma_{\mathbf{x}}^2}} \tag{3.1}$$

where the ratio $\sigma_{\mathbf{x}} / \sigma_t = c$ has the dimension of a velocity. This approximation implies a choice for σ_t which allows to disregard contributions from the infinite past or future as being negligibly small. The spatial derivatives with respect to y and t can be written analogously. Similarly, we introduce the following representation for the first component u of the optical flow :

$$u(\mathbf{x},t) = \left(u(0,0) + u_x(0,0)\,x + u_y(0,0)\,y + u_t(0,0)\,t \right) \frac{c}{\left(\sigma_{\mathbf{x}}^2\, 2\pi\right)^{3/2}}\, e^{-\frac{x^2+y^2+c^2t^2}{2\sigma_{\mathbf{x}}^2}} \tag{3.2}$$

and an analogous one for the second component v. It should be noted that eq. (3.2) comprises partial derivatives of u and v with respect to time in addition to those with respect to the spatial coordinates x and y.

In order to simplify subsequent derivations, we introduce a more compact notation where the arguments (0,0) of the partial derivatives have been dropped : $G_x = (g_x , g_{xx} , g_{xy} , g_{xt})^T$;

$G_y = (g_y\ ,\ g_{yx}\ ,\ g_{yy}\ ,\ g_{yt}\ )^T$; $G_t = (g_t\ ,\ g_{tx}\ ,\ g_{ty}\ ,\ g_{tt}\ )^T$. In analogy we introduce $U = (u\ ,u_x\ ,u_y\ ,u_t\ )^T$ and $V = (v\ ,v_x\ ,v_y\ ,v_t\ )^T$. Similarly, we define

$$X = \frac{c}{\left(\sigma_x^2 2\pi\right)^{3/2}} \begin{pmatrix} 1 \\ x \\ y \\ t \end{pmatrix} e^{-\frac{x^2+y^2+c^2t^2}{2\sigma_x^2}} \tag{3.3}$$

Using these conventions, we may write the modified BCCE with explicated dependency on **x**, on t, and on the extent of the environment :

$$\left(G_x^T X\right)\left(U^T X\right) + \left(G_y^T X\right)\left(V^T X\right) + \left(G_t^T X\right) = 0 \tag{3.4}$$

Rather than requiring that this form of the BCCE should be valid at each space-time location (**x**,t), we postulate that the integral of the square of eq. (3.4) should be minimized :

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ \left[G_x^T X \bullet U^T X + G_y^T X \bullet V^T X + G_t^T X \right]^2 \right\} => minimum \tag{3.5}$$

The trivariate Gaussian in X will enforce convergence of the integral without the necessity to introduce sharp boundaries for the integration region. It appears reasonable to estimate the spatio-temporal derivatives G_x , G_y , and G_t by a convolution of g(**x**,t) with the corresponding derivatives of the trivariate Gaussian in X. The choice of σ_x and σ_t thus determines the extent of the spatio-temporal domain which contributes significantly to the integral in eq. (3.5). 'Local estimation' of U and V is understood to refer to the domain around the origin determined by σ_x and σ_t .

4. Equations for the unknown components of U and V

The partial derivatives contained in G_x , G_y , and G_t , taken at the origin of the coordinate system, are observed constant values. The explication of the dependency on **x** and t through the expression for X defined by eq. (3.3) implies that the components of U and V are considered to be constant unknown values associated with the origin (**0**,0) of the local coordinate system. Setting the partial derivatives of the integral in eq. (3.5) with respect to the unknown components of U and V to zero results in the following system of eight linear equations, written here as a set of two (4x1)-vector equations :

$$\int\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ G_x^T X \bullet G_x^T X \bullet XX^T \right\} U + \int\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ G_y^T X \bullet G_x^T X \bullet XX^T \right\} V = -\int\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ G_t^T X \bullet G_x^T X \bullet X \right\} \tag{4.1a}$$

and

$$\int\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ G_x^T X \bullet G_y^T X \bullet XX^T \right\} U + \int\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ G_y^T X \bullet G_y^T X \bullet XX^T \right\} V = -\int\int_{-\infty}^{+\infty} d\mathbf{x}dt \left\{ G_t^T X \bullet G_y^T X \bullet X \right\} \tag{4.1b}$$

The coefficients of this system of linear equations in U and V are integrals with each integrand consisting of a trivariate Gaussian, multiplied by monomials in x, y and t of up to fourth order. The expansion of the coefficient integrals yields for the general case on the left hand side with $\eta, \zeta \in \{x, y\}$ and $\{G_\eta^T X \bullet X^T G_\zeta \bullet XX^T\} = \{(G_\eta^T XX^T G_\zeta) \bullet XX^T\}$:

$$\left\{ \frac{c^2}{(\sigma_x^2 2\pi)^3} \bullet \begin{pmatrix} (G_\eta^T XX^T G_\zeta)\bullet 1 & (G_\eta^T XX^T G_\zeta)\bullet x & (G_\eta^T XX^T G_\zeta)\bullet y & (G_\eta^T XX^T G_\zeta)\bullet t \\ (G_\eta^T XX^T G_\zeta)\bullet x & (G_\eta^T XX^T G_\zeta)\bullet x^2 & (G_\eta^T XX^T G_\zeta)\bullet xy & (G_\eta^T XX^T G_\zeta)\bullet xt \\ (G_\eta^T XX^T G_\zeta)\bullet y & (G_\eta^T XX^T G_\zeta)\bullet yx & (G_\eta^T XX^T G_\zeta)\bullet y^2 & (G_\eta^T XX^T G_\zeta)\bullet yt \\ (G_\eta^T XX^T G_\zeta)\bullet t & (G_\eta^T XX^T G_\zeta)\bullet tx & (G_\eta^T XX^T G_\zeta)\bullet ty & (G_\eta^T XX^T G_\zeta)\bullet t^2 \end{pmatrix} \bullet e^{-\frac{2(x^2+y^2+c^2t^2)}{2\sigma_x^2}} \right\} \tag{4.2}$$

The evaluation of the integral over each component of this 4x4 matrix presents no problems. Space limits prevent an illustration of intermediate results for this evaluation. The integrands on the right hand side of eqs. (4.1) can be treated analogously.

Using these coefficients, it is straightforward to write the system of linear equations (4.1) for the unknown entities u, v, and their partial derivatives. It turns out, however, that attempts at the symbolic solution of this system of equations result in rather involved expressions. In order to provide some insight into the structure of this system of linear equations, a smaller system will be discussed here. It is obtained by omitting the derivative with respect to time t in the Taylor expansion of eqs. (3.1) as well as (3.2) and by omitting the Gaussian weight function depending on t. The system of linear equations resulting from these steps are formally equal to those in eqs. (4.1) although the evaluation of the integrals yields slightly different values for the coefficients.

5. Equations for u, v, and their partial derivatives with respect to x and y

Since there will be no Gaussian weighting function with respect to time and thus no σ_t, the distinction between σ_t and σ_x will not be necessary. Henceforth, σ without a subscript will be used. Only u, v, u_x, u_y, v_x, and v_y are retained as unknowns. The integrals are evaluated in analogy to the procedure discussed previously.

In order to simplify the subsequent presentation, it is assumed that the local coordinate system has been aligned with the directions of principal curvatures of g(x,t), i. e. $g_{xy} = 0$. Moreover, in all expressions it turns out that whenever a second partial derivative of g(x,t) appears, it does so always in combination with a factor $\sigma/2$. It thus is advantageous to introduce the convention - again with $\zeta \in \{x, y\}$ - that $g_{\zeta\zeta}^* = (\sigma/2)\, g_{\zeta\zeta}$ and $g_{t\zeta}^* = (\sigma/2)\, g_{t\zeta}$. By this definition, the second partial derivatives with an asterisk have the same dimension - namely [gray value]$\bullet$[length]$^{-1}$ or [gray value]$\bullet$[time]$^{-1}$, respectively, - as the first partial derivatives. This can be useful while checking some of the formulas. For simplicity, the asterisk is henceforth dropped and all second partial derivatives are taken to carry implicitly a factor σ/2 . Similarly, all first partial derivatives of u and v are taken to carry implicitly a factor σ/2 . Using these conventions, the linear system of equations for the components of U and V can be written as follows, after having exchanged some rows and columns in order to group u and v as well as their first derivatives together:

$$\begin{pmatrix} g_x^2+g_{xx}^2 & g_xg_y & 2\frac{\sigma}{2}g_xg_{xx} & 0 & g_yg_{xx} & g_xg_{yy} \\ g_xg_y & g_y^2+g_{yy}^2 & g_yg_{xx} & g_xg_{yy} & 0 & 2g_yg_{yy} \\ 2g_xg_{xx} & g_yg_{xx} & g_x^2+3g_{xx}^2 & 0 & g_xg_y & g_{xx}g_{yy} \\ 0 & g_xg_{yy} & 0 & g_x^2+g_{xx}^2 & g_{xx}g_{yy} & g_xg_y \\ g_yg_{xx} & 0 & g_xg_y & g_{xx}g_{yy} & g_y^2+g_{yy}^2 & 0 \\ g_xg_{yy} & 2g_yg_{yy} & g_{xx}g_{yy} & g_xg_y & 0 & g_y^2+3g_{yy}^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ u_x \\ u_y \\ v_x \\ v_y \end{pmatrix} = -\frac{32}{9}\pi\sigma^2 \begin{pmatrix} \frac{3}{4}g_tg_x+g_{tx}g_{xx} \\ \frac{3}{4}g_tg_y+g_{ty}g_{yy} \\ g_tg_{xx}+g_xg_{tx} \\ g_xg_{ty} \\ g_yg_{tx} \\ g_tg_{yy}+g_yg_{ty} \end{pmatrix} \quad (5.1)$$

It turns out that the matrix in eq. (5.1) has only rank five rather than the required full rank of six. The eigenvector of this matrix corresponding to its eigenvalue 0 has the form

$$e_0 = (g_y, -g_x, 0, g_{yy}, -g_{xx}, 0)^T \quad (5.2)$$

A test using the components of e_0 as factors reveals that not only the row vectors of the coefficient matrix, but both left and right hand sides of eqs. (5.1) are linearly dependent, i. e. either the first, second, fourth, or fifth equation can be deleted as redundant. Deletion of, say, the fifth equation leaves a system of five equations with six unknowns, i. e. five unknowns can be determined up to a linear function of the sixth one, say v_x .

Assume that a special solution is known for the unknowns. Since the eigenvector e_0 is orthogonal to all row vectors of the coefficient matrix of eq. (5.1), we may define the general solution of this system of equations by adding a multiple w of e_0 to this special solution, i. e.

$$(\hat{u}, \hat{v}, \hat{u}_x, \hat{u}_y, \hat{v}_x, \hat{v}_y)^T = (u, v, u_x, u_y, v_x, v_y)^T + w\,e_0 . \quad (5.3)$$

If we now treat the newly defined entities on the left hand side of eq. (5.3) as unknowns, we can set $\hat{v}_x = v_x - wg_{xx} = 0$ by choosing w appropriately. This removes the fifth column from the left hand side of eq. (5.1) and enables us to solve for the five remaining variables $\hat{u}$, $\hat{v}$, $\hat{u}_x$, $\hat{u}_y$, and $\hat{v}_y$. Since the resulting expressions in the derivatives of g are rather lengthy, space limitations force us to restrict the discussion of the results to a particularly interesting case.

Eq. (5.3) in combination with eq. (5.2) shows that the solutions for u_x and v_y do not depend on the free parameter w. It is possible, therefore, to determine the divergence of **u** directly in terms of the spatio-temporal derivatives of the grayvalue function g(**x**,t) :

$$div\,\mathbf{u} = u_x + v_y = \frac{-8\pi\sigma}{9\left[(g_x^2 g_{yy}^2 + g_y^2 g_{xx}^2)^2 + 4\,g_{xx}^4 g_{yy}^4\right]} \Bigg\{ 8\,(g_{tx} g_x g_{yy} + g_{ty} g_y g_{xx}) \left[g_{xx}^3 (g_y^2 - g_{yy}^2) + g_{yy}^3 (g_x^2 - g_{xx}^2) \right]$$

$$+ g_t g_{xx} \left[(g_y^2 + g_{yy}^2)\,(g_x^2 g_{yy}^2 + g_y^2 g_{xx}^2) + 2\,g_{yy} (g_x^2 g_{yy}^3 + g_y^2 g_{xx}^3) \right]$$

$$+ g_t g_{yy} \left[(g_x^2 + g_{xx}^2)\,(g_x^2 g_{yy}^2 + g_y^2 g_{xx}^2) + 2\,g_{xx} (g_x^2 g_{yy}^3 + g_y^2 g_{xx}^3) \right] \Bigg\} \quad (5.4)$$

In this expression, the implicit factor of $\sigma/2$ in the definition of u_x and v_y has already been made explicit again and has been compensated against the same factor on the right hand side of eq. (5.4).

Another possibility to cope with the degeneracy of the linear system of eq. (5.1) consists in supplementing the minimization problem by a regularization term. Then, the inverse of the correspondingly supplemented coefficient matrix according to eq. (5.1) exists. Using a symbolic algebra programming system like MAPLE, it can be computed without problem. The resulting expressions, however, are very lengthy and are not immediately amenable to significant simplifications. Therefore, they will not be presented here.

Alternatively, one could fix either one or a linear combination of the unknowns by an additional assumption. Since it is desirable to introduce any additional assumption in a manner invariant to rotations of the coordinate system, it is suggested to demand that the shear tensor should vanish.

$$shear(\mathbf{u}) \;=\; \frac{1}{2} \begin{pmatrix} u_x - v_y & u_y + v_x \\ u_y + v_x & -(u_x - v_y) \end{pmatrix} \;=\; 0 \quad (5.5)$$

This implies two additional constraint equations, leaving only four unknowns. The postulate expressed by eq. (5.5) can be incorporated into the minimization problem by adding the following terms to the integrand :

$$\mu^2\,(u_x - v_y)^2 \;+\; \nu^2\,(u_y + v_x)^2 \quad (5.6)$$

where μ^2 and ν^2 represent Lagrange multipliers. The contribution of these terms drop out of the equations obtained by forming the partial derivatives of the expanded minimization problem with respect to U and V since they contain a factor of either $(u_x - v_y)$ or $(u_y + v_x)$ which vanishes. One obtains a system of five equations of rank 5 for four unknowns. A pseudo-inverse formalism can be used to solve for the unknowns. The resulting expressions are involved.

During a recent discussion, I learned that Koenderink and coworkers also investigate the direct estimation of optical flow and its partial derivatives, but taking into account higher than second order spatio-temporal derivatives of the gray value distribution. In both cases, a Gaussian is used to localize the estimation procedure (*Koenderink 89*).

6. Comparison with the definition of optical flow by Girosi et al. 89

The following discussion concentrates on the direct estimation of u and v. The BCCE in itself does not provide sufficient constraints in order to estimate both u and v. If, however, the gray values vary sufficiently as a function of x and y, it could be shown that - by taking into account higher order spatial derivatives of g(x,t) - both components of **u** can be estimated directly (*Nagel 83*).

Recently, *Girosi et al. 89* discussed another possibility to directly estimate u and v which may be considered to be a special case of the approach investigated, for example, in *Nagel 83+87*. The difference consists in the fact that *Girosi et al. 89* define optical flow not by something like eq. (1.1) but as the solution of the vector equation

$$\frac{d}{dt}\nabla g = \begin{pmatrix} \frac{d}{dt} g_x \\ \frac{d}{dt} g_y \end{pmatrix} = \begin{pmatrix} g_{xx}\frac{dx}{dt} + g_{xy}\frac{dy}{dt} + g_{xt} \\ g_{yx}\frac{dx}{dt} + g_{yy}\frac{dy}{dt} + g_{yt} \end{pmatrix} = 0 \tag{6.1}$$

which results in the following system of equations :

$$g_{xx}u^* + g_{xy}v^* = -g_{xt} \qquad and \qquad g_{xy}u^* + g_{yy}v^* = -g_{yt} \tag{6.2}$$

Here, $\mathbf{u}^* = (u^*, v^*)^T$ has been used in order to emphasize the difference between the definition of optical flow $\mathbf{u}^*$ according to eq. (6.1) and the one of **u** according to eq. (1.1). The solution to this system of equations is

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = -\frac{1}{g_{xx}^2 g_{yy}^2} \begin{pmatrix} g_{yy}^2 & 0 \\ 0 & g_{xx}^2 \end{pmatrix} \begin{pmatrix} g_{xt} \\ g_{yt} \end{pmatrix} \tag{6.3}$$

where the convention of setting $g_{xy} = 0$ has been used. This approach captures only part of the situations which allow to estimate locally both components of the optical flow **u**, namely only those situations with a non-singular Hessian. For curved lines of maximum gray value slope, in particular for 'gray value corners' characterized as points of maximum curvature in such locus lines of maximum slope, the gradient is maximum which implies that the corresponding second partial derivative vanishes. In such cases, the Hessian becomes singular and the approach expressed by eq. (6.1) breaks down, whereas the one of *Nagel 83* provides a useful estimate in such cases.

7. Conclusion

The extension of the oriented smoothness constraint into the temporal domain is expected to facilitate a better detection and localization of discontinuities in the optical flow field.

It has been shown that, by appropriate modeling of the local gray value variation, it becomes possible - at least in theory - to estimate not only both components of optical flow, but in addition some linear combination of its partial derivatives with respect to x and y. In particular, it becomes possible to estimate div(u) directly from spatio-temporal gray value variations. If the isotropic smoothness term introduced by *Horn and Schunck 81* is included into the model developed here, one does not need to make the assumption of vanishing shear(u) in order to determine all spatial partial derivatives of u and v. Obviously, experimental investigations are needed in order to test the reliability of these approaches.

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