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Extending the Transport Theorem to Rough Domains of Integration

Transport theorems, such as that named after Reynolds, are an important tool in the field of continuum physics. Recently, Seguin and Fried used Harrison's theory of differential chains to establish a transport theorem valid for evolving domains that may become irregular. Evolving irregular domains occur in many different physical settings, such as phase transitions or fracture. Here, emphasizing concepts over technicalities, we present Harrison's theory of differential chains and the results of Seguin and Fried in a way meant to be accessible to researchers in continuum physics. We also show how the transport theorem applies to three concrete examples and approximate the resulting terms numerically. Furthermore, we discuss how the transport theorem might be used to weaken certain basic assumptions underlying the description of continua and the challenges associated with doing so. [DOI: 10.1115/1.4026910]

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1 Introduction

Most students of mechanics are aware of the Reynolds [1] transport theorem, which can be stated as follows: Let $\mathcal{R}$ be a spatial region that convects with a moving body whose evolution is described by a velocity field $\mathbf{v}$, and let φ be a time-dependent scalar field defined on $\mathcal{R}$. The derivative with respect to time of the integral of φ over $\mathcal{R}$ is then given by

$$\overline{\int_{\mathcal{R}} \varphi \, \mathrm{dv}}^{\cdot} = \int_{\mathcal{R}} \varphi' \, \mathrm{dv} + \int_{\partial\mathcal{R}} \varphi \mathbf{v} \cdot \boldsymbol{\nu} \, \mathrm{da}, \tag{1}$$

where the superposed dot denotes the time-derivative, the prime denotes the partial time-derivative, and $\boldsymbol{\nu}$ is the exterior unit-normal to the boundary $\partial\mathcal{R}$ of $\mathcal{R}$. Eq. (1) is known as the Reynolds transport identity. The proof of the Reynolds [1] transport theorem is not inherently difficult and can be found in many places, such as the book by Gurtin et al. [2]. Among other things, this theorem is used in the conversion of integral statements of balance to local field equations.

For a time-dependent control volume $\mathcal{P}$ with boundary $\partial\mathcal{P}$ that migrates with a scalar normal-velocity $V_{\partial\mathcal{P}}$, in the direction of its exterior unit-normal $\boldsymbol{\nu}$, potentially different from the normal component $\mathbf{v} \cdot \boldsymbol{\nu}$ of the restriction of velocity $\mathbf{v}$ of the underlying body to $\partial\mathcal{P}$, the counterpart of Eq. (1) is

$$\overline{\int_{\mathcal{P}} \varphi \, \mathrm{dv}}^{\cdot} = \int_{\mathcal{P}} \varphi' \, \mathrm{dv} + \int_{\partial\mathcal{P}} \varphi V_{\partial\mathcal{P}} \, \mathrm{da}. \tag{2}$$

If $\mathcal{P}$ is a convecting region, $V_{\partial\mathcal{P}} = \mathbf{v} \cdot \boldsymbol{\nu}$ and Eq. (2) reduces to Eq. (1). The Reynolds transport identity (1) is therefore a special case of Eq. (2). To establish Eq. (2), first choose a "virtual velocity" $\mathbf{v}$ such that $\mathbf{v} \cdot \boldsymbol{\nu} = V_{\partial\mathcal{P}}$. Using this virtual velocity, the argument used to establish Eq. (1) can then be emulated to obtain Eq. (2). However, it is important to keep in mind that the virtual velocity associated with $V_{\partial\mathcal{P}}$ is not unique. That there are infinitely many such velocities with the same regularity as $V_{\partial\mathcal{P}}$ follows from a standard argument relying on the tubular neighborhood theorem and bump functions (see, for example, Lee [3].)

More generally, there is a transport theorem for a k-dimensional domain—that is, a k-dimensional manifold—evolving in an n-dimensional Euclidean space. In this situation, a distinction can, and should, be made between a convecting domain, such as that appearing in Eq. (1), and a domain whose evolution is described intrinsically, such as that appearing in Eq. (2). To describe the evolution of a k-dimensional domain intrinsically when $k < n$, it is necessary to specify not only the scalar normal-velocity of its boundary in the direction of its exterior unit-normal but also the (vector) normal velocity of the domain in the direction normal to the domain—meaning that at each point on the domain, the (vector) normal velocity is orthogonal to the tangent space of the domain at that point. When $k = n$, only the scalar normal-velocity is required. The transport theorem for a k-dimensional domain contains three terms. Of these, two are analogous to those appearing in Eq. (1) and Eq. (2) and the remaining one involves the extrinsic curvature of the manifold—that is, the curvature of the manifold associated with how it sits in the ambient Euclidean space. Equations (1) and (2) only have two terms because the extrinsic curvature vanishes when the dimensions of the manifold and the ambient space coincide. For a convecting k-dimensional domain evolving in an n-dimensional Euclidean space, the relevant transport theorem was established by Betounes [4]. Of particular interest to workers in continuum physics are the cases $k = 1, 2, 3$ with $n = 3$. Gurtin et al. [5] established the case analogous to Eq. (2) for $k = 2$ and $n = 3$ and it appears that an extension of their proof is possible for any evolving k-dimensional domain in an n-dimensional Euclidean space.

The discussion thus far pertains to evolving domains that are both smooth and smoothly evolving. However, it is easy to imagine situations where smoothness breaks down. For example, a domain might become perforated, split into pieces, or have a boundary that develops corners or cusps. Even worse, the fractal dimension of a domain might change with time. A transport theorem for such domains could be useful in various settings. Gurtin's [6] book on configurational forces illustrates applications of the transport theorem involving interfaces and other defects. Catalan et al. [7] discuss rough and fractal domain walls in thin films of ferromagnets, ferroelectrics, and multiferroics. Stanley [8] explores how fractals appear in physics and biology. In the calculus of variations, it is sometimes necessary to compute the variation of an integral in which both the integrand and the domain of integration vary (with a scalar parameter that may be construed

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with time); for a survey of variational problems that involve minimizing over a class of domains, see Bucur and Buttazzo [9].

Seguin and Fried [10] recently used Harrison's [11] theory of differential chains to establish a transport theorem for evolving domains that exhibit the kind of irregularities described in the previous paragraph. The goal of the present paper is to present the most salient features of that theorem together with simple applications, and to thereby demonstrate its utility. The paper is organized as follows. In Sec. 2, we review the essential features of Harrison's [11] theory of differential chains. Specifically, after describing the most important properties of differential chains, we explain how the space of differential chains is constructed. Various differential operators on the space are then introduced. In Sec. 3, we introduce the space of regularly evolving chains and present the generalized transport theorem of Seguin and Fried [10]. To illustrate the key steps in obtaining this transport theorem, a proof is given in one space dimension. Section 4 considers a special class of regularly evolving chains, called convecting chains, and states the form of the transport theorem for such chains. Then, in Sec. 5, we discuss various classical transport theorems that are consequences of the general transport theorem. In Sec. 6, three concrete examples are used to demonstrate applications of the transport theorem. The first example involves calculating the area of an evolving region whose boundary is a fractal set. The terms that arise upon applying the transport theorem are approximated numerically and compared with the values obtained by approximating the time-derivative of the area directly using two different numerical schemes. The second example involves calculating the rate of change of the circulation around a convecting fractal curve. Once again, the terms encountered upon applying the transport theorem are approximated numerically. The third and final example involves computing the rate of change of the energy of a compressible neo-Hookean solid undergoing a deformation that results in a change of topology—more specifically, the formation of a hole. The rate of change of the energy is obtained by applying the generalized transport theorem and the resulting terms are approximated numerically. Finally, in Sec. 7, we discuss the possible application of the roughened transport theorem to the study of nonsmooth bodies in continuum mechanics and the difficulties associated with this problem.

2 The Theory of Differential Chains

Accessible surveys of the theory of differential chains, as presented in Harrison's [11] comprehensive treatise, are provided by Marzocchi [12] and Rodnay and Segev [13]. Here, we present a heuristic account of only those aspects of the theory necessary to understand the statement of the transport theorem of Seguin and Fried [10].

We begin with a discussion of the relationship between differential chains and domains of integration, leaving the definition of the space of differential chains to the second subsection. In the third subsection, we discuss the B^r norm that is used to define the space of differential chains. In the final subsection, several operators on the space of differential chains that are needed later are introduced.

2.1 Differential Chains and Domains. A differential chain generalizes the notion of a domain of integration. This feature of differential chains will be highlighted throughout this review. A differential chain J has both a dimension and a class. The set of all differential chains of dimension k and class B^r is denoted by $\widehat{\mathcal{B}}_k^r$. Here both k and r are natural numbers. The dimension of a differential chain has a meaning similar in spirit to that of the dimension of a manifold. In particular, we will see that differential chains can represent manifolds in a particular sense. The class of a differential chain describes the extent to which a domain may be irregular, which increases with the value of r. Elements of $\widehat{\mathcal{B}}_k^r$ are called differential k-chains of class B^r, differential k-chains or, sometimes, when there is no basis for confusion, chains.

The space $\widehat{\mathcal{B}}_k^r$ is not only a linear space but is also a Banach space. Although this property is inconsistent with the idea that chains are like domains of integration, it forms the basis for one of the great strengths of the theory of differential chains. The norm on the space $\widehat{\mathcal{B}}_k^r$, called the B^r norm, is discussed in detail in Subsec. 2.3. This norm is very important in that it allows the theory to work. There have been other attempts to find a Banach space, whose elements can be thought of as domains of integration, such as the spaces of sharp and flat chains developed by Whitney [14], but these spaces have various drawbacks that are not shared by the space of differential chains.[1] It is necessary to consider differential chains of various dimensions as well as classes and, hence, to avoid restricting attention to any particular values of k and r.

Since $\widehat{\mathcal{B}}_k^r$ is a Banach space, we may consider its (topological) dual $\mathcal{B}_k^r := (\widehat{\mathcal{B}}_k^r)'$, which is called the space of cochains. Harrison [11] showed that this space is isometric to the space of differential k-forms of a certain regularity, the regularity depending on the value of r—specifically, the larger the value of r, the smoother the differential form. Since the isometry between cochains and differential forms is natural, we do not explicitly denote the isometry that associates a differential form with a cochain and treat $\widehat{\mathcal{B}}_k^r$ as a space of differential forms. For $r \geq 1$, $\widehat{\mathcal{B}}_k^r$ consists of differential forms of class C^{r-1} whose $r-1$ directional derivatives are Lipschitz continuous. The space $\widehat{\mathcal{B}}_k^0$ consists of bounded differential k-forms.

In keeping with the idea that chains represent domains of integration, we denote the dual pairing between a differential form and a chain with an integral symbol decorated by a slash:

$$\int\!\!\!\!\!\!\diagup_J \omega := \omega(J) \quad \text{for all } J \in \widehat{\mathcal{B}}_k^r,\ \omega \in \mathcal{B}_k^r \tag{3}$$

The left-hand side of Eq. (3) should be read as "the integral of ω over J."

The notion that chains correspond to domains of integration can be made precise. Specifically, Harrison [11] proved that given any compact, Lipschitz manifold $\mathcal{M}$ of dimension k, there is a k-chain of class B^1 that represents $\mathcal{M}$ in the sense that

$$\int_{\mathcal{M}} \omega = \int\!\!\!\!\!\!\diagup_J \omega \quad \text{for all } \omega \in \mathcal{B}_k^1 \tag{4}$$

where the integral over $\mathcal{M}$ is to be interpreted in the classical sense of Riemann. This result can be interpreted as saying that chains may represent domains in the context of integration.

It is important to realize that not all chains represent classical domains—that is, manifolds. For example, Harrison [15] showed that chains may represent fractals, such as the Sierpinski triangle.

2.2 Constructing the Space of Chains. We have thus far discussed how chains generalize the notion of a domain of integration. Although a detailed account of the technical features of the construction of the space of chains is beyond the scope of the present work, a rough picture of the characteristics of that space is not only necessary but also provides insight. Moreover, differential chains are constructed in a very geometric way and, thus, are visualizable. Here, we outline the main ideas underlying the construction, always keeping in mind the connection between chains and domains.

The construction of the space $\widehat{\mathcal{B}}_k^r$ norm of differential k-chains of class B^r can be broken into four stages beginning with the introduction of simple chains, which are subsequently used to build

[1] For a discussion of these drawbacks, see the introduction of Harrison [11].

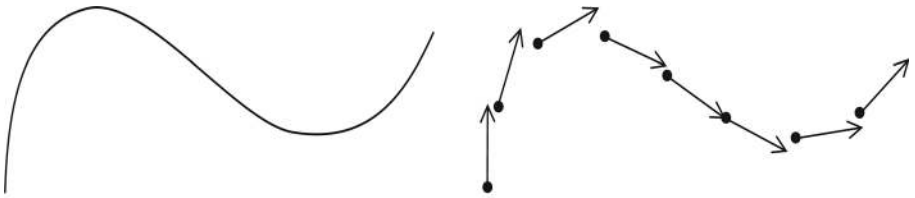

Fig. 1 The curve γ and an approximation of it by a Dirac chain. (*a*) The curve γ. (*b*) A depiction of A_3, which approximates γ.

more complicated objects until, at the final stage, the full space of differential chains emerges. At each stage, we will emphasize how the chains involved can be thought of as domains.

We work in an n-dimensional Euclidean (point) space $\mathcal{E}$ with associated vector space $\mathcal{V}$. A simple k-chain consists of a point p in $\mathcal{E}$ and a skew k-form α and is designated by the pair $(p; \alpha)$. The pair $(p; \alpha)$ should be thought of as the skew form α attached to the point p. Even this simple object can be thought of as a domain, an *infinitesimal* domain. This can be seen by considering specific cases of k. If $k = 1$, α can be identified with a vector, which represents an infinitesimal line element. If $k = 2$, α represents an infinitesimal area element. More generally, a skew k-form represents an infinitesimal k-dimensional content element.[2] Although these infinitesimal elements are oriented, we will not stress this fact here. Thus, the simple chain $(p; \alpha)$ can be thought of as an infinitesimal k-dimensional domain located at p.

Since $(p; \alpha)$ is a chain, it can be paired with a differential form. Put another way, it is possible to integrate, in the sense of Eq. (3), a differential form ω over $(p; \alpha)$, giving

$$\int_{(p;\alpha)} \omega = \omega(p) \cdot \alpha \tag{5}$$

where the dot denotes the normalized inner-product between skew k-forms. Specifically, the inner-product is normalized so that, for example, in the case $n = 3$ and $k = 2$, if $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis for $\mathcal{V}$, then $\{\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1\}$ is an orthonormal basis for the space of skew 2-forms.

The next step in the construction of the space of differential chains involves considering linear combinations

$$A = \sum_{i \in I} (p_i; \alpha_i) \tag{6}$$

of simple k-chains. Here I is a finite index set and $(p_i; \alpha_i)$ is, for each i in I, a simple chain. It is assumed that chains are linear in their second argument, meaning that

$$a(p; \alpha) + b(p; \beta) = (p; a\alpha + b\beta) \tag{7}$$

where a and b are scalars and α and β are skew k-forms, which is why no loss of generality results by choosing each coefficient in the linear combination (6) to equal unity. Chains constructed in accord with Eq. (6) are called Dirac k-chains. The best way to visualize a Dirac chain is to consider it as a finite collection of simple chains—that is, a finite collection of points in space to each of which a skew k-form is attached. A Dirac chain can also be thought of as a discrete approximation of a classical domain. To see this, consider a smooth curve in $\mathcal{E}$ of length L; see Fig. 1(*a*). Let $\gamma : [0, L] \to \mathcal{E}$ be an arc-length parameterization of said curve. Fix a natural number m. For each $i = 0, \ldots, 2^m - 1$, the simple chain $(\gamma(iL2^{-m}); \dot{\gamma}(iL2^{-m})L2^{-m})$ can be thought of as an infinitesimal line element attached to the point $\gamma(iL2^{-m})$ that is tangent to the curve. Thus, the Dirac 1-chain

$$A_m := \sum_{i=0}^{2^m-1} (\gamma(iL2^{-m}); L2^{-m}\dot{\gamma}(iL2^{-m})) \tag{8}$$

can be thought of as a discrete approximation of the curve; see Fig. 1(*b*).

Since the action of a differential form on chains is linear, it follows easily from Eq. (5) that the action of a differential form ω on a Dirac chain of the form Eq. (6) yields

$$\int_A \omega = \sum_{i \in I} \omega(p_i) \cdot \alpha_i \tag{9}$$

If the Dirac chain under consideration represents a discrete approximation of a domain, then the sum on the right-hand side of Eq. (9) can be thought of as a Riemann sum. In fact, on recalling the sequence $\{A_m, m = 1, 2, \ldots\}$ of Dirac chains A_m defined in Eq. (8), the limit given by

$$\lim_{m\to\infty} \int_{A_m} \omega = \lim_{m\to\infty} \sum_{i=0}^{2^m-1} L2^{-m}\omega(\gamma(iL2^{-m})) \cdot \dot{\gamma}(iL2^{-m}) \tag{10}$$

is seen to be a limit of Riemann sums and, therefore, converges to the classical integral of ω over the curve parameterized by γ.

The third step in the construction of the space of differential chains involves introducing a norm on the linear space of Dirac chains. For each natural number r, it is possible to define the B^r norm. In the next subsection, we provide some insight regarding the nature of this norm.

Finally, the completion of the linear space of Dirac k-chains with respect to the B^r norm is the space $\widehat{\mathcal{B}}_k^r$ of differential k-chains of order B^r. As indicated previously, chains may represent classical domains—in the context of integration, as described by Eq. (4)—and nonclassical domains, such as fractals. Although a chain may represent a domain, it is important to keep in mind that a chain is more than a domain. Domains are sets, chains are not. Although it is possible to associate to each chain a set, called its support, this set does not completely characterize the chain.[3] In fact, given any compact set Ω, there are infinitely many chains with support Ω.

2.3 On the B^r Norm. As in the definition of any Banach space, the norm used to define $\widehat{\mathcal{B}}_k^r$ is of pivotal importance. The subject of geometric integration theory has a history of using norms that do not easily lend themselves to interpretation, as is evident on considering the flat, sharp, and natural norms introduced by Federer [16] and Whitney [14]. As Rodnay and Segev [13] commented, actual computations using these norms can be complicated. Unfortunately, the B^r norm is no different. In this subsection, the B^r norm is specified explicitly for small values of r and a strategy for generalizing the $\mathcal{B}^r$ norm for larger r is described. An alternate characterization of this norm is also provided and the

[2]Also known as a k-dimensional volume element.

[3]The support of a chain is always a compact set.

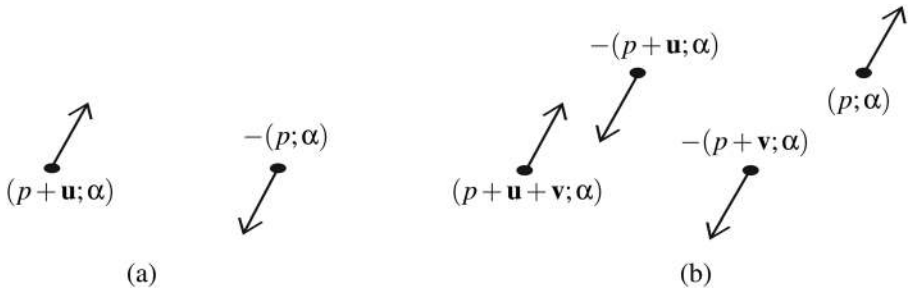

Fig. 2 (*a*) Depiction of $\Delta_{\mathbf{u}}(p;\alpha)$. The arrow on the right-hand side illustrates $-(p;\alpha)$, the opposite of the original chain $(p;\alpha)$, and the arrow on the left-hand side illustrates $(p+\mathbf{u};\alpha)$, which is a translation of $(p;\alpha)$. (*b*) Depiction of $\Delta_{\mathbf{v}}\Delta_{\mathbf{u}}(p;\alpha)$. The arrow on the upper right depicts the original chain $(p;\alpha)$ and the other arrows illustrate chains obtained from this one by translation and inversion.

utility of this characterization for establishing results concerning differential chains is demonstrated in the context of an example.

To begin, the notion of the mass norm $|\alpha|$ of a skew k-form α is required. This norm takes the form

$$|\alpha| := \inf\left\{\sum_{i\in I}|\alpha_i|_{\mathrm{e}} \mid \alpha=\sum_{i\in I}\alpha_i,\ \alpha_i \text{ simple}\right\} \tag{11}$$

where $|\alpha_i|_{\mathrm{e}}$ denotes the Euclidean norm of α_i and I is a finite index set. The B^0 norm of a simple chain A is given by

$$\|A\|_{B^0} := \inf\left\{\sum_{i\in I}|\alpha_i| \mid A=\sum_{i\in I}(p_i;\alpha_i)\right\} \tag{12}$$

It is important to emphasize that the infimum in Eq. (12) is taken over all possible ways of representing A as a finite sum of simple chains. The B^0 norm coincides with the mass norm defined on chains (see Whitney [14]). This norm is insensitive to the points p_i, with i in I, used to define A (see Eq. (6))—insensitive in the sense that if

$$A'=\sum_{i\in I}(p_i';\alpha_i) \tag{13}$$

is another Dirac chain, then $\|A\|_{B^0}=\|A'\|_{B^0}$. Because of this, the B^0 norm does not appear to be very useful.

The B^1 norm, on the other hand, is sensitive to the points used to define A. To specify this norm, first define for every vector $\mathbf{u}$

$$\Delta_{\mathbf{u}}(p;\alpha) := (p+\mathbf{u};\alpha)-(p;\alpha) \tag{14}$$

This operation, called the difference through $\mathbf{u}$, on the simple chain $(p,\ \alpha)$ is depicted in Fig. 2(*a*) and can be defined on any Dirac chain by requiring that it be linear. One possible way to think of $\Delta_{\mathbf{u}}(p,\alpha)$ is as a dipole. The B^1 norm of a simple chain A is defined by

$$\|A\|_{B^1} := \inf\left\{\sum_{i_0\in I_0}|\alpha_{i_0}|+\sum_{i_1\in I_1}|\mathbf{u}_{i_1}||\alpha_{i_1}| \mid A=\sum_{i_0\in I_0}(p_{i_0};\alpha_{i_0})+\sum_{i_1\in I_1}\Delta_{\mathbf{u}_{i_1}}(p_{i_1};\alpha_{i_1})\right\} \tag{15}$$

That the foregoing defines a norm is a nontrivial result. To appreciate the difference between the B^0 norm and the B^1 norm, notice that

$$\|\Delta_{\mathbf{u}}(p,\alpha)\|_{B^0}=2|\alpha| \tag{16}$$

$$\|\Delta_{\mathbf{u}}(p,\alpha)\|_{B^1}\le|\mathbf{u}||\alpha| \tag{17}$$

and, hence, that as $\mathbf{u}\to\mathbf{0}$, $\Delta_{\mathbf{u}}(p;\alpha)$ goes to zero in the B^1 norm but not in the B^0 norm.

The definition of the B^r norm for $r\ge 2$ involves higher-order differences. For example, in the definition of the B^2 norm an infimum over representations of A involving not only terms of the form $\Delta_{\mathbf{u}}(p;\alpha)$ but also terms of the form

$$\Delta_{\mathbf{v}}\Delta_{\mathbf{u}}(p;\alpha)=\Delta_{\mathbf{v}}((p+\mathbf{u};\alpha)-(p;\alpha)) \tag{18}$$

$$=(p+\mathbf{u}+\mathbf{v};\alpha)-(p+\mathbf{u};\alpha)-(p+\mathbf{v};\alpha)+(p;\alpha) \tag{19}$$

which can be viewed as a quadrupole, as seen in Fig. 2(*b*). The explicit formula for the B^r norm for higher values of r is not provided here. The interested reader is encouraged to consult Harrison [11] for further details.

It is also possible to characterize the B^r norm by using a norm on the space of differential forms. This norm is the dual norm on $\mathcal{B}_k^r$ induced by the B^r norm and is given by

$$\|\omega\|_{B^r} := \sup\left\{\frac{|\omega(J)|}{\|J\|_{B^r}} \mid J\in\widehat{\mathcal{B}}_k^r, J\neq 0\right\} \tag{20}$$

This norm has two important properties. For all ω in $\mathcal{B}_k^r$

(P1) $\sup_{x\in\mathcal{E}}|\omega(x)|\le\|\omega\|_{B^r}$, and
(P2) the Lipschitz constants of all directional derivatives of ω up to order $r-1$ are bounded above by $\|\omega\|_{B^r}$.

Moreover, by the Hahn–Banach theorem, the B^r norm on $\mathcal{B}_k^r$ determines the B^r norm on $\widehat{\mathcal{B}}_k^r$ through the formula

$$\|J\|_{B^r}=\sup\left\{\frac{|\omega(J)|}{\|\omega\|_{B^r}} \mid J\in\widehat{\mathcal{B}}_k^r, J\neq 0\right\} \tag{21}$$

It transpires that Eq. (21) and the properties (P1) and (P2) are sufficient to prove a number of things about differential chains. In support of this assertion, we sketch a proof that the sequence of Dirac chains defined in Eq. (8) converges to the curve γ in the B^1 norm. The first step toward this involves showing that the sequence defined in Eq. (8) is Cauchy in the B^1 norm. To achieve this, first notice that for natural numbers m and n and ω in $\mathcal{B}_1^1$

$$|\omega(A_{m+n}-A_n)|=\Big|\sum_{j=0}^{2^{n+m}-1}L2^{-m-n}\omega(\gamma(jL2^{-m-n}))\cdot\dot\gamma(jL2^{-m-n})-\sum_{i=0}^{2^n-1}L2^{-n}\omega(\gamma(iL2^{-n}))\cdot\dot\gamma(iL2^{-n})\Big| \tag{22}$$

$$= |\sum_{i'=0}^{2^m-1}\sum_{i=0}^{2^n-1} L2^{-m-n}\omega(\gamma((i2^m+i')L2^{-m-n}))\cdot\dot\gamma((i2^m+i')L2^{-m-n})$$
$$- \sum_{i'=0}^{2^m-1}\sum_{i=0}^{2^n-1} L2^{-m-n}\omega(\gamma(iL2^{-n}))\cdot\dot\gamma(iL2^{-n})|$$
$$= \sum_{i'=0}^{2^m-1}\sum_{i=0}^{2^n-1} L2^{-m-n}|\omega(\gamma((i2^m+i')L2^{-m-n}))\cdot\dot\gamma((i2^m+i')L2^{-m-n})$$
$$- \omega(\gamma(iL2^{-n}))\cdot\dot\gamma(iL2^{-n})| \tag{23}$$

Adding and subtracting the term

$$\omega(\gamma(iL2^{-n}))\cdot\dot\gamma((i2^m+i')L2^{-m-n}) \tag{24}$$

within the absolute value of the summand in Eq. (23) and using property (P1) and the fact that γ is an arc-length parameterization allows us to bound the absolute value in the summand of Eq. (23) by

$$|\omega(\gamma((i2^m+i')L2^{-m-n})) - \omega(\gamma(iL2^{-n}))|$$
$$+ \|\omega\|_{B^1}|\dot\gamma((i2^m+i')L2^{-m-n}) - \dot\gamma(iL2^{-n})| \tag{25}$$

Substituting Eq. (25) into Eq. (23) and using Property (P2) yields

$$|\omega(A_{m+n} - A_n)|$$
$$\le L\|\omega\|_{B^1}\sup_{i,i'}|\gamma((i2^m+i')L2^{-m-n}) - \gamma(iL2^{-n})|$$
$$+ L\|\omega\|_{B^1}\sup_{i,i'}|\dot\gamma((i2^m+i')L2^{-m-n}) - \dot\gamma(iL2^{-n})| \tag{26}$$

Since γ is continuously differentiable, it follows from Eq. (21) and Eq. (26) that

$$\lim_{n\to\infty}\|A_{m+n} - A_n\|_{B^1} = 0 \tag{27}$$

and, hence, that Eq. (8) defines a Cauchy sequence. This leads to the conclusion that A_m converges to some element J in $\widehat{\mathcal{B}}_1^1$ as m tends to infinity. To see that J represents γ in the sense of Eq. (4), it suffices to consult Eq. (10) and the surrounding discussion.

2.4 Operators on Differential Chains. Since differential chains of a fixed dimension and class form a Banach space, it is possible to define continuous linear mappings—or operators—that act on them. While some of these operators are analogous to operators defined on sets, others are not.

Two operators that have no classical analog are required to understand the generalized transport theorem. The first of these, called the prederivative, resembles the directional derivative. Fix a vector $\mathbf{a}$ and consider the expression

$$P_{\mathbf{a}}(p;\alpha) := \lim_{h\to 0}\frac{(p+h\mathbf{a};\alpha) - (p;\alpha)}{h} \tag{28}$$

where the limit is taken in the B^2 norm. That the limit in Eq. (28) exists is nontrivial. Here, the mapping $P_{\mathbf{a}}$, called the prederivative in the direction $\mathbf{a}$, is only defined on simple chains; however, it can be extended to any chain—first, by linearity, to Dirac chains and then, by continuity, to a general chain. Once this has been achieved, the prederivative becomes an operator of the form

$$P_{\mathbf{a}}: \widehat{\mathcal{B}}_k^r \to \widehat{\mathcal{B}}_k^{r+1} \tag{29}$$

Thus, the prederivative of a chain of dimension k and class B^r leaves the dimension unchanged but raises the class to B^{r+1}. The notation

$$(p;\mathbf{a}\otimes\alpha) := P_{\mathbf{a}}(p;\alpha) \tag{30}$$

is sometimes useful. Let $\nabla_{\mathbf{a}}\omega$ denote the directional derivative of ω in the direction $\mathbf{a}$—that is

$$\nabla_{\mathbf{a}}\omega(x) := \lim_{h\to 0}\frac{\omega(x+h\mathbf{a}) - \omega(x)}{h} \quad \text{for all } x\in\mathcal{E} \tag{31}$$

It can be shown that

$$\int_{(p;\mathbf{a}\otimes\alpha)}\omega = \int_{(p;\alpha)}\nabla_{\mathbf{a}}\omega \quad \text{for all } \omega\in\mathcal{B}_k^{r+1} \tag{32}$$

Some insight as to why the prederivative increases the class of a chain (see Eq. (29)) emerges on considering Eq. (32), which states that the dual of the prederivative is the partial derivative operator on differential forms. Since taking a partial derivative decreases the regularity class—that is, from C^r to C^{r-1}—of a differential form by one, by duality, the prederivative must increase the class by one.

Another valuable operator is called the extrusion. Given a vector $\mathbf{a}$, the extrusion $E_{\mathbf{a}}$ in the direction $\mathbf{a}$ of a simple chain $(p;\alpha)$ is defined by

$$E_{\mathbf{a}}(p;\alpha) := (p;\mathbf{a}\wedge\alpha) \tag{33}$$

Just as with the prederivative, the extrusion can be extended to Dirac chains by linearity and then to all differential chains by continuity so that

$$E_{\mathbf{a}}: \widehat{\mathcal{B}}_k^r \to \widehat{\mathcal{B}}_{k+1}^r \tag{34}$$

Thus, the extrusion of a chain of dimension k and class $\mathcal{B}^r$ raises the dimension to $k+1$ but leaves the class unchanged.

One way to think about the extrusion is as follows. Let J be the differential 2-chain that represents a 2D square in three dimensions and let $\mathbf{a}$ be a vector that is perpendicular to the square. The differential 3-chain $E_{\mathbf{a}}J$ can be interpreted as a thickened version of the square with infinitesimally small width in the $\mathbf{a}$ direction.

It is possible to define the extrusion in the case where the vector $\mathbf{a}$ is replaced by a vector field $\mathbf{v}$ of sufficient regularity. Under these circumstances, Eq. (33) becomes

$$E_{\mathbf{v}}(p;\alpha) := (p;\mathbf{v}(p)\wedge\alpha) \tag{35}$$

Moreover, similar to Eq. (32)

$$\int_{E_{\mathbf{v}}(p;\alpha)}\omega = \int_{(p;\alpha)} i_{\mathbf{v}}\omega \quad \text{for all } \omega\in\mathcal{B}_{k+1}^r \tag{36}$$

where $i_{\mathbf{v}}\omega$ is the interior product of ω with respect to $\mathbf{v}$.

The prederivative operator can be used to construct the boundary operator. Consider a simple k-chain $(p;\alpha)$, where α is simple, so that there are vectors $\mathbf{a}_i$ such that

$$\alpha = \bigwedge_{i=1}^{k}\mathbf{a}_i \tag{37}$$

Put $\hat\alpha_i := \wedge_{j=1,j\neq i}^{k}\mathbf{a}_j$. The boundary of $(p;\alpha)$ is defined by

$$\partial(p;\alpha) := \sum_{j=1}^{n}\sum_{i=1}^{k}(-1)^{i+1}(p;\mathbf{e}_j\otimes(\mathbf{a}_i\cdot\mathbf{e}_j)\hat\alpha_i) \tag{38}$$

It is possible to extend the boundary to an operator on all differential chains by continuity so that

$$\partial: \widehat{\mathcal{B}}_k^r \to \widehat{\mathcal{B}}_{k-1}^{r+1} \tag{39}$$

Notice that the boundary of a chain of dimension k and class r is a chain of dimension $k-1$, as might be expected, and class B^{r+1}.

Recall that the class of a chain can be thought of as a measure of the regularity of the chain—the larger the value of r, the greater the potential irregularity of the chain. To see why the boundary of a chain might increase the irregularity of a chain, consider the following example. A unit square in two dimensions is a smooth manifold and, hence, admits a representation as a 2-chain. The boundary of the square is not a smooth manifold, but only the union of (four) smooth manifolds. Hence, the boundary of a smooth domain may be a nonsmooth domain. A more extreme example is the following. Consider an open set of a Euclidean space. Being a manifold, that set admits a representation as a chain. The boundary of the open set can be highly irregular, but the boundary of the corresponding chain is well-defined and can be "integrated over" in the sense that it can be paired with a differential form.

That the boundary operator agrees with the classical notion of boundary is clear from the following. Consider a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. If J represents $\mathcal{M}$, in the sense made precise in Eq. (4), then ∂J represents $\partial\mathcal{M}$.

On using d to denote the exterior derivative, the boundary operator ∂ satisfies the relation

$$\int_{\partial J} \omega = \int_J d\omega \quad \text{for all } \mathrm{J} \in \widehat{\mathcal{B}}^{\mathrm{r}}_{\mathrm{k}}, \quad \omega \in \mathcal{B}^{\mathrm{r}+1}_{\mathrm{k}-1} \tag{40}$$

which generalizes Stokes' theorem. Since the integrals in Eq. (40) are dual pairings, Eq. (40) embodies the notion that the adjoint of the boundary operator ∂ on chains is the exterior derivative on forms.

For readers familiar with currents, there is nothing new to the observation that Stokes' theorem is a statement pertaining to adjoints. However, there is a subtle but important difference between the ways in which the boundaries of currents and chains are defined. Whereas the boundary of a current is defined via the adjoint of the exterior derivative, the boundary of a differential chain is defined independently and the adjoint of the boundary is shown to be the exterior derivative. Although it holds by definition for currents, Stokes' theorem must be proven for chains.

Another operator that is related to an operation that can be applied to a classical domain is the push-forward. Given a smooth function $f : \mathcal{E} \to \mathcal{E}$, it is possible to define the push-forward

$$f_* : \widehat{\mathcal{B}}^r_k \to \widehat{\mathcal{B}}^r_k \tag{41}$$

induced by f. If a chain represents a domain, then its push-forward represents the image of the domain under the mapping f. The push-forward f_*J of a chain J can therefore be considered as the image of the chain under f.

There are many other interesting operators that act on differential chains, but the operators mentioned in this section are the only ones that are needed to understand the transport theorem of Seguin and Fried [10]. The interested reader is encouraged to consult the work of Harrison [11] for more information concerning operators on differential chains.

3 The Generalized Transport Theorem

In this section, the generalized transport theorem of Seguin and Fried [10] is presented. Fundamentally, a transport theorem yields an expression for the rate of change of an integral in which both the domain of integration and the integrand may depend on time. Since domains of integration admit representations as differential chains, the main idea is to represent the evolving domain with a time-dependent differential chain. Thus, a transport theorem should yield an expression for

$$\overline{\int_J \omega} \tag{42}$$

where J is a time-dependent differential k-chain and ω is a time-dependent skew k-form of matching class. If both J and ω are differentiable in time, then, since the integral in Eq. (42) is a dual pairing, Harrison's [11] "generalized Leibniz rule"

$$\overline{\int_J \omega} = \int_J \dot{\omega} + \int_j \omega \tag{43}$$

follows as a consequence of the product rule.

Notice that, due to the absence of a term involving the boundary, the result (43) differs from the transport identities (1) and (2) mentioned in the introduction. Moreover, it is unclear how a boundary term might be extracted from the right-hand side of Eq. (43). For these reasons, it seems appropriate to refer to the generalized Leibniz rule as a proto-transport theorem. To deduce a version of Eq. (43) involving the boundary requires additional assumptions on the time-dependent chain J. We consider a subspace of the space of all time-dependent chains defined using a specific norm. The reason for using this particular norm will be explained later in this section. The elements of this subspace are called regularly evolving differential chains and the collection of all such objects is denoted by $\widehat{\mathcal{B}}^r_k[I]$. Here, I denotes a compact interval of $\mathbb{R}$ consisting of more than one point which represents the expanse of time over which the chain evolves.

To illustrate the salient points in the proof of the generalized transport theorem and the construction of the space $\widehat{\mathcal{B}}^r_k[I]$, we, for the moment, restrict ourselves to the case $n = 1$ and $k = 1$. Since $n = 1$, the ambient Euclidean space $\mathcal{E}$ is taken to be the space $\mathbb{R}$ of real numbers.

Let $\mathbf{e}$ be the unit vector pointed in the positive direction. A simple 1-chain can be written as

$$(x; a\mathbf{e}) \tag{44}$$

where x and a are numbers. An evolving simple 1-chain has the form Eq. (44) except that x and a are C^1 functions of time defined on the interval I. In an analogous way, $(x; a)$ denotes an evolving simple 0-chain. Moreover, a time-dependent differential 1-form can be written as

$$f\mathbf{e} \tag{45}$$

where f is a time-dependent scalar-valued function. In this one dimensional setting, evaluating Eq. (43) at time t with the choices $J = (x; a\mathbf{e})$ and $\omega = f\mathbf{e}$ gives

$$\overline{\int_{(x;a\mathbf{e})} f\mathbf{e}}(t) = \int_{(x(t);a(t)\mathbf{e})} \dot{f}(t)\mathbf{e} + \int_{\overline{(x;a\mathbf{e})}(t)} f(t)\mathbf{e} \tag{46}$$

The second term on the right-hand side of Eq. (46) involves the expression

$$\overline{(x; a\mathbf{e})}(t) := \lim_{\tau \to 0} \frac{(x(t+\tau); a(t+\tau)\mathbf{e}) - (x(t); a(t)\mathbf{e})}{\tau} \tag{47}$$

where the limit is taken in the B^2 norm. This derivative satisfies

$$\overline{(x; a\mathbf{e})}(t) = (x(t); \dot{x}(t)\mathbf{e} \otimes a(t)\mathbf{e}) + (x(t); \dot{a}(t)\mathbf{e}) \tag{48}$$

To establish Eq. (48), it must be shown that

$$\frac{1}{|\tau|} \| (x(t+\tau); a(t+\tau)\mathbf{e}) - (x(t); a(t)\mathbf{e}) - \tau(x(t); \dot{x}(t)\mathbf{e} \otimes a(t)\mathbf{e}) - \tau(x(t); \dot{a}(t)\mathbf{e}) \|_{B^2} \tag{49}$$

goes to zero as τ goes to zero. Toward this end, add and subtract the terms

$$(x(t)+\tau\dot{x}(t)\mathbf{e};a(t)\mathbf{e}) \tag{50}$$

$$(x(t+\tau);a(t)\mathbf{e}) \tag{51}$$

$$(x(t+\tau);\tau\dot{a}(t)\mathbf{e}) \tag{52}$$

within the norm in Eq. (49) and invoke the triangle inequality to find that Eq. (49) is bounded above by

$$\frac{1}{|\tau|}\|(x(t)+\tau\dot{x}(t)\mathbf{e};a(t)\mathbf{e})-(x(t);a(t)\mathbf{e}) - \tau(x(t);\dot{x}(t)\mathbf{e}\otimes a(t)\mathbf{e})\|_{B^2} \tag{53}$$

$$+\frac{1}{|\tau|}\|(x(t+\tau);a(t)\mathbf{e})-(x(t)+\tau\dot{x}(t)\mathbf{e};a(t)\mathbf{e})\|_{B^2} \tag{54}$$

$$+\frac{1}{|\tau|}\|(x(t+\tau);\tau\dot{a}(t)\mathbf{e})-\tau(x(t);\dot{a}(t)\mathbf{e})\|_{B^2} \tag{55}$$

$$+\frac{1}{|\tau|}\|(x(t+\tau);a(t+\tau)\mathbf{e}-a(t)\mathbf{e}-\tau\dot{a}(t)\mathbf{e})\|_{B^2} \tag{56}$$

The term in Eq. (53) tends to zero by Eq. (28), the term in Eq. (54) tends to zero by the differentiability of x, the term in Eq. (55) tends to zero by the continuity of x, and the term in Eq. (56) tends to zero by the differentiability of a. This establishes Eq. (48).

On the right-hand side of Eq. (48), the first term involving the prederivative can be thought of as the derivative of the first argument of $(x;a\mathbf{e})$ with respect to time and the second term can be thought of as the derivative of the second argument of $(x;a\mathbf{e})$ with respect to time. The relation (48) therefore possesses the characteristics of a product rule. The two terms on the right-hand side of Eq. (48) are important enough to be given their own symbols. Define

$$P_t(x;a\mathbf{e}) := (x(t);\dot{x}(t)\mathbf{e}\otimes a(t)\mathbf{e}) \tag{57}$$

$$G_t(x;a\mathbf{e}) := (x(t);\dot{a}(t)\mathbf{e}) \tag{58}$$

It is also useful to define a similar mapping

$$E_t(x;a) := (x(t);\dot{x}(t)a(t)\mathbf{e}) \tag{59}$$

on evolving 0-chains. It is interesting that P_t is related to the boundary operator ∂ through

$$P_t(x;a\mathbf{e}) = E_t\partial(x;a\mathbf{e}) \tag{60}$$

This identity holds because using Eq. (39) and Eq. (59) yields

$$E_t\partial(x;a\mathbf{e}) = E_t(x;\mathbf{e}\otimes a) = (x(t);\dot{x}(t)\otimes a(t)\mathbf{e}) = P_t(x;a) \tag{61}$$

In view of Eqs. (48), (57), (58), and (60), the one-dimensional proto-transport identity (46) becomes a transport identity for the evolving simple chain $(x;a\mathbf{e})$ with the form

$$\dot{\overline{\int_{(x;a\mathbf{e})} f\mathbf{e}}}(t) = \int_{(x(t);a(t)\mathbf{e})}\dot{f}(t)\mathbf{e} + \int_{E_t\partial(x;a\mathbf{e})} f(t)\mathbf{e} + \int_{G_t(x;a\mathbf{e})} f(t)\mathbf{e} \tag{62}$$

Versions of the transport identity (62) for more complicated evolving chains can also be established. To achieve this, start by considering linear combinations

$$A = \sum_{i\in I}(x_i;a_i\mathbf{e}) \tag{63}$$

of evolving simple chains, which are known as evolving Dirac chains. Despite the superficial resemblance between Eqs. (6) and (63), it is important to realize that, aside from the fact that n and k in Eq. (6) are arbitrary, the quantity A determined by Eq. (63) is time dependent, in contrast to its counterpart appearing in Eq. (6). By linearity, the mappings P_t, G_t, and E_t, each of which is defined on evolving simple chains, can be defined on evolving Dirac chains. It follows that Eq. (60) holds with $(x;a\mathbf{e})$ replaced by A and, hence, that there is a transport identity for Dirac chains of the form

$$\dot{\overline{\int_A f\mathbf{e}}}(t) = \int_{A(t)}\dot{f}(t)\mathbf{e} + \int_{E_t\partial A} f(t)\mathbf{e} + \int_{G_tA} f(t)\mathbf{e} \tag{64}$$

The final step required to obtain the generalized transport theorem is to introduce the C^1_r norm on the vector space of Dirac 1-chains. The superscript in C^1_r is not related to the dimension of the chain but rather is related to the fact that any Dirac 1-chains can be built up using C^1 functions. The underlying details of this norm are not discussed here. The essential idea is to define the norm in a way that ensures that the mappings P_t, G_t, and E_t are continuous. The completion of the space of Dirac 1-chains defined on the interval of time I with respect to the C^1_r norm is denoted by $\widehat{\mathcal{B}}^r_1[I]$ and the elements of this space are called regularly evolving differential chains. By construction, given J in $\widehat{\mathcal{B}}^r_1[I]$, there is a sequence A_m of evolving Dirac 1-chains of the form Eq. (63) such that

$$A_m \to J \tag{65}$$

in the C^1_r norm as m goes to infinity.

To obtain a transport theorem for a regularly evolving differential chain J, begin by finding A_m such that Eq. (65) holds. Since Eq. (64) holds with A replaced by A_m, taking the limit as m goes to infinity yields

$$\dot{\overline{\int_J f\mathbf{e}}}(t) = \int_{J(t)}\dot{f}(t)\mathbf{e} + \int_{E_t\partial J} f(t)\mathbf{e} + \int_{G_tJ} f(t)\mathbf{e} \tag{66}$$

To summarize, for $n=1$ and $k=1$ the following four main steps are needed to arrive at the generalized transport theorem:

(1) Prove that $\dot{\overline{(x;a\mathbf{e})}}(t) = P_t(x;a\mathbf{e}) + G_t(x;a\mathbf{e})$.
(2) Prove that $P_t = E_t\partial$.
(3) Establish the transport identity (62) for evolving simple chains.
(4) Introduce the C^1_r norm and take linear combinations and limits in Eq. (62) to obtain Eq. (66).

To obtain the generalized transport theorem for arbitrary n and k, the same four steps are required with one exception. Step 2 changes to proving that

$$P_t = \partial E_t + E_t\partial \tag{67}$$

The reason for this change is that $E_t(x;a\mathbf{e}) = 0$, whereby in the case $n=1$ and $k=1$, Eq. (67) reduces to $P_t = E_t\partial$. With this slight change, the generalized transport theorem can be stated as follows.

Theorem 3.1. *Consider an evolving chain J in $\widehat{\mathcal{B}}^r_k[I]$ and a time-dependent differentiable skew k-form $\omega: I \to \mathcal{B}^{r+1}_k$. Then, for all t in I,*

$$\dot{\overline{(\int_J \omega)}}(t) = \int_{J(t)}\dot{\omega}(t) + \int_{E_tJ} d\omega(t) + \int_{E_t\partial J}\omega(t) + \int_{G_tJ}\omega(t) \tag{68}$$

The elements of the space $\widehat{\mathcal{B}}_k^r[I]$ can be thought of as time-dependent chains. From this point onward, only regularly evolving chains are considered; for this reason, we often suppress the modifier "regularly" when referring to such chains. Since chains may represent domains, evolving chains may represent evolving domains. Importantly, Seguin and Fried [10] showed that evolving chains may represent domains that may become perforated, rip into pieces, or lose regularity (as would occur, for instance, were the boundary to develop corners or cusps). Seguin and Fried [10] also showed that evolving chains may represent evolving fractal domains. In particular, a domain that transitions from smooth to fractal may also be represented by a regularly evolving chain.

4 Convecting Chains

The variety of evolving domains representable by regularly evolving differential chains is very diverse. A very common type of evolving chain is what we call a convecting chain. To generate a convecting chain, it is necessary to have a reference chain J_R, that is independent of time, and a smooth mapping,

$$f : \mathcal{E} \times I \to \mathcal{E} \tag{69}$$

called a convection. It is useful to define

$$f_t : \mathcal{E} \to \mathcal{E} \quad \text{by} \quad f_t(x) := f(x,t) \quad \text{for all } x \in \mathcal{E} \tag{70}$$

We assume that f_t is invertible for each t in I. Associated with the convection is a velocity $\mathbf{v} : \mathcal{E} \times I \to \mathcal{V}$, defined by

$$\mathbf{v}(x,t) := \lim_{\tau \to 0} \frac{f(f_t^{-1}(x), t+\tau) - f(f_t^{-1}(x), t)}{\tau} \quad \text{for all } (x,t) \in \mathcal{E} \times I \tag{71}$$

Similar to Eq. (70), we need the notation

$$\mathbf{v}_t : \mathcal{E} \to \mathcal{V} \tag{72}$$

where

$$\mathbf{v}_t(x) := \mathbf{v}(x,t) \quad \text{for all } (x,t) \in \mathcal{E} \times I \tag{73}$$

Using the two objects f and J_R and the push-forward described in the paragraph containing Eq. (41), we may define a time-dependent chain J by

$$J(t) := (f_t)_* J_R \quad \text{for all } t \in I \tag{74}$$

If J_R is of class B^r and f is smooth enough, Seguin and Fried [10] showed that J is in $\widehat{\mathcal{B}}_k^r[I]$, where k is the dimension of J_R. If the ambient space $\mathcal{E}$ is three-dimensional, so that $n = 3$, and J_R represents a regular region in $\mathcal{E}$, then the convecting chain defined by Eq. (74) represents the motion of the regular region under f. This is the situation most often encountered in continuum physics.

When the evolving chain in Theorem 3.1 is the convecting chain J, as defined in Eq. (74), Eq. (68) takes on a different form. For such chains, Seguin and Fried [10] showed that

$$E_t J = E_{\mathbf{v}_t} J(t) \tag{75}$$

and

$$(E_t \partial + G_t) J = E_{\mathbf{v}_t} \partial J(t) \tag{76}$$

Upon using Eqs. (36), (75), and (76), Eq. (68) becomes

$$\overline{\left(\int_J \omega\right)}(t) = \int_{J(t)} (\dot{\omega}(t) + i_{\mathbf{v}_t} d\omega(t)) + \int_{\partial J(t)} i_{\mathbf{v}_t} \omega(t) \tag{77}$$

for all t in I. Here, it is important to keep in mind that $\dot{\omega}$ is not the material time-derivative of ω, but rather is the derivative of the function $\omega : I \to \mathcal{B}_k^{r+1}$ with respect to time. Instead of viewing ω as a time-dependent differential form, it is possible to view it as a mapping that assigns to each point x and time t the skew k-form $\omega(x, t)$. When this view is adopted the partial time-derivative ω' of ω becomes meaningful. In this case, Seguin and Fried [10] demonstrated that $\omega' = \dot{\omega}$. For this reason, from now on we will use the partial time-derivative notation. For example, Eq. (77) can be rewritten, suppressing the dependence on time, as

$$\overline{\int_J \omega} = \int_J (\omega' + i_{\mathbf{v}} d\omega) + \int_{\partial J} i_{\mathbf{v}} \omega \tag{78}$$

The result (78), which is closer in form to the classical results (1) and (2) mentioned in the introduction, was proven independently by Harrison [11] and Seguin and Fried [10].

It might initially seem surprising that the generalized transport identity (68) contains one more term than the transport identity (78) for convecting chains. This difference originates with the last term on the right-hand side of Eq. (68). As is evident from Eq. (58), this term describes how the (oriented) density of the chain—that is, the $a\mathbf{e}$ part of Eq. (44)—changes in time. To see this, consider a (time independent) differential chain $J_\circ$ and define a time-dependent chain J by

$$J(t) := a(t) J_\circ \tag{79}$$

where a is a smooth scalar-valued function of time. It can be shown that the J defined in Eq. (79) is a regularly evolving differential chain and that for any time independent differential form $\omega_\circ$

$$\overline{\int_J \omega_\circ}(t) = \int_{\dot{a}(t) J_\circ} \omega_\circ \tag{80}$$

Of course, it is impossible for an evolving chain like Eq. (79) to be a convecting chain unless $k = 0$. For a convecting chain in which $k \neq 0$, the change in density is slaved to the manner in which the chain moves through the ambient space because of Eq. (74) and the way the push-forward of a differential chain is defined. For additional detail regarding this issue, see Seguin and Fried [10].

5 Classical Transport Theorems

Theorem 3.1 can be specialized to yield classical results involving regular domains and, thus, is a generalization of other well-known transport theorems. These results follow from corollary (78) rather than directly from Theorem 3.1. There are two similar but different transport theorems for classical domains: one for convecting domains and another for domains that evolve independent of the motion of an underlying body, as with a phase interface, crack, or other defect.

The setup for a convecting domain is similar to that of a convecting chain. Let $\mathcal{M}_R$ be a reference k-dimensional domain and let f be a convection, as described in Sec. 4. These two objects induce a convecting domain $\mathcal{M}$ defined by

$$\mathcal{M}(t) := f_t(\mathcal{M}_R) \quad \text{for all } t \in I \tag{81}$$

If J_R is the differential chain that represents $\mathcal{M}_R$, then $J(t)$, defined by Eq. (74), represents $\mathcal{M}(t)$ for all t in I. Thus, Eq. (78) can be used to prove that

$$\overline{\int_{\mathcal{M}} \omega} = \int_{\mathcal{M}} (\omega' + i_{\mathbf{v}} d\omega) + \int_{\partial \mathcal{M}} i_{\mathbf{v}} \omega \tag{82}$$

where $\mathbf{v}$ is the velocity of the convection defined in Eqs. (72) and (73). The details leading to this result are provided by Seguin and Fried [10]. Also, for a direct proof of Eq. (82) without recourse to chains, see Flanders [17].

It is important to remember that the differential forms considered so far are defined on all of Euclidean space $\mathcal{E}$. Some applications, however, involve differential forms that are defined only on an evolving domain. In this event, Eq. (82) cannot be used directly since the derivative ω' is undefined. However, Eq. (82) can be used to obtain a transport theorem involving such differential forms. To state this result we need to introduce a few additional concepts. Let $\mu_{\mathcal{M}}$ denote the (time-dependent) content form for the convecting domain $\mathcal{M}$. Notice that for every differential k-form ω defined on $\mathcal{M}$, there is a scalar function φ such that $\omega = \varphi\mu_{\mathcal{M}}$. We may introduce two different time derivatives for φ. The convection-dependent time-derivative $\delta\varphi/\delta t$ of φ is defined by

$$\frac{\delta\varphi}{\delta t}(x,t) := \lim_{\tau\to 0}\frac{\varphi(f_{t+\tau}(f_t^{-1}(x)),t+\tau)-\varphi(x,t)}{\tau} \quad \text{for all } x \text{ and } t \text{ such that } x\in\mathcal{M}(t) \tag{83}$$

In continuum physics, when f is a motion (or time-dependent deformation), this derivative is the material time-derivative. The normal time-derivative $\overset{\square}{\varphi}$ of φ is given by

$$\overset{\square}{\varphi} := \frac{\delta\varphi}{\delta t} - \nabla^{\mathcal{M}}_{\mathbf{v}_{\tan}}\varphi \tag{84}$$

where $\nabla^{\mathcal{M}}$ is the covariant gradient on $\mathcal{M}$ and $\mathbf{v}_{\tan}$ is the component of the velocity $\mathbf{v}$ that is tangent to $\mathcal{M}$.[4] On using this notation and suppressing explicit dependence on time, the transport theorem for a differential form only defined on $\mathcal{M}$ can be written as

$$\dot{\int_{\mathcal{M}}\varphi\mu_{\mathcal{M}}} = \int_{\mathcal{M}}(\overset{\square}{\varphi} - k\varphi(\mathbf{v}\cdot\mathbf{h}))\mu_{\mathcal{M}} + \int_{\partial\mathcal{M}}\varphi(\mathbf{v}\cdot\boldsymbol{\nu})\mu_{\partial\mathcal{M}} \tag{85}$$

where $\mathbf{h}$ is the mean-curvature vector for $\mathcal{M}$, $\mu_{\partial\mathcal{M}}$ is the content form for $\partial\mathcal{M}$, and $\boldsymbol{\nu}$ is the outward unit-normal to $\partial\mathcal{M}$. For a proof of Eq. (85), see Betounes [4].

It should be emphasized that there are infinitely many references and convections that generate any particular convecting domain. Sometimes, there may be a preferred reference or convection. However, this is not always the case. For example, in continuum physics, an evolving body is a convecting domain, whose reference is called the reference placement and whose convection mapping is called the motion of the body. Material surfaces are also convecting domains since their evolution is described by the motion of the body. The velocity of the surface in the direction of its unit normal is given by $(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$, where $\mathbf{v}$ is the velocity of the motion of the body. A nonmaterial surface, such as the interface between two phases, generally evolves independently of the underlying body. Specifically, suppose that we have an interface $\mathcal{S}$ whose evolution is described by the zero level-sets of a function $\psi:\mathcal{E}\times I\to\mathbb{R}:\mathcal{S}(t)=\{x\in\mathcal{E}\,|\,\psi(x,t)=0\}$. Then, the velocity $\mathbf{v}_{\mathcal{S}}$ of the interface in the normal direction is given by

$$\mathbf{v}_S = -\frac{\psi'}{|\nabla\psi|}\mathbf{n} \tag{86}$$

In general, $\mathbf{v}_{\mathcal{S}}\neq(\mathbf{v}\cdot\mathbf{n})\mathbf{n}$. Because of this, it is desirable to have a transport theorem for evolving domains that does not involve a convection and, hence, a way of describing the evolution of a domain without relying upon a convection.

The evolution of a smooth domain can be described by two mappings: a (vector-valued) normal velocity $\mathbf{v}_{\mathcal{M}}$ defined on $\mathcal{M}$ and a scalar normal-velocity $V_{\partial\mathcal{M}}$ defined on $\partial\mathcal{M}$. Fix t in I. The normal velocity $\mathbf{v}_{\mathcal{M}}(x,t)$ at a point x belonging to $\mathcal{M}(t)$ is a vector that is perpendicular to the manifold $\mathcal{M}(t)$ at x and describes the evolution of $\mathcal{M}(t)$ at x in the normal direction. The scalar normal-velocity $V_{\partial\mathcal{M}}(x,t)$ at a point x belonging to $\partial\mathcal{M}(t)$ describes the evolution of $\partial\mathcal{M}(t)$ in the outward-normal direction. It can be shown that the normal time-derivative of a scalar field can be defined in this case as well. For a proof of this in the case $n=3$ and $k=2$, see Cermelli et al. [19]. The general case has an analogous proof. For an evolving domain described with $\mathbf{v}_{\mathcal{M}}$ and $V_{\partial\mathcal{M}}$, the relevant transport identity takes the form

$$\dot{\int_{\mathcal{M}}\varphi\mu_{\mathcal{M}}} = \int_{\mathcal{M}}(\overset{\square}{\varphi} - k\varphi(\mathbf{v}_{\mathcal{M}}\cdot\mathbf{h}))\mu_{\mathcal{M}} + \int_{\partial\mathcal{M}}\varphi V_{\partial\mathcal{M}}\mu_{\partial\mathcal{M}} \tag{87}$$

A proof of this result based on the roughened transport identity (68) is provided by Seguin and Fried [10].

There are several special cases of Eq. (87) that are of particular interest to researchers in continuum physics, all of which arise for $n=3$.

- If $k=3$, so that $\mathcal{M}(t)$ is a smooth region at each time t, then

$$\dot{\int_{\mathcal{M}}\varphi\,\mathrm{dv}} = \int_{\mathcal{M}}\varphi'\,\mathrm{dv} + \int_{\partial\mathcal{M}}V_{\partial\mathcal{M}}\,\mathrm{da} \tag{88}$$

- If $k=2$, so that $\mathcal{M}(t)$ is a smooth surface at each time t, then

$$\dot{\int_{\mathcal{M}}\varphi\,\mathrm{da}} = \int_{\mathcal{M}}(\overset{\square}{\varphi} - 2H\varphi\mathbf{v}_{\mathcal{M}}\cdot\mathbf{n})\,\mathrm{da} + \int_{\partial\mathcal{M}}\varphi V_{\partial\mathcal{M}}\,\mathrm{dl} \tag{89}$$

  where $\mathbf{n}$ is a unit normal to the surface and H is the (scalar) mean-curvature.
- If $k=1$, so that $\mathcal{M}(t)$ is a smooth curve for all for time t, then

$$\dot{\int_{\mathcal{M}}\varphi\,\mathrm{dl}} = \int_{\mathcal{M}}(\overset{\square}{\varphi} - \kappa\varphi\mathbf{v}_{\mathcal{M}}\cdot\mathbf{n})\,\mathrm{dl} + \varphi V_{\partial\mathcal{M}}|_p^q \tag{90}$$

  where $\mathbf{n}$ is the unit normal, κ is the curvature, and p and q are the endpoints of the curve.

While Eq. (88) is a generalization of Reynolds [1] transport theorem, (89) is known as the surface transport theorem and was first proven by Gurtin et al. [5]. The surface transport theorem has drawn considerable interest. See, for example, the work of Cermelli et al. [19], Fosdick and Tang [20], and Lidström [21]. Another proof of the surface transport theorem, using the theory of distributions, was provided by Estrada and Kanwal [22]. The $k=1$ case, for a planar curve, was proven by Angenent and Gurtin [23].

There are other transport theorems that follow from Eq. (87) and are of use in continuum physics. One such theorem arises on choosing the density φ so that the integrand involves a flux. Consider a time-dependent vector field $\mathbf{w}$.

- If $k=2$, so that $\mathcal{M}(t)$ is a smooth surface at each time t, then

$$\dot{\int_{\mathcal{M}}\mathbf{w}\cdot\mathbf{n}\,\mathrm{da}} = \int_{\mathcal{M}}(\overset{\square}{\mathbf{w}}\cdot\mathbf{n} - \mathbf{w}\cdot\nabla^{\mathcal{M}}V_{\mathcal{M}} - 2H(\mathbf{w}\cdot\mathbf{n})V_{\mathcal{M}})\,\mathrm{da} + \int_{\partial\mathcal{M}}(\mathbf{w}\cdot\mathbf{n})V_{\partial\mathcal{M}}\,\mathrm{dl} \tag{91}$$

[4]The normal time-derivative, under a different name and in a more specific setting, was first introduced by Thomas [18].

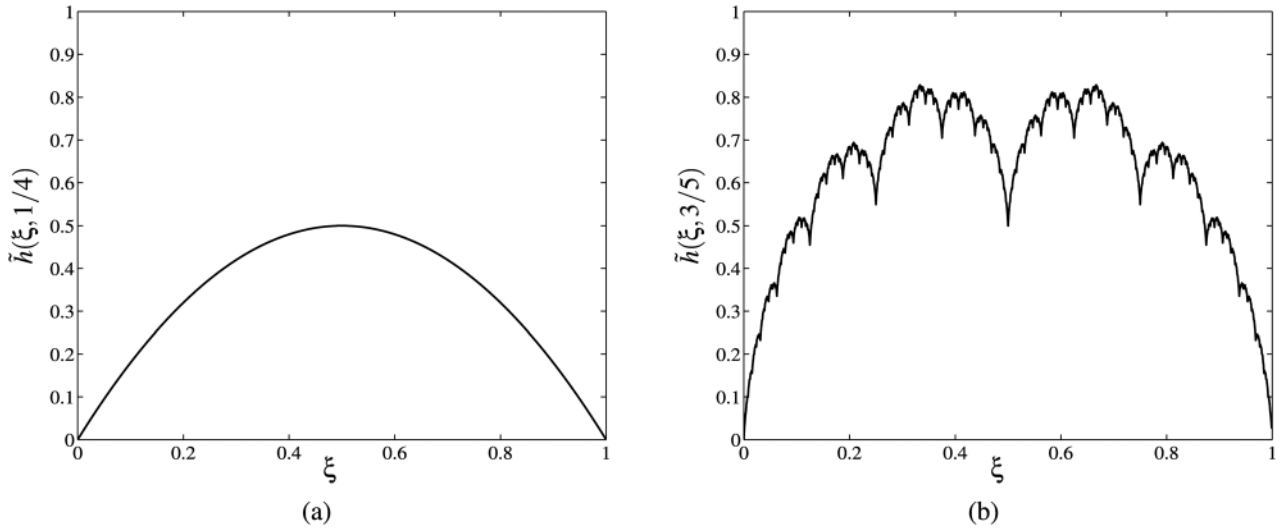

Fig. 3 Graph of $\tilde{h}$ at various times. These graphs also appear in Seguin and Fried [10]. (*a*) Graph of $\tilde{h}(\cdot, 1/4)$. (*b*) Graph of $\tilde{h}(\cdot, 3/5)$.

where $V_{\mathcal{M}} := \mathbf{v}_{\mathcal{M}} \cdot \mathbf{n}$ and H is the (scalar) mean-curvature and the relation $\overset{\square}{\mathbf{n}} = -\nabla^{\mathcal{M}} V_{\mathcal{M}}$ has been used.

- If $k = 1$, so that $\mathcal{M}(t)$ is a smooth curve for all for time t, then

$$\overset{\cdot}{\int_{\mathcal{M}} \mathbf{w} \cdot \mathbf{t}\, \mathrm{dl}} = \int_{\mathcal{M}} (\overset{\square}{\mathbf{w}} \cdot \mathbf{t} + \mathbf{w} \cdot (\nabla^{\mathcal{M}}_{\mathbf{t}} \mathbf{v}_{\mathcal{M}} + \kappa(\mathbf{v}_{\mathcal{M}} \cdot \mathbf{n})\mathbf{n}) - \kappa(\mathbf{w} \cdot \mathbf{t})\mathbf{v}_{\mathcal{M}} \cdot \mathbf{n})\, \mathrm{dl} + (\mathbf{w} \cdot \mathbf{t}) V_{\partial\mathcal{M}}|_p^q \tag{92}$$

where $\mathbf{t}$ is the unit tangent, p and q are the endpoints of the curve, and κ is the curvature and the relation

$$\overset{\square}{\mathbf{t}} = \nabla^{\mathcal{M}}_{\mathbf{t}} \mathbf{v}_{\mathcal{M}} + \kappa(\mathbf{v}_{\mathcal{M}} \cdot \mathbf{n})\mathbf{n} \tag{93}$$

has been used.

6 Explicit Examples Evaluated Numerically

We previously noted that evolving chains may represent evolving domains and that these domains may be highly irregular. We next give three explicit examples of evolving chains that represent evolving domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. We denote the associated standard basis vectors by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

6.1 The Rate of Change of an Area of a Region With Fractal Boundary. Consider the function $h : [0, x_\circ] \times [0, 3t_\circ/4] \to \mathbb{R}$ defined by

$$h(x,t) := h_\circ \sum_{n=0}^{\infty} (\frac{t}{t_\circ})^n s(\frac{2^n x}{L}) \quad \text{for all } (x,t) \in [0, x_\circ] \times [0, 3t_\circ/4] \tag{94}$$

where $s(y) := \min_{n \in \mathbb{N}} |y - n|$ for all $y \in \mathbb{R}$, $x_\circ$ and $h_\circ$ are characteristic lengths, and $t_\circ$ is a characteristic time. To nondimensionalize the space and time variables, we define a new function $\tilde{h} : [0,1] \times [0, 3/4] \to \mathbb{R}$ by

$$\tilde{h}(\xi, \tau) := \frac{h(x_\circ \xi, t_\circ \tau)}{h_\circ} \quad \text{for all } (\xi, \tau) \in [0,1] \times [0, 3/4] \tag{95}$$

The graph of $\tilde{h}(\cdot, \tau)$ is, for $1/2 \le \tau \le 3/4$, called a Takagi–Landsberg curve and has fractal dimension of $2 + \log_2 \tau$, as explained by Mandelbrot [24]. (See Fig. 3.) For all τ in [0, 3/4], any natural number n, and $0 \le i, j \le 2^n - 1$, define q, c, and ar such that

$$q(i,j,\tau;n) := (i2^{-n}, \tilde{h}(i2^{-n}, \tau) j 2^{-n}) \tag{96}$$

$$c(i,j,\tau;n) := \tilde{h}'(i2^{-n}, \tau) j 2^{-n} \tag{97}$$

$$\mathrm{ar}(i,\tau;n) := \tilde{h}(i2^{-n}, \tau) 2^{-2n} \tag{98}$$

Seguin and Fried [10] show that the limit

$$J = \lim_{n \to \infty} \sum_{i,j=0}^{2^n - 1} (q(i,j,\cdot;n); \mathrm{ar}(i,\cdot;n)\mathbf{e}_1 \wedge \mathbf{e}_2) \tag{99}$$

of evolving chains converges (in the C^1_2 norm)[5] to the evolving chain that represents the domain bounded by the ξ-axis and the graph of $\tilde{h}$. It can also be shown that

$$E_\tau J = 0 \tag{100}$$

$$E_\tau \partial J = \lim_{n \to \infty} \sum_{i,j=0}^{2^n - 1} (q(i,j,\tau;n); \mathrm{ar}(i,\tau;n) c(i,j,\tau;n) \mathbf{e}_2 \otimes \mathbf{e}_1 \wedge \mathbf{e}_2)$$

$$G_t J = \lim_{n \to \infty} \sum_{i,j=0}^{2^n - 1} (q(i,j,\tau;n); \mathrm{ar}'(i,\tau;n)\mathbf{e}_1 \wedge \mathbf{e}_1) \tag{101}$$

where ar′ is the derivative of ar with respect to τ. Since $\int_J \mathbf{e}_1 \wedge \mathbf{e}_2$ gives the (dimensionless) area of the domain bounded by the ξ-axis and the graph of $\tilde{h}$, Eq. (68) can be used to compute the rate of change of this integral as the graph of $\tilde{h}$ evolves. Using Eqs. (5) and (32), we find that

$$\overset{\cdot}{(\int_J \mathbf{e}_1 \wedge \mathbf{e}_2)}(\tau) = \int_{J(\tau)} (\mathbf{e}_1 \wedge \mathbf{e}_2)' + \int_{E_\tau J} d(\mathbf{e}_1 \wedge \mathbf{e}_2) + \int_{E_\tau \partial J} \mathbf{e}_1 \wedge \mathbf{e}_2 + \int_{G_t J} \mathbf{e}_1 \wedge \mathbf{e}_2 \tag{102}$$

$$= \lim_{n \to \infty} \sum_{i,j=0}^{2^n - 1} \mathrm{ar}'(i,\tau;n) \tag{103}$$

$$= \lim_{n \to \infty} \sum_{i=0}^{2^n - 1} \tilde{h}'(i2^{-n}, \tau) 2^{-n} \tag{104}$$

[5]This norm is mentioned very briefly, without details, in Sec. 3.

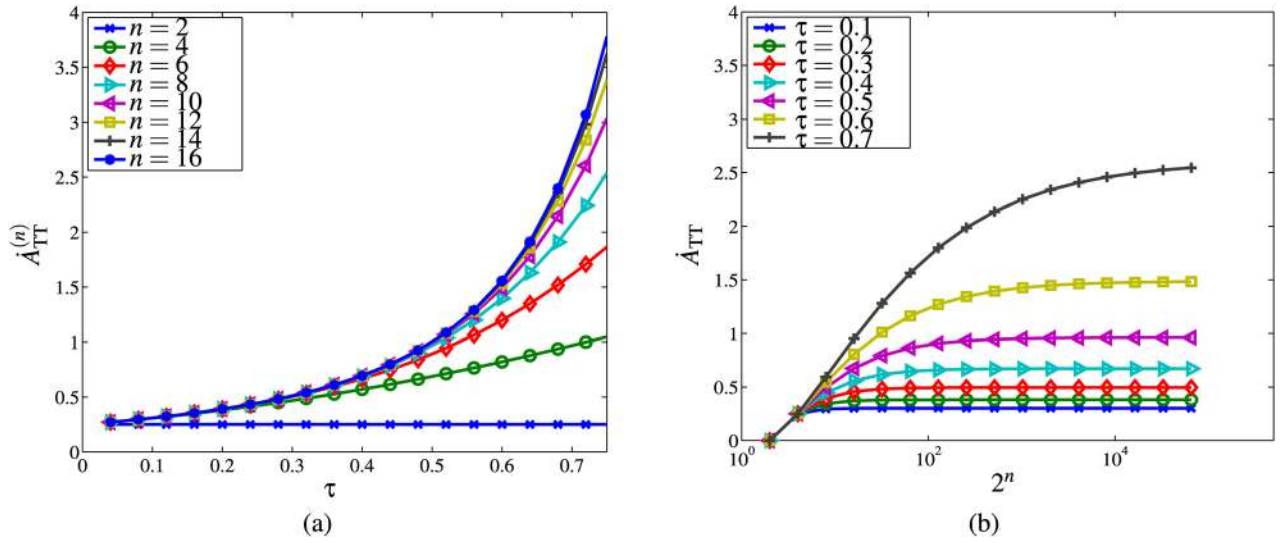

Fig. 4 Convergence of the rate of change of area as given by the transport identity in the right-hand side of Eq. (104). (*a*) Plot of the evolution of $\dot{A}_{\mathrm{TT}}^{(n)}$ for different values of *n*, which correspond to the different approximations of the right-hand side of Eq. (104). (*b*) Convergence of the rate of change of the area $\dot{A}_{\mathrm{TT}}$ for different snapshots in time. Notice the logarithmic abscissa in (*b*).

For future reference, let

$$\dot{A}_{\mathrm{TT}}^{(n)} := \sum_{i=0}^{2^n-1} \tilde{h}'(i2^{-n},\tau)2^{-n} \tag{105}$$

For any given value of τ, it is straightforward to find an approximate value for the limit on the right-hand side of Eq. (104) by computing $\dot{A}_{\mathrm{TT}}^{(n)}$ for several different values of n and examining whether its values appear to converge with increasing n. Results of this kind are depicted in Fig. 4. Figures 4(*a*) and 4(*b*) indicate that for τ small and less than 1/2, in which case the boundary of the region is not fractal, a relatively small n is sufficient to achieve convergence. However, Figs. 5(*a*) and 5(*b*) also show that for τ closer to 3/4 the value of n necessary to achieve convergence increases significantly.

For comparison, the left-hand side of Eq. (102) can be determined directly by first calculating the area of the bounded region at each value of τ and then finding the rate of change of this area. Two numerical approaches have been used to achieve this. Let $\dot{A}_{\mathrm{Rie,c}}^{(n)}$ denote the approximation of the left-hand side of Eq. (102) using first a Riemann sum determined by using 2^n equally spaced grid points to approximate the integral and then a central-Euler scheme to approximate the time-derivative, and let $\dot{A}_{\mathrm{Sim,c}}^{(n)}$ denote the approximation of the left-hand side of Eq. (102) using first Simpson's rule determined by using 2^n equally spaced grid points to approximate the integral and then a central-Euler scheme to approximate the time-derivative. The convergence of the left-hand side of Eq. (102) using these two approaches is depicted in Fig. 5. In Fig. 6, the values of $\dot{A}_{\mathrm{TT}}^{(8)}, \dot{A}_{\mathrm{Rie,c}}^{(8)}$, and $\dot{A}_{\mathrm{Sim,c}}^{(8)}$ are compared to that of $\dot{A}_{\mathrm{TT}}^{(16)}$, which, as is evident from Fig. 4(*a*), provides the closest approximation of the right-hand side of Eq. (104). That $\dot{A}_{\mathrm{TT}}^{(8)}$ and $\dot{A}_{\mathrm{Rie,c}}^{(8)}$ agree so well justifies the assertion (appearing immediately after Eq. (9)) that the pairing between a differential chain and a differential form can be thought of as a Riemann sum. The value $\dot{A}_{\mathrm{Sim,c}}^{(8)}$ is closer to $\dot{A}_{\mathrm{TT}}^{(16)}$ than is the value of $\dot{A}_{\mathrm{Rie,c}}^{(8)}$, which is consistent with the status of Simpson's rule as a second-order scheme.

6.2 The Rate of Change of the Circulation Around an Evolving Fractal Curve. We now consider a second example involving a convecting differential 1-chain. First, consider the boundary of the region bounded by h_T and h_B defined such that

$$\left.\begin{aligned}\tilde{h}_T(\xi) &:= \tilde{h}(\xi - 1/2, 3/5),\\ \tilde{h}_B(\xi) &:= -\tilde{h}(\xi - 1/2, 3/5)\end{aligned}\right\} \quad \text{for all } \xi \in [1/2, 3/2] \tag{106}$$

For all natural numbers n and $0 \le i,j \le 2^n - 1$, let q and ar be defined such that

$$q(i,j;n) := (i2^{-n} + 1/2, \tilde{h}(i2^{-n}, 3/5)(2j2^{-n} - 1)) \tag{107}$$

$$\mathrm{ar}(i;n) := 2\tilde{h}(i2^{-n}, 3/5)2^{-2n} \tag{108}$$

Notice that Eqs. (107) and (108) supersede Eqs. (96) and (98), respectively. The boundary of the region defined with the functions $\tilde{h}_T$ and $\tilde{h}_B$ is represented by the differential 1-chain

$$C_R = \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} (q(i,j;n); (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)\mathrm{ar}(i;n)) \tag{109}$$

We next consider the convecting chain generated by the reference C_R and convection f defined by

$$\begin{aligned} f((\xi,\zeta),\tau) := (&\xi\cos\hat{\theta}_\tau(\xi,\zeta) - \zeta\sin\hat{\theta}_\tau(\xi,\zeta),\\ &\xi\sin\hat{\theta}_\tau(\xi,\zeta) + \zeta\cos\hat{\theta}_\tau(\xi,\zeta)) \end{aligned} \tag{110}$$

where $\hat{\theta}_\tau$ and $\mathbf{v}_\tau$ are defined such that

$$\left.\begin{aligned}\hat{\theta}_\tau(\xi,\zeta) &:= \int_0^\tau \frac{|\mathbf{v}_\tau(\xi,\zeta)|}{\sqrt{\xi^2+\zeta^2}}\mathrm{d}\sigma,\\ \mathbf{v}_\tau(\xi,\zeta) &:= \frac{\Gamma\left(1-\mathrm{e}^{-\frac{\xi^2+\zeta^2}{4\nu\tau}}\right)}{2\pi(\xi^2+\zeta^2)}(\xi\mathbf{e}_2 - \zeta\mathbf{e}_1),\end{aligned}\right\} \tag{111}$$

$$\text{for all } (\xi,\zeta) \in \mathbb{R}^2, \tau \in [0,T]$$

As the notation suggests, $\mathbf{v}$ denotes the velocity field associated with the convection f. Notice that in Eq. (111) the notation introduced in Eqs. (72) and (73) is used, but with t replaced by τ. This convection is associated with the Lamb–Oseen [25, 26] vortex and Γ and ν are dimensionless circulation strength and viscosity, respectively. The evolving chain generated by C_R and f is (see Seguin and Fried [10])

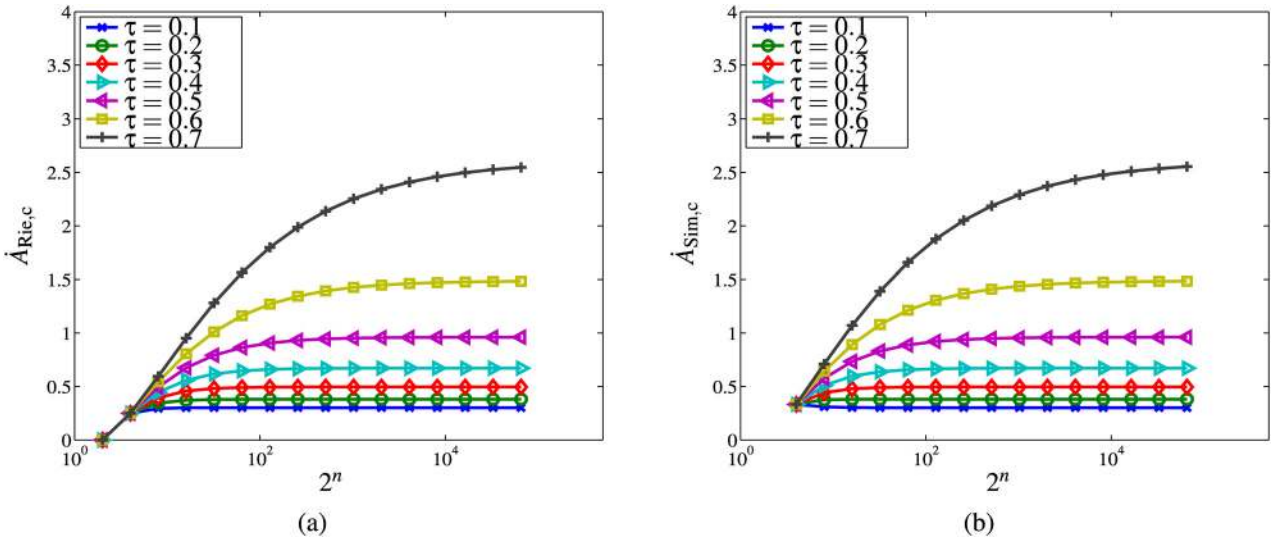

Fig. 5 Convergence of the rate of change of the area for different snapshots in time using (*a*) Riemann sums and (*b*) Simpson's rule. Notice the logarithmic abscissas in (*a*) and (*b*).

$$C = \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} (f(q(i,j;n),\cdot);(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)\mathrm{ar}(i;n)) \tag{112}$$

The transport theorem, in the form of Eq. (78), can be used to compute the rate of change of the circulation of the velocity $\mathbf{v}$ around the convecting loop C. After setting $\mathbf{w} := \mathbf{v}' + i_{\mathbf{v}} d\mathbf{v} = \mathbf{v}' + (\nabla\mathbf{v} - \nabla\mathbf{v}^\top)\mathbf{v}$ and using Eqs. (5) and (32), we have

$$\dot{(\int_C \mathbf{v})}(\tau) = \int_{C(\tau)} (\mathbf{v}' + i_{\mathbf{v}} d\mathbf{v})(\tau) + \int_{\partial C(\tau)} (i_{\mathbf{v}}\mathbf{v})(\tau) \tag{113}$$

$$= \int_{C(\tau)} \mathbf{w}_\tau \tag{114}$$

$$= \lim_{n\to\infty} \sum_{i,j=0}^{2^n-1} (\mathbf{w}_{\tau 2,1}(f(q(i,j;n),\tau)) \tag{115}$$

$$- \mathbf{w}_{\tau 1,2}(f(q(i,j;n),\tau)))\mathrm{ar}(i;n) \tag{116}$$

Here $\mathbf{w}_{\tau i,j}$ is the directional derivative of the ith component of $\mathbf{w}_\tau$ in the direction $\mathbf{e}_j$, $i,j = 1, 2$. For future reference, let

$$\dot{C}_{\mathrm{TT}}^{(n)} := \sum_{i,j=0}^{2^n-1} (\mathbf{w}_{\tau 2,1}(f(p(i,j;n),\tau)) - \mathbf{w}_{\tau 1,2}(f(p(i,j;n),\tau)))\mathrm{ar}(i;n) \tag{117}$$

We numerically compute the rate of change of circulation $\dot{C}_{\mathrm{TT}}^{(n)}$ for different values of n, up to $n = 8$. We choose the dimensionless viscosity $\nu = 1$ and dimensionless circulation strength $\Gamma = 10$. To test the sensitivity of the results on the size of the integration time step, computations are performed with different choices of the dimensionless integration time step $\Delta\tau$. We find that $\Delta\tau = 0.001$ is sufficiently small to yield time-step-size independent results. We thus fix $\Delta\tau = 0.001$. The original fractal curve at $\tau = 0$, the deformed fractal curve at $\tau = 2$ for $n = 8$, and streamlines of the velocity field of the Lamb–Oseen [25, 26] vortex are shown in Fig. 7(*b*).

Since the velocity field is initially curl-free and, as is evident from Eq. (111), slowly diverges from this state, the rate of change of circulation $\dot{C}_{\mathrm{TT}}^{(n)}$ is 0 at $\tau = 0$, as shown in Fig. 7(*a*). Due to the viscous damping of the Lamb–Oseen [25, 26] vortex, the circulation and, thus, the rate of change of circulation $\dot{C}_{\mathrm{TT}}^{(n)}$ changes in time. The quantity $\dot{C}_{\mathrm{TT}}^{(n)}$ is found to have a peak around $\tau \approx 0.04$ as shown in the inset of Fig. 7(*a*). As $\tau \to \infty$, $\dot{C}_{\mathrm{TT}}^{(n)} \to 0$. The choice of n associated with the spatial discretization—that is, the truncation of the infinite sum—is found to have the most influence on the peak region, with smaller choices of n resulting in lower, and slightly delayed peak values of $\dot{C}_{\mathrm{TT}}^{(n)}$. The results show convergent behavior for increasing n. The values of $\dot{C}_{\mathrm{TT}}^{(n)}$ after the peak region are only marginally influenced by the choice of n.

6.3 The Rate of Change of the Energy of a Neo-Hookean Material Undergoing a Discontinuous Deformation. Select numbers $x_\circ > 0$, $y_\circ > 0$, and $z_\circ > 0$ and consider the rectangular reference domain

$$\mathcal{D}_R := [-x_\circ, x_\circ] \times [-y_\circ, y_\circ] \times [-z_\circ, z_\circ] \tag{118}$$

The goal is to formulate a deformation of $\mathcal{D}_R$ in which a crack initiates and propagates. More specifically, it is desired that at time t a crack has opened up along the surface

$$\mathcal{S}_t := [-v_\circ t, v_\circ t] \times [-z_\circ, z_\circ] \tag{119}$$

in the (x, z)-plane, where $v_\circ > 0$ is the given steady speed of propagation of a crack that forms in the reference domain $\mathcal{D}_R$. Toward this end, we utilize the function ϕ defined consistent with

$$\phi(x,t) = \begin{cases} 0.2 v_\circ t \exp(-\dfrac{x^2}{v_\circ^2 t^2 - x^2}) & \text{if} \quad |x| < v_\circ t, \\ 0 & \text{if} \quad |x| \geq v_\circ t \end{cases} \tag{120}$$

For each t, $\phi(\cdot, t)$ is a bump function. The deformation f of $\mathcal{D}_R$ is given by

$$f((x,y,z),t) = \begin{cases} (x, \phi(x,t) + (1 - \phi(x,t)/y_\circ)y, z) & \text{if} \quad y > 0, \\ (x, -\phi(x,t) + (1 - \phi(x,t)/y_\circ)y/, z) & \text{if} \quad y < 0 \end{cases} \tag{121}$$

Let $\mathcal{D}_t$ denote the spatial region occupied by the body at time t. Where defined, we set

$$I_1 := \mathrm{tr}(\nabla f^\top \nabla f) \tag{122}$$

$$I_3 := \det \nabla f \tag{123}$$

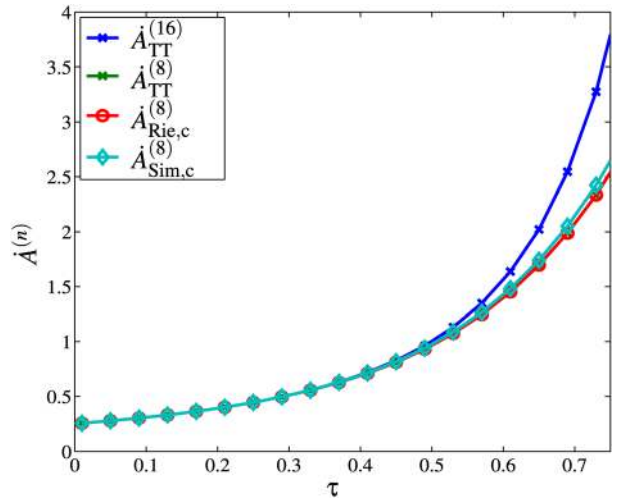

Fig. 6 Comparison of $\dot{A}_{TT}^{(8)}$, $\dot{A}_{Rie,c}^{(8)}$, and $\dot{A}_{Sim,c}^{(8)}$ with $\dot{A}_{TT}^{(16)}$.

Assuming a compressible neo-Hookean response, the free energy-density ψ measured per unit volume in the spatial configuration is given by

$$\psi = \frac{\mu}{2I_3}(I_1 - 3 - 2\log I_3) + \frac{\kappa}{2I_3}(I_3 - 1)^2 \tag{124}$$

where μ is the shear modulus and κ is the bulk modulus, as found in the book by Ogden [27]. The net free-energy $E(t)$ of the system at time t is given by

$$E(t) = \int_{\mathcal{D}_t} \psi((x,y,z),t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\mathcal{S}_t} \Gamma\,\mathrm{d}x\mathrm{d}z \tag{125}$$

where the integral over $\mathcal{S}_t$ is the contribution to the energy due to crack formation and Γ is the energy required to extend the crack by a unit length, measured per unit area in the spatial configuration. The form of this energy is based on Griffith's [28] model for fracture, as formulated by Francfort and Marigo [29].

To render the energy dimensionless, we choose $x_\circ$, $v_\circ$, and Γ as three characteristic quantities. Dimensionless space (ξ, ζ, η) and time τ variables may then be defined by

$$\xi = \frac{x}{x_\circ}, \qquad \zeta = \frac{y}{x_\circ}, \qquad \eta = \frac{z}{x_\circ} \tag{126}$$

and

$$\tau = \frac{v_\circ t}{x_\circ} \tag{127}$$

In terms of these dimensionless quantities, the reference domain becomes

$$\widetilde{\mathcal{D}}_R := [-1,1] \times [-y_\circ/x_\circ, y_\circ/x_\circ] \times [-z_\circ/x_\circ, z_\circ/x_\circ] \tag{128}$$

and the crack surface at time τ is

$$\widetilde{\mathcal{S}}_\tau := [-\tau,\tau] \times [-z_\circ/x_\circ, z_\circ/x_\circ] \tag{129}$$

It is also convenient to define a dimensionless counterpart $\tilde{f}$ of the deformation f by

$$\tilde{f}((\xi,\zeta,\eta),\tau) := \frac{f((x_\circ\xi, x_\circ\zeta, x_\circ\eta), x_\circ\tau/v_\circ)}{x_\circ} \tag{130}$$

The effect of this deformation on the reference domain $\widetilde{\mathcal{D}}_R$ is depicted in Fig. 8.

Bearing in mind that I_1 and I_3 are unaffected by the nondimensionalization, it is useful to define a dimensionless free-energy density $\tilde{\psi}$ by

$$\tilde{\psi} = \frac{\mu x_\circ}{2I_3\Gamma}(I_1 - 3 - 2\log I_3) + \frac{\kappa x_\circ}{2I_3\Gamma}(I_3 - 1)^2 \tag{131}$$

in which case the dimensionless net free-energy at dimensionless time τ is given by

$$\tilde{E}(\tau) = \int_{\widetilde{\mathcal{D}}_\tau} \tilde{\psi}((\xi,\zeta,\eta),t)\,\mathrm{d}\xi\,\mathrm{d}\zeta\,\mathrm{d}\eta + \int_{\widetilde{\mathcal{S}}_\tau} \mathrm{d}\xi\,\mathrm{d}\eta \tag{132}$$

Notice that $\tilde{E}$ and E are related by

$$\tilde{E}(\tau) = \frac{E(x_\circ\tau/v_\circ)}{x_\circ^2\Gamma} \tag{133}$$

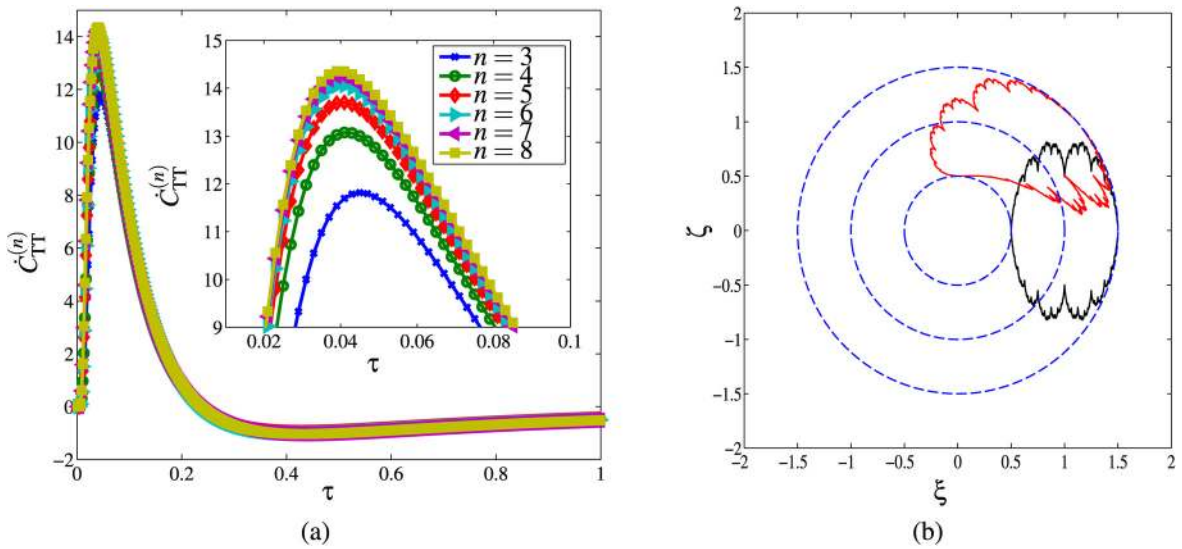

Fig. 7 (*a*) Rate of change of the circulation computed with the transport identity (117) for different *n* with dimensionless integration time step $\Delta\tau = 0.001$, dimensionless kinematic viscosity $\nu = 1$, and dimensionless circulation $\Gamma = 10$. The inset in (*a*) shows a detailed view of the region around the peak of maximal rate of change of the circulation. (*b*) Original fractal curve at $\tau = 0$ and deformed fractal curve at $\tau = 2$ for $n = 8$ and $\Gamma = 10$ along with streamlines of the velocity field of the Lamb–Oseen [25, 26] vortex (dashed lines).

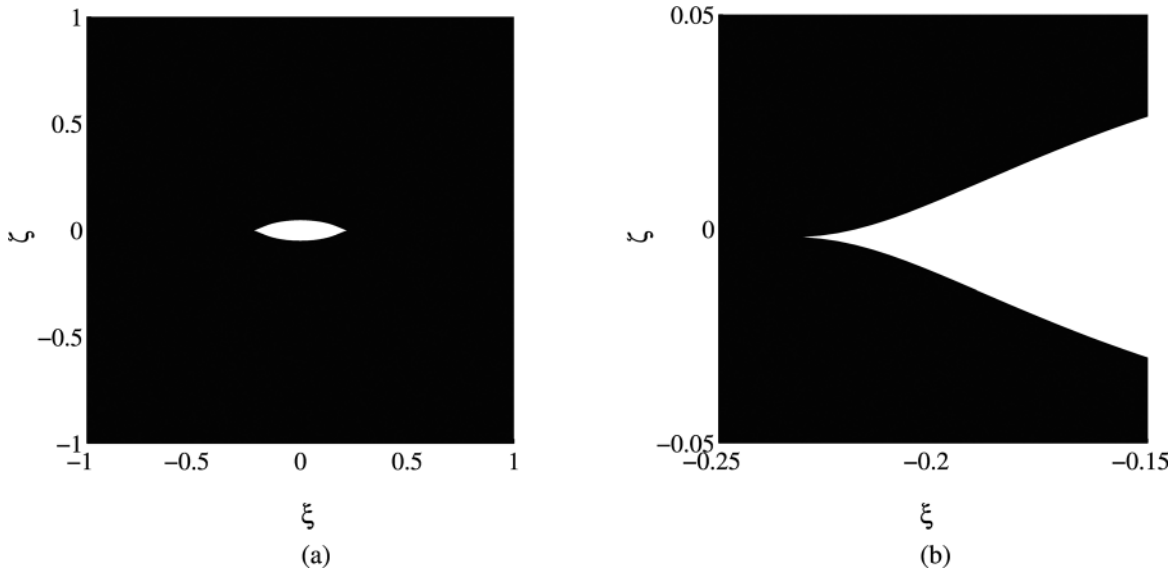

Fig. 8 Snapshots of a cross section of the domain $\widetilde{\mathcal{D}}_R$ in the (ξ, ζ)-plane at $\tau = 0.25$. (*a*) The entire cross section $[-1, 1] \times [-1, 1]$. (*b*) A detailed view $[-0.25, -0.15] \times [-0.25, -0.15]$ of the cross section showing the crack tip.

The important dimensionless parameters that appear in Eqs. (132) and (133) are

$$\frac{y_\circ}{x_\circ}, \qquad \frac{z_\circ}{x_\circ}, \qquad \frac{\mu x_\circ}{\Gamma}, \qquad \text{and} \qquad \frac{\kappa x_\circ}{\Gamma} \tag{134}$$

For all τ in $[0, 1]$, any natural number n, and $0 \le i, j, k \le 2^n - 1$, define ξ, ζ, η, c, and ar such that

$$\xi(i;n) := -1 + i2^{1-n} \tag{135}$$

$$\zeta(i,j,\tau;n) := \begin{cases} -\phi(\xi(i;n),\tau) - [y_\circ/x_\circ - \phi(\xi(i;n),\tau)](2^{n-1} - j)2^{-n+1} & \text{if} \quad 0 \le j \le 2^{n-1} - 1, \\ \phi(\xi(i;n),\tau) + [y_\circ/x_\circ - \phi(\xi(i;n),\tau)](j - 2^{n-1})2^{-n+1} & \text{if} \quad 2^{n-1} \le j \le 2^n - 1 \end{cases} \tag{136}$$

$$\eta(k;n) := -z_\circ/x_\circ + 2z_\circ i 2^{-n}/x_\circ \tag{137}$$

$$c(i,j,\tau;n) := \begin{cases} -\phi'(\xi(i;n),\tau) j 2^{-n+1} & \text{if} \quad 0 \le j \le 2^{n-1} - 1, \\ \phi'(\xi(i;n),\tau)(1 - (j - 2^{n-1})2^{-n+1}) & \text{if} \quad 2^{n-1} \le j \le 2^n - 1 \end{cases} \tag{138}$$

$$\mathrm{ar}(i,\tau;n) := 8\frac{z_\circ}{x_\circ}[y_\circ/x_\circ - \phi(x(i;n),t)]2^{-3n} \tag{139}$$

Moreover, define q and α by

$$q(i,j,k,\tau;n) := (\xi(i;n), \zeta(i,j,\cdot;n), \eta(k;n)) \tag{140}$$

$$\alpha(i,\tau;n) := \mathrm{ar}(i,\tau;n)\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \tag{141}$$

It can be shown that the limit

$$J = \lim_{n\to\infty} \sum_{i,j,k=0}^{2^n-1} (q(i,j,k,\cdot;n); \alpha(i,\cdot;n)) \tag{142}$$

of evolving chains converges (in the C_2^1 norm) to the evolving chain that represents the evolving domain. It can also be shown that

$$E_\tau \partial J = \lim_{n\to\infty} \sum_{i,j,k=0}^{2^n-1} (q(i,j,k,\tau;n); c(i,j,\tau;n)\mathbf{e}_2 \otimes \alpha(i,\tau;n))$$

$$G_\tau J = \lim_{n\to\infty} \sum_{i,j,k=0}^{2^n-1} (q(i,j,k,\tau;n); \alpha'(i,\tau;n))$$

Using the generalized transport theorem, Eq. (5), and Eq. (32), we find that

$$\dot{\tilde{E}}(\tau) = \int_{J(\tau)} \tilde{\psi}'_\tau \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \int_{E_\tau J} d(\tilde{\psi}_\tau \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) + \int_{E_\tau \partial J} \tilde{\psi}_\tau \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \int_{G_\tau J} \tilde{\psi}_\tau \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 + \frac{4z_\circ}{x_\circ} \tag{143}$$

$$= \lim_{n\to\infty} \sum_{i,j,k=0}^{2^n-1} \Big[\tilde{\psi}'(q(i,j,k,\tau;n),\tau)\mathrm{ar}(i,\tau;n) + \tilde{\psi}_{,2}(q(i,j,k,\tau;n),\tau)c(i,j,\tau;n)\mathrm{ar}(i,\tau;n) + \tilde{\psi}(q(i,j,k,\tau;n),\tau)\mathrm{ar}'(i,\tau;n)\Big] + 4z_\circ/x_\circ \tag{144}$$

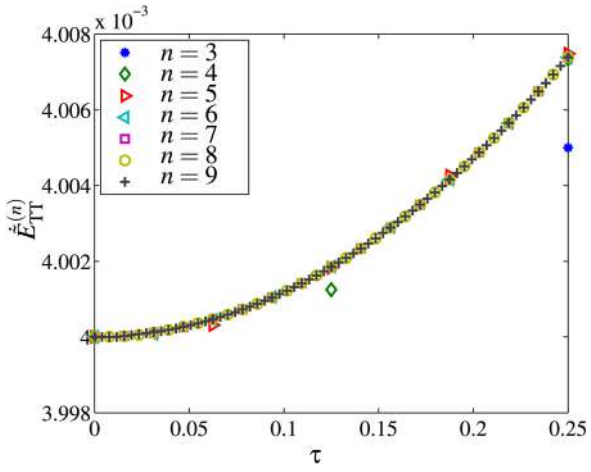

Fig. 9 Comparison of $\dot{\tilde{E}}^{(n)}_{TT}$ over nondimensional time τ for different resolutions n

where $\tilde{\psi}_{,2}$ is the partial derivative of $\tilde{\psi}$ with respect to ζ. For future reference, let

$$\begin{aligned}\dot{\tilde{E}}^{(n)}_{TT} := \sum_{i,j,k=0}^{2^n-1} \Big[& \tilde{\psi}'(q(i,j,k,\tau;n),\tau)\mathrm{ar}(i,\tau;n) \\ & + \tilde{\psi}_{,2}(q(i,j,k,\tau;n),\tau)c(i,j,\tau;n)\mathrm{ar}(i,\tau;n) \\ & + \tilde{\psi}(q(i,j,k,\tau;n),\tau)\mathrm{ar}'(i,\tau;n)\Big] + 4z_\circ/x_\circ \end{aligned} \tag{145}$$

The rate of change of the dimensionless energy (145) is computed numerically for different values of n up to $n=9$. We choose the referential dimensions in accord with $x_\circ = 1.0\,\mathrm{m}$, $y_\circ = 1.0\,\mathrm{m}$, and $z_\circ = 1.0 \cdot 10^{-3}\mathrm{m}$, so that the dimensionless parameters $y_\circ/x_\circ$ and $z_\circ/x_\circ$ in Eq. (134) are respectively equal to 1.0 and $1.0 \cdot 10^{-3}$. Following Anand [30] and Groutos et al. [31], we choose material parameters characteristic for rubber-like substances:

$$\left.\begin{aligned} \mu &= 0.3\mathrm{MPa}, \\ \kappa &= 2.0 \cdot 10^3 \mathrm{MPa}, \\ \Gamma &= 4.0 \cdot 10^3 \mathrm{J/m^2} \end{aligned}\right\} \tag{146}$$

In combination with the choice $x_\circ = 1.0\,\mathrm{m}$, the choices of μ, κ, and Γ in Eq. (146) yield the values of $7.5 \cdot 10^{-5}$ and 0.5 for the remaining dimensionless parameters $\mu x_\circ/\Gamma$ and $\kappa x_\circ/\Gamma$ in Eq. (134). Notice that the (nontruncated) rate of change (144) in this example is exact in the sense that it can be evaluated at arbitrary times without computing previous time steps, since it does not depend on the time-history.

However, in the truncated case, there is a natural connection between the space discretization and the time-discretization. In particular, the size $2x_\circ 2^{-n}$ of one element in space, the crack propagation speed v_0, and the time step Δt are related by

$$\Delta t = \frac{2x_\circ 2^{-n}}{v_\circ} \tag{147}$$

Given a crack propagation speed $v_\circ$, the criterion (147) can be found from the requirement that physically reasonable information may only be obtained at time steps for which the crack propagates one mesh spacing further. That is, the discrete time step Δt is the time the crack needs to propagate from one discrete space point to the next discrete space point. Notice that technically, through the choice of (120), the expression (145) allows for the computation of $\dot{\tilde{E}}^{(n)}_{TT}$ for arbitrary combinations of the space and time steps, however, results for $\Delta t < 2x_\circ 2^{-n}/v_\circ$ may contain spurious behavior.

Figure 9 shows snapshots of $\dot{\tilde{E}}^{(n)}_{TT}$ at discrete time steps in agreement with Eq. (147). The results have been validated with the corresponding computations in the referential description. The rate of change shows convergence with increasing n. For $n=6$ and larger, the results at the corresponding time steps are practically indistinguishable, as can be seen from Fig. 9. However, in view of the coupling (147) for increasing time resolution of $\tilde{E}$, it is also necessary to add spatial points correspondingly. Interestingly, if no particular resolution in time is required, $n=6$ is sufficient to predict the rate of change at the corresponding time steps with very good accuracy. Notice that the good convergence at a relatively small n compared to the n necessary in the previously discussed examples can be attributed to the absence of fractal boundaries in the present example.

7 Looking Forward

We next discuss some possible applications of the roughened transport theorem. For each of these examples, we state how the transport theorem might be used and discuss some attendant difficulties. This section is devoted more to posing interesting questions than to answering those questions.

One topic that is of interest to researchers in continuum physics is the weakening of the regularity assumptions underlying the characterization of continuous bodies. In the middle of the 20th century, a continuous body was assumed, usually tacitly, to occupy regions of space that were open with regular (in some sense) boundaries. If pressed, researchers usually stated that such a boundary should consist of the union of a finite number of smooth surfaces. Later, Gurtin et al. [32] and Noll and Virga [33] used geometric measure theory to specify classes of sets that are appropriate for occupation by continuous bodies.[6] The main idea was to consider sets with boundaries that, while not necessarily smooth, are required to have an exterior unit-normal almost everywhere (with respect to the Lebesgue areal measure). This condition allows for the computation of fluxes across the boundary and the application of the divergence theorem. Recently, there has been interest in considering even rougher sets to represent continuous bodies and obtaining balance laws for such bodies. See, for example, the work of Marzocchi [12], Rodnay and Segev [13], Šilhavý [34], and Schuricht [35]. Marzocchi [12], at the end of his work, used a weak formulation to sketch a path to writing balance laws for a body represented by a differential chain. His balance laws should hold for bodies with fractal boundaries that evolve in a "nice" way—that is, by a convection. We hope that the roughened transport theorem presented and discussed in the present work will enable the formulation of balance laws for bodies that do not evolve in such a way or may have rough boundaries.

7.1 Balance Laws for Rough Bodies. To illustrate the challenges associated with the formulation of balance laws for rough bodies, it is instructive to begin by considering the principle of conservation of mass. The other basic balance laws can be discussed in a similar way. Let $\mathcal{B}_R$ be an open subset of a three-dimensional Euclidean space and let $f : \mathcal{E} \times I \to \mathcal{E}$ be a continuous mapping that is differentiable in time. We do not assume that f is differentiable in space. Hence, f need not be a convection of the type defined in Sec. 4. The open set $\mathcal{B}_R$ represents a reference for the body and f describes a deformation of the body. For all open sets $\mathcal{Q}_R$ contained in $\mathcal{B}_R$, let $\mathcal{Q}_t := f_t(\mathcal{Q}_R)$. Since f is only continuous, the boundary of $\mathcal{Q}_t$ can be very irregular even if $\mathcal{Q}_R$ is smooth. We denote by Q_t the differential chain corresponding to the open set $\mathcal{Q}_t$. Since regularly evolving chains may represent regions with irregular boundaries, it seems possible to represent $Q := t \mapsto Q_t$ as a regularly evolving differential chain. Whether or

[6] These are more suitable since they are, for example, stable under interactions.

not this is actually possible is an interesting open question. If the answer to this question is negative, it would be useful to determine minimal assumptions on f necessary to ensure that Q is a regularly evolving differential chain. To proceed, assume that Q is a regularly evolving differential chain so we can apply the transport theorem. Let a scalar-valued function ρ of space and time represent the mass density of the body. The principle of conservation of mass states that for all parts $\mathcal{Q}_R$ of $\mathcal{B}_R$, and hence all corresponding evolving chains Q,

$$\dot{\overline{(\int_Q \rho)}}(t) = 0 \quad \text{for all } t \in I \tag{148}$$

By the roughened transport identity Eq. (68), Eq. (148) becomes

$$\int_{Q_t} \rho_t{}' + \int_{E_t \partial Q} \rho_t + \int_{G_t Q} \rho_t = 0 \quad \text{for all } t \in I \tag{149}$$

where $\rho_t(x) = \rho(x,t)$ for any x in Q_t. If the deformation f were smooth then $\int_{E_t \partial Q} \rho_t + \int_{G_t Q} \rho_t$ could be replaced by $\int_Q \operatorname{div}(\rho_t \mathbf{v}_t)$, where $\mathbf{v}$ is the velocity corresponding to f and, consistent with Eq. (72), $\mathbf{v}_t(x) = \mathbf{v}(x,t)$ for any x in Q_t. As f is only continuous in space, the divergence of the velocity is undefined and this substitution cannot be made. However, since Eq. (149) must hold for all evolving parts Q, some restriction on the density ρ is clearly implied. The nature of this restriction is presently unknown. To apply the theory of differential chains and the roughened transport theorem to obtain local balance laws from global balance laws, additional work must be done to investigate the condition(s) that ρ must satisfy for Eq. (149) to hold for all Q.

7.2 Balances Laws for Evolving Surfaces. Besides obtaining balances for evolving regions, it is also useful to obtain balances for evolving interfaces. The integral form of a surface balance law for an evolving smooth surface $\mathcal{S}$ contains a term of the form

$$\dot{\overline{\int_{\mathcal{A}} \phi}} \tag{150}$$

where ϕ is a density defined on the surface and $\mathcal{A}$ is an arbitrary subsurface of $\mathcal{S}$. Often the surface $\mathcal{A}$ does not evolve according to a flow, as in situations where $\mathcal{S}$ corresponds to a shock or phase interface. Moreover, there are settings where the surface is rough and evolves in a nonsmooth fashion, as is clear from the works of Gurtin [6], Catalan et al. [7], and Stanley [8]. It is tempting to replace $\mathcal{A}$ in Eq. (150) with an evolving 2-chain A and the Riemann integral with the modified integral associated with the symbol $\int$, but there is a problem with proceeding that way. In general, it is not possible to integrate a density over a differential 2-chain; only 2-forms can be integrated. However, there are some 2-chains over which it is possible to integrate a density. The task is to find an appropriate subspace of chains with this property. To expand a term of the form (150) when $\mathcal{A}$ evolves irregularly, it is therefore necessary to modify the roughened transport theorem so that it holds for evolving chains with additional properties and, thus, make it is possible to integrate a density over the chain at each time. Exactly what subspace of evolving chains would be appropriate here is an open question. However, in her work on the Plateau problem, Harrison [36] proposes a subspace that might prove valuable in this context.

Although the roughened transport theorem has potential applications in the formulation of balance laws in continuum physics, further developments in the theory of evolving chains will be necessary before tangible progress can be made. We hope that this paper will inspire researchers to address the underlying challenges.

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