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EXTENSION OF A THEOREM OF LAGUERRE TO ENTIRE FUNCTIONS OF EXPONENTIAL TYPE II

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1. Introduction. A domain whose boundary is a circle or a straight line is called a circular domain. The following theorem of Laguerre [**6**, pp. 56-63] which we state in a form used in [**12**, p. 33] does not only play an important role in the location of critical points of polynomials [**7**] but also allows one to deduce Bernstein's inequality for polynomials on the unit disk and various of its refinements [**3**; **10**, Chapters 1 and 4].

THEOREM A. *Let $p(z)$ be a polynomial of degree $n \geq 1$. If $p(z) \neq 0$ in a (closed or open) circular domain K, then*

$$n\, p(z) - (z - \zeta)p'(z) \neq 0 \quad \text{for } z \in K,\ \zeta \in K$$

which in the case $\zeta = \infty$ means that $p'(z) \neq 0$ for $z \in K$.

With the object of getting a result of similar scope for entire functions of exponential type we recently [**11**] proved the following

THEOREM 1. *Let f be an entire function of exponential type $\tau > 0$ such that*

$$h_f(\pi/2) := \limsup_{r \to \infty} \frac{\log |f(re^{i\pi/2})|}{r} = 0$$

and denote by H the (closed or open) upper half-plane. If $f(z) \neq 0$ for $z \in H$, then

$$\tau f(z) + i(1 - \zeta)f'(z) \neq 0 \text{ for } z \in H \text{ and } |\zeta| \leq 1.$$

We showed that the assumptions of this theorem cannot be relaxed and it constitutes an extension of Theorem A. Furthermore, we deduced

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various old and new results on entire functions of exponential type from it. The purpose of this note is to present further applications of Theorem 1 which include a new short proof of Bernstein's inequality for entire functions of exponential type and a necessary condition for the stability problem of delay-equations. Together with the results given in [**11**] one can see that Theorem 1 indeed provides a useful tool for entire functions of exponential type.

2. Bernstein's inequality and refinements. We first show that the famous inequality of Bernstein [**2**] for entire functions of exponential type is a simple consequence of Theorem 1.

COROLLARY 1. *Let f be an entire function of exponential type $\tau > 0$ such that $|f(x)| \leq 1$ for $x \in R$. Then*

$$|f'(x)| \leq \tau \text{ for } x \in R. \tag{1}$$

More precisely, for a given $x_0 \in \mathbf{R}$,

$$|f'(x_0)|^2 + \{\tau^2 - |f'(x_0)|^2 \sin^2 \alpha\}|f(x_0)|^2 \leq \tau^2, \tag{2}$$

where $\alpha := \arg\{f(x_0)/f'(x_0)\}$ if $f(x_0), f'(x_0)$ are both different from zero; otherwise α can be any real number.

PROOF. For every $\lambda \in C$ such that $|\lambda| > 1$ the function

$$F(z) := e^{i\tau z} f(z) - \lambda$$

is of exponential type 2τ such that $h_F(\pi/2) = 0$. Furthermore, $|f(x)| \leq 1$ for $x \in \mathbf{R}$ implies (see [**11**]) that $|e^{i\tau z} f(z)| \leq 1$ for all z in the closed upper half-plane H. Hence Theorem 1 applies to F with τ replaced by 2τ and we obtain

$$e^{i\tau x}\{\tau f(x) + if'(x) + \zeta(\tau f(x) - if'(x)\} - 2\tau\lambda \neq 0$$

for all $x \in R$, $|\lambda| > 1$ and $|\zeta| \leq 1$. This implies

$$|\tau f(x) + if'(x)| + |\tau f(x) - if'(x)| \leq 2\tau. \tag{3}$$

Now (1) follows on noting that, by the triangle inequality, the left hand side of (3) cannot be smaller than $2|f'(x)|$.

For (2) we write

$$|\tau f(x_0) \pm i f'(x_0)| = \{|f'(x_0)|^2 + \tau^2|f(x_0)|^2 \pm 2\tau|f(x_0)f'(x_0)\sin\alpha|\}^{1/2}$$

in (3) (with $x = x_0$) and square the two sides obtaining thereby

$$\{(|f'(x_0)|^2 + \tau^2|f(x_0)|^2)^2 - 4\tau^2|f(x_0)f'(x_0)|^2\sin^2\alpha\}^{1/2} \\ \leq 2\tau^2 - |f'(x_0)|^2 - \tau^2|f(x_0)|^2.$$

This is readily seen to be equivalent to (2). □

REMARK. It was proved by Duffin and Schaeffer [**5**] that if f is an entire function of exponential type $\tau > 0$ such that $|f(x)| \leq 1$ for $x \in \mathbf{R}$ then

$$|f'(x)|^2 + \tau^2|f(x)|^2 \leq \tau^2 \tag{4}$$

for $x \in \mathbf{R}$ provided $f(x)$ is real for real x. Our inequality (2) shows that, for (4) to hold at a given point x_0 of the real axis, it is enough that one of the following conditions be satisfied:

(i) $f(x_0) = 0$,

(ii) $f'(x_0) = 0$,

(iii) $f(x_0)/f'(x_0)$ is real.

As another consequence of Theorem 1 we have

COROLLARY 2. [**9**, inequality (3.12)]. *Let f be an entire function of exponential type τ such that $h_f(\pi/2) \leq 0$ and set $\omega(z) = e^{i\tau z}f(z)$. If $|f(x)| \leq 1$ for $x \in \mathbf{R}$, then*

$$|f'(x)| + |\omega'(x)| \leq \tau \text{ for } x \in \mathbf{R}. \tag{5}$$

PROOF. Let λ be any complex number such that $|\lambda| > 1$. Then Theorem 1 applies to $F(z) := f(z) - \lambda$ and yields

$$\tau f(x) + if'(x) - i\zeta f'(x) \neq \lambda\tau$$

for all $x \in \mathbf{R}$, $|\lambda| > 1$ and $|\zeta| \leq 1$. Hence

$$|f'(x)| + |\tau f(x) + if'(x)| \leq \tau \text{ for } x \in \mathbf{R}$$

which is (5).

3. A representation theorem. Yet another obvious consequence of Theorem 1 is the following analogue of a representation theorem of Dieudonné for the logarithmic derivative of a polynomial [**4**, p. 7].

COROLLARY 3. *Let f be an entire function of exponential type $\tau > 0$ such that $h_f(\pi/2) = 0$. If all the zeros of f lie in the (closed or open) lower half-plane G, then, for $z \in C\backslash G$,*

$$\frac{f'(z)}{f(z)} = \frac{i\tau}{1 - 1/\varphi(z)},$$

where φ is holomorphic and $|\varphi(z)| < 1$.

4. Stability of delay-equations. Consider a function f given by

$$f(z) = \sum_{\nu=0}^{n} P_\nu(z)e^{-\lambda_\nu z}, \tag{6}$$

where $P_\nu(z)$ are polynomials and λ_ν are real numbers such that

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n =: \tau.$$

For stability of the solutions of certain delay-equations one would like to know under what conditions such a function f has all its zeros in the open left half-plane (see [**1**, Chapter 12]). For this question we may obviously assume that $P_0(z) \not\equiv 0$. Then f is an entire function of exponential type τ such that $h_f(0) = 0$. In this situation we may utilize the following equivalent form of Theorem 1 obtained by a rotation of the complex plane.

THEOREM 1′. *Let f be an entire function of exponential type $\tau > 0$ such that*

$$\limsup_{x\to\infty} \frac{\log|f(x)|}{x} = 0$$

and denote by $\mathcal{R}$ the (open or closed) right half-plane. If $f(z) \neq 0$ for $z \in \mathcal{R}$, then

$$\tau f(z) + (1-\zeta) f'(z) \neq 0 \text{ for } z \in \mathcal{R} \text{ and } |\zeta| \leq 1.$$

Now we may proceed as follows. Suppose that, for the function f in (6), we have $f(z) \neq 0$ for $z \in \mathcal{R}$ and denote by k the degree of $P_n(z)$. Then, by Theorem 1′, also

$$f_1(z) := \tau f(z) + f'(z) \neq 0 \text{ for } z \in \mathcal{R}.$$

Obviously f_1 is of the form

$$f_1(z) = \sum_{\nu=0}^{n} \tilde{P}_\nu(z) e^{-\lambda_\nu z}$$

with polynomials $\tilde{P}_\nu(z)$ of the same degree as $P_\nu(z)$ except for $\tilde{P}_n(z)$ whose degree has become $k-1$. Defining

$$f_{j+1}(z) := \tau f_j(z) + f_j'(z) \; (j = 1, \ldots, k)$$

we may apply Theorem 1′ repeatedly. After k steps we arrive at an entire function f_{k+1} of the form

$$f_{k+1}(z) = \sum_{\nu=0}^{n-1} Q_\nu(z) e^{-\lambda_\nu z}$$

with polynomials $Q_\nu(z)$ of the same degree as $P_\nu(z)$ which does not vanish in $\mathcal{R}$, according to Theorem 1′. Now we may continue the procedure with τ replaced by λ_{n-1}, and so on. Reducing the number of exponential terms this way we finally end up with a polynomial $P(z)$ of the same degree as $P_0(z)$ which should have all its zeros in $C\backslash\mathcal{R}$. If this turns out to be false (for example, by using the Hurwitz-Routh test) it follows that $f(z)$ is not different from zero in $\mathcal{R}$. The polynomial $P(z)$ can be explicitly written as

$$P(z) = (\lambda_1 + D)^{k_1}(\lambda_2 + D)^{k_2} \dots (\lambda_n + D)^{k_n} P_0(z)$$

where $D := \frac{d}{dz}$ and

$$k_\nu := 1 + \text{ degree } P_\nu(z) \quad (\nu = 1, 2, \dots, n).$$

As a simple example (see also [**8**, p. 18]) let us consider

$$f(z) = a + z + be^{-\lambda z} \ (\lambda > 0). \tag{7}$$

Then

$$P(z) = (\lambda + D)(a + z) = \lambda(a + z) + 1$$

and hence

$$\text{Re } a > -\frac{1}{\lambda}$$

is a necessary condition for (7) to have all its zeros in the open left half-plane.

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