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Extension of KMS States and Chemical Potential

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Abstract. We present an algebraic description of the concept of chemical potential for a general compact gauge group G, as a first step in the classification of thermodynamical equilibrium states of a given temperature. Adopting first the usual setting of a field algebra $\mathscr{F}$ containing the observable algebra $\mathfrak{A}$ as its gauge invariant part, we establish the following results (i) the existence and uniqueness, up to gauge, of τ-weakly clustering states ϕ of $\mathscr{F}$ extending a given such state ω of $\mathfrak{A}$ (τ an asymptotically abelian automorphism group of $\mathscr{F}$ commuting with G) (ii) in the case of an ω faithful and β-KMS for a time evolution commuting with τ, and of a time-invariant ϕ, the fact that ϕ is β-KMS for a one-parameter group of time and gauge whose gauge part lies in the center of the stabilizer G_ϕ of ϕ. (iii) a description of the general case where ϕ is neither time invariant nor faithful: ϕ is then in general vacuum-like in directions of the gauge space governed by an "asymmetry subgroup". We further analyze the representations and von Neumann algebras determined by ω, ϕ and the gauge average $\bar{\omega}$ of ϕ. The covariant representation generated by $\bar{\omega}$ is shown to be obtained by inducing up from G_ϕ to G the representation generated by ϕ. Finally we present, for the case where G is an n-dimensional torus, an intrinsic description of the chemical potential in terms of cocycle Radon-Nicodym derivatives of the state ω w.r.t. its (quasi equivalent) transforms by localized automorphisms of $\mathfrak{A}$. Our main result (ii) is established using two independant techniques, the first making systematic use of clustering properties, the second relying on the analysis of representations. Both proofs are basically concerned with Tannaka duality—the second with a version thereof formulated in Robert's theory of Hilbert spaces in a von Neumann algebra.

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I. Introduction

The physical motivation of this study arose from the following basic question: given an infinitely extended dynamical quantum system (an infinite medium) how does one define its thermodynamic equilibrium states? Traditionally this question is approached by considering first finite systems; for these one argues that an equilibrium state may be described by an appropriate Gibbs ensemble; then one increases the size and takes the "thermodynamic limit" (infinite extension). If one starts from a grand canonical ensemble, characterized by the inverse temperature β and the chemical potentials μ_i of the various components (species of particles in the system) then the thermodynamic limit is most simply described: one keeps μ_i, β fixed and just increases the volume of the system. If the infinite volume limit of the expectation values of local observables exists, then it defines a state ϕ_{β,μ_i} for the infinite medium and one demonstrates that this satisfies (due to its construction from grand canonical ensembles) the Kubo-Martin-Schwinger (KMS) condition with parameter β for a one parameter subgroup of the symmetry group [1]. The latter consists of time translation and gauge transformations (one parameter for each species of particles), so it is a $(k+1)$-parametric Abelian Lie group, if k is the number of species of particles (chemical components). If we denote the generators of time translations and gauge transformations in the Lie algebra of the symmetry group respectively by H and N_i then the one-parameter subgroup mentioned above is generated by

$$X = H - \sum_i \mu_i N_i \tag{I.1}$$

and one may summarize the conclusion mathematically by saying that an equilibrium state of the infinite system is a state whose modular automorphism group [2, 3] is generated by $-\beta X$, i.e. it is a KMS-state with respect to this subgroup. The equilibrium parameters β, μ_i determine the selection of the subgroup.

An alternative way to characterize equilibrium states of an infinite system has been described in [4]. It does not use Gibbs ensembles of finite systems but characterizes the pure phase equilibrium states of an infinite medium directly as those states which are stationary (i.e. invariant under time translation), primary (i.e. not centrally decomposable)[1] and stable against any small local[2] perturbation of the dynamics. It is then shown in [4] that such a state satisfies the KMS-condition for the time translation group with some value of β[3] (unless it is a ground state). In other words, its modular automorphism group is generated by $-\beta X$ with $\mu_i = 0$. Conversely it is known that any KMS state is stable under local perturbations of the dynamics [5]; and that an extremal (pure phase) KMS state is primary (see e.g. [6] Theorem 10.3). So the requirements on a state to be stationary, primary and locally stable are essentially equivalent to requiring that it be an extremal KMS-state with respect to time translation for some value of β.

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[1] This is taken as an expression of the "pure phase" requirement

[2] The word "local" is used in the physical sense: the perturbations considered are changes of the dynamical law in a finite space region

[3] Actually in deriving the KMS condition from stability in [4] the absolute integrability of correlation functions has been assumed

In order to see how the chemical potentials enter one has to look at the rôle of gauge invariance more closely. The setting of the problem may be described as follows. There is a C^*-algebra $\mathscr{F}$, the algebra of quasilocal operations, which we shall also call the "*field algebra*" by analogy with the situation in Quantum Field Theory. From the point of view of the physicist this algebra is thought of as being generated by the creation operators $a_i^*(f)$ (and annihilators $a_i(f)$) of the various particle types $(i=1,\ldots,k)$ at finite times in essentially finite space volumes (f symbolizes the wave functions of the created particle). To remind us of this background we denote elements of $\mathscr{F}$ by small Roman characters $a, b \ldots$

The gauge group G is represented by a strongly continuous automorphism group on $\mathscr{F}$. In the example envisaged so far G is the k-dimensional torus; an element $g \in G$ is parametrized by k angles $0 \leqq \phi_i < 2\pi$ and the action of the corresponding automorphism is explicitly

$$\gamma_g(a_i^*(f)) = e^{i\phi_i} a_i^*(f) \,; \quad \gamma_g(a_i(f)) = e^{-i\phi_i} a_i(f) \,. \tag{I.2}$$

We shall, however, consider in the sequel also more general gauge groups.

On $\mathscr{F}$ we have also an action of the groups of time translation and space translation by automorphisms α_t, respectively $\alpha_{\mathbf{x}}$. These automorphism groups commute with the gauge transformations:

$$\alpha_t \gamma_g = \gamma_g \alpha_t \,; \quad \alpha_{\mathbf{x}} \gamma_g = \gamma_g \alpha_{\mathbf{x}} \,; \tag{I.3}$$

and $\mathscr{F}$ is asymptotically Abelian in the sense[4]

$$\lim_{|t| \to \infty} \| [a, \alpha_t(b)] \| = 0 \,; \quad \lim_{|\mathbf{x}| \to \infty} \| [a, \alpha_{\mathbf{x}}(b)] \| = 0 \,. \tag{I.4}$$

Actually we shall use in the following only asymptotic Abelianness with respect to some subgroup τ^n of automorphisms and one may then interpret τ^n either as a sequence of space translations or time translations moving to infinity.

The set of all gauge invariant elements of $\mathscr{F}$ is a C^*-subalgebra of $\mathscr{F}$ which we denote by $\mathfrak{A}$ and call the *algebra of observables*. For its elements we use capital Roman letters $A, B \ldots$

Because of (1.3) the automorphisms $\alpha_t, \alpha_{\mathbf{x}}, \tau^n$ of $\mathscr{F}$ define corresponding automorphisms of $\mathfrak{A}$, the restrictions of the former to $\mathfrak{A}$. Ultimately the physical information contained in an equilibrium state $\phi_{\beta\mu_i}$ concerns only the expectation values for *observables* i.e. it concerns only the state ω_{β,μ_i} which is the restriction of ϕ_{β,μ_i} to $\mathfrak{A}$. Since the gauge transformations act trivially on $\mathfrak{A}$ a KMS state of $\mathscr{F}$ with respect to the generator (1.1) will become in restriction to $\mathfrak{A}$ a KMS state with respect to time translation and parameter β.

Thus, irrespective of whether one starts with grand canonical ensembles and performs the thermodynamic limit to obtain the states ϕ_{β,μ_i} of $\mathscr{F}$ and restricts them again to the states ω_{β,μ_i} of $\mathfrak{A}$ or whether, alternatively, one considers the dynamical system $(\mathfrak{A}, \alpha_t)$ and singles out equilibrium states by the stability requirement under local perturbations (from $\mathfrak{A}$) of the dynamics, one reaches the same conclusion: the equilibrium states ω are KMS states of $\mathfrak{A}$ with respect to α_t, and parameter β; in short they are (α_t, β)-KMS states.

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[4] In (I.4) the bracket [,] denotes the generalized commutator. If $\mathscr{F}$ contains Fermi-type elements then the bracket between two such elements means the anticommutator

However, since the μ_i have observable significance, the ω_{β,μ_i} are different states of $\mathfrak{A}$ for different values of the μ_i i.e. there is a k-parametric family of extremal (α_t, β)-KMS states of $\mathfrak{A}$. In the traditional approach they are distinguished by looking at their extensions to $\mathscr{F}$ which correspond to different modular automorphisms (different directions of the generator (1.1) in the Lie algebra of the symmetry group).

We shall therefore study the relation between (α_t, β)-KMS states of $\mathfrak{A}$ and their extensions to $\mathscr{F}$. The setting will be more general than in the above discussion in so far as we allow for G an arbitrary compact group. The case where G is not just a torus group may be of some physical interest in connection with (exact) non-Abelian internal symmetries and their spontaneous symmetry breaking.

In carrying out the analysis it appears that the essential structural requirement is asymptotic Abelianness of $\mathscr{F}$ under some subgroup of automorphisms τ^n (for its interpretation see above). One then restricts the attention to states which are extremal invariant (weakly clustering) with respect to τ^n. The results of Section II which for simplicity we state here in a specialized form are:

i) Any τ^n-extremal invariant state of $\mathfrak{A}$ has at least one extension to a τ^n-extremal invariant state of $\mathscr{F}$. If ϕ_1 and ϕ_2 are two such extensions then $\phi_1 = \phi_2 \circ \gamma_g$ i.e. ϕ_1 and ϕ_2 differ by a fixed gauge transformation.

This means in particular that the restrictions of states ϕ_{β,μ_i} belonging to different generators (1.1) cannot coincide on $\mathfrak{A}$ i.e. the ω_{β,μ_i} are different states of $\mathfrak{A}$ for different μ_i.

ii) Let ϕ be an extremal invariant state with respect to time translation[5] of $\mathscr{F}$ whose restriction to $\mathfrak{A}$ is an (α_τ, β)-KMS state ω. Denote by G_ϕ the greatest closed subgroup of G under which ϕ is invariant. There may exist a closed, *normal* subgroup N_ϕ of G_ϕ for which ϕ is a "ground state" i.e. the spectrum of the generators of N_ϕ in the representation of $\mathscr{F}$ induced by ϕ is one-sided. If we divide that part out by considering only the restriction of ϕ to the algebra $\mathscr{F}^{N_\phi}$ (the subalgebra of $\mathscr{F}$ which is N_ϕ-invariant) then $\phi|_{\mathscr{F}^{N_\phi}}$ is a KMS state with parameter β for a modified one-parameter group $\tilde{\alpha}_t = \alpha_t \beta_t$ where β_t is a one-parameter subgroup in the center of G_ϕ / N_ϕ.

In the statistical mechanics of a k-component system the subgroup N_ϕ has no great practical significance. It arises if some of the μ_i tend to $-\infty$ which means that the density of these components is zero. So we may forget these components together with the part N_ϕ of the gauge group. Let us assume now that N_ϕ is trivial. If G_ϕ has no central one-parameter subgroup (for instance if G_ϕ is trivial i.e. if the gauge symmetry is completely broken or if G_ϕ is a semi simple group) then ϕ is an (α_t, β)-KMS state and, since it is extremal α_t-invariant it is also an extremal KMS state; hence it is primary and strongly clustering (mixing). If G_ϕ has a central one-parameter subgroup (e.g. when G is a torus and ϕ does not completely break the gauge symmetry) then there is the possibility that ϕ may not be *extremal* KMS (with respect to $\tilde{\alpha}_t$). In that case each of the extremal $(\tilde{\alpha}_t, \beta)$-KMS states into which ϕ may be decomposed is again an extension of the state ω but these extensions are no longer α_t-invariant; the application of α_t to such a state will produce the rotation of

[5] This corresponds to the specialization of Theorem II.4 obtained by choosing τ^n to be time translations (in which case the α'_t of Section II coincides with α_t)

some phase angle in the gauge group[6]. This case may again be considered as a breaking of the gauge invariance but one which becomes visible only when we look at the strongly clustering extensions of ω.

This situation may appear in the superfluid or superconducting state (though not in the Bose-Einstein condensation of the free gas). In the example of chemical potentials (Ga torus) the normal situation is the one where $G_\phi = G$ and the extension ϕ is already extremal KMS with respect to $\tilde{\alpha}_t$.

In Section III the representation of $\mathscr{F}$ and $\mathfrak{A}$ generated by the states described above are studied. This gives a more detailed and deeper insight into the mathematical structure of the extension problem and also provides an alternative way for deriving the results of Section II. A description of these questions and results is given at the beginning of Section III.

The general setting studied in the present paper (involving $\mathscr{F}, G, \mathfrak{A}, \alpha_t$, their properties and relations) is very similar to the one arising in the study of superselection rules and charge numbers in Quantum Field Theory [7]. There the structure is, of course, tighter due to relativistic invariance, locality etc. but in some respects one may regard the present analysis as an adaptation of the questions treated in [7] to the case of equilibrium states with finite temperature. In that spirit one may replace the field algebra $\mathscr{F}$ by localized morphisms acting on $\mathfrak{A}$ as done in [7]. The difference is now that while in [7] these morphisms are applied to the ground state representation of $\mathfrak{A}$ where they produce inequivalent representations of $\mathfrak{A}$ and thereby lead to the charge quantum numbers and associated superselection rules, in the present case, where we apply them to thermodynamic states ω_{β,μ_i}, they lead to quasiequivalent representations. This is intuitively easily understood since these morphisms change the charge (or in our case the particle number) by a finite amount and this will have no drastic effect if the mean particle number of the state is already infinite. Instead of a discrete set of superselection rules labeled by charges we have here a continuous set, labeled by the μ_i. But in order to change μ_i we would need an infinite change in particle number i.e. an infinite product of the localized morphisms mentioned. This will not be considered here. However we shall see in Section IV that localized morphisms of $\mathfrak{A}$ may be used to describe directly the significance of the chemical potentials in the observable algebra (without considering the extension of the state to $\mathscr{F}$).

This work arose from a coalescence of independent efforts to understand the algebraic background of the chemical potential [8,9]. The problem had been initially approached through a conditional stability assumption (sketched in [8]). It was later realized that the chemical potential can be inferred merely by looking at extensions of (α_t, β)-KMS states of $\mathfrak{A}$ to the field algebra, as is shown in [9] and here using different assumptions and techniques[7].

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[6] Note that Theorem II.4 in its general formulation affords a direct treatment of such states non invariant in time but homogeneous in space (by taking τ^n to be space translations)

[7] The "conditional stability" requires the states of the field algebra to be stable under local gauge-invariant perturbations. Since this entails the (α_t, β)-KMS property in restriction to $\mathfrak{A}$ (cf. [4]) conditional stability is a priori stronger than our present requirements. However the latter imply in turn conditional stability as an easy consequence of Theorem II.4

II. Extension of States

II.1. Assumptions and Results

(a) Assumptions. We consider a C*-algebra $\mathscr{F}$ (the algebra of fields), a continuous[8] one-parameter group

$$t \in R \to \alpha_t \in \operatorname{Aut} \mathscr{F}$$

of *-automorphisms of $\mathscr{F}$ (the time evolution), a compact group G (gauge group), a continuous representation γ of G by *-automorphisms of $\mathscr{F}$

$$g \in G \to \gamma_g \in \operatorname{Aut} \mathscr{F}$$

and a *-automorphism

$$\tau \in \operatorname{Aut} \mathscr{F}$$

such that $\alpha_t (t \in R)$, $\gamma_g (g \in G)$ and τ mutually commute and $\mathscr{F}$ is *asymptotically Abelian*[9] relative to τ: For any $a, b \in \mathscr{F}$,

$$\lim_n \| [\tau^n a, b] \| = 0 . \tag{II.1.1}$$

(The case where Fermion operators are present will be treated in Section II.8.)

The G-fixed point subalgebra of $\mathscr{F}$ (the *algebra of observables*) is denoted by $\mathfrak{A}$:

$$\mathfrak{A} = \{ A \in \mathscr{F} ; \gamma_g A = A \quad \text{for all} \quad g \in G \} . \tag{II.1.2}$$

Then $\mathfrak{A}$ is a C^*-subalgebra of $\mathscr{F}$ globally invariant under $\alpha_t (t \in R)$ and τ.

Definition II.1. A state ϕ of $\mathscr{F}$ is said to be *weakly τ-clustering* if $\phi(\tau a) = \phi(a)$ for all $a \in \mathscr{F}$ (τ-invariance) and if

$$M_n \{ \phi(a \tau^n n) \} = \phi(a) \phi(b) \tag{II.1.3}$$

for all $a, b \in \mathscr{F}$, where the (Cesaro) mean $M_n(f_n)$ of a function f_n of $n \in N$ is defined by

$$M_n \{ f_n \} = \lim_{n \to \infty} N^{-1} \sum_{n=1}^{N} f_n . \tag{II.1.4}$$

(b) Uniqueness of Weakly Clustering Extensions

Theorem II.1. (1) *If ϕ_1 and ϕ_2 are weakly τ-clustering states of $\mathscr{F}$ and if their restrictions to $\mathfrak{A}$ are equal:*

$$\phi_1(A) = \phi_2(A) , \qquad A \in \mathfrak{A} ,$$

then there exists $g \in G$ such that for all a

$$\phi_2(a) = \phi_1(\gamma_g a) .$$

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[8] We assume that, for each $a \in \mathscr{F}$, the function $\alpha_t(a)$ of t is continuous relative to the norm topology of $\mathscr{F}$ (the strong continuity)

[9] For the sake of definiteness, we have chosen the asymptotic Abelian property relative to a group $\{\tau^n ; nZ\}$. All our results and proofs remain valid if the group $\{\tau^n\}$ is replaced by an amenable group of *-automorphisms of $\mathscr{F}$ [10]. For example, for a one-parameter group τ_λ, our proofs remain valid if $M_n\{f_n\}$ in (II.1.4) is replaced by

$$M_\lambda \{ f_\lambda \} \equiv \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} f_\lambda d\lambda$$

(2) *If $\mathscr{F}$ is asymptotically Abelian relative to τ and if the state ω of $\mathfrak{A}$ is weakly τ-clustering, then there exists an extension of ω to a weakly τ-clustering state of $\mathscr{F}$.*

Remark II.1. This theorem shows the existence and uniqueness of a weakly clustering extension up to gauge. Assuming the asymptotic τ-Abelian property is not needed for the part (1) of Theorem II.1. For this theorem, the dynamical group α_t is irrelevant.

Remark II.2. If ω is an extremal KMS state of $\mathfrak{A}$, then it is primary [6]. If ω is τ-invariant in addition, it is (strongly and hence) weakly τ-clustering.

Definition II.2. The subgroup

$$G_\phi = \{g \in G\ ;\ \phi(\gamma_g a) = \phi(a) \quad \text{for all} \quad a \in \mathscr{F}\} \tag{II.1.5}$$

is called the *stabilizer of* ϕ. The subgroups

$$N(G_\phi, G) = \{g \in G\ ;\ g G_\phi g^{-1} = G_\phi\} \tag{II.1.6}$$

and

$$Z(G_\phi, G) = \{g \in G\ ;\ gh = hg \quad \text{for all} \quad h \in G_\phi\}$$

are respectively called the *normalizer* and *centralizer* of G_ϕ.

Theorem II.2. *Assume that G is separable. If ϕ is a weakly τ-clustering state of $\mathscr{F}$ and if its restriction to $\mathfrak{A}$ is invariant under time translations:*

$$\phi(\alpha_t(A)) = \phi(A), \quad A \in \mathfrak{A}, \quad t \in R$$

then there exists a continuous one-parameter subgroup

$$t \in R \to \varepsilon_t \in Z(G_\phi, G)$$

such that ϕ is invariant under (the modified time translation)

$$\alpha'_t(a) = \alpha_t \gamma_{\varepsilon_t}(a), \quad a \in \mathscr{F}. \tag{II.1.8}$$

Remark. II.3. The asymptotic Abelianness is not needed for this Theorem. If G is not separable, one finds ε_t in $N(G_\phi, G)/G_\phi$.

(c) Spectrum. In the cyclic representation $(\mathfrak{H}_\phi, \pi_\phi, \Omega_\phi)$ associated with ϕ, (GNS-construction), a continuous unitary representation of G_ϕ is canonically defined by

$$U_\phi(g)\pi_\phi(a)\Omega_\phi = \pi_\phi(\gamma_g a)\Omega_\phi, \quad a \in \mathscr{F}, \quad g \in G_\phi. \tag{II.1.9}$$

The restriction of U_ϕ to a subgroup H of G_ϕ is denoted by U_ϕ^H.

Theorem II.3. *Let H be a closed subgroup of the stabilizer G_ϕ of a weakly τ-clustering state ϕ of $\mathscr{F}$. Then U_ϕ^H has the following semi-group property: if the finite dimensional continuous unitary representations U_1 and U_2 are contained in U_ϕ^H, then $U_1 \otimes U_2$ is also contained in U_ϕ^H.*

Definition II.3. For a closed subgroup H of G_ϕ, the set Σ_ϕ^H of all irreducible representations p of H contained in U_ϕ^H is called the *H-spectrum of* ϕ. The H-

spectrum of ϕ is said to be *one-sided* if it is contained in a set Σ which has the following two properties:

(i) If p_1 and p_2 are in Σ, then any irreducible component of $p_1 \otimes p_2$ is in Σ (semi-group property).

(ii) The simultaneous occurence of p and its conjugate representation $\bar{p}$ in Σ implies that p is the identity representation: $p=1$. (Total asymmetry.)

(d) Extensions of KMS States. The following is the main theorem in Section II.

Theorem II.4. *Assume that G is separable. Let ϕ be a weakly τ-clustering state of $\mathscr{F}$, whose restriction to $\mathfrak{A}$ is an extremal (α_t, β)-KMS state. Then there exists a closed normal subgroup N_ϕ of G_ϕ, a continuous one-parameter subgroup ε_t of $Z(G_\phi)$ and a continuous one-parameter subgroup ζ_t of G_ϕ such that*

(i) *the N_ϕ-spectrum of ϕ is one-sided,*

(ii) *the restriction of ϕ to the fixed point algebra under N_ϕ:*

$$\mathscr{F}^{N_\phi} = \{a \in \mathscr{F}\,;\ \gamma_g a = a \quad \text{for all} \quad g \in N_\phi\} \tag{II.1.10}$$

is an $(\tilde{\alpha_t}, \beta)$-KMS state[10] *for $\tilde{\alpha_t} = \alpha_t \gamma_{\varepsilon_t \zeta_t}$, and*

(iii) *the image $\xi_t = \zeta_t N_\phi$ of ζ_t in G_ϕ / N_ϕ is in the center of G_ϕ / N_ϕ.*

Remark II.4. N_ϕ is called the *asymmetry subgroup of ϕ*. It is trivial if ϕ is faithful on $\mathfrak{A}$. If N_ϕ is trivial, then (ii) implies that ϕ is separating [i.e. Ω_ϕ separates the weak closure of $\pi_\phi(\mathscr{F})$].

Remark II.5. Let ξ_t and η_t be commuting continuous one-parameter subgroups of G such that there exists a β-KMS state $\phi_{\mu,\lambda}$ of $\mathscr{F}$ relative to

$$\alpha_t \gamma_{\xi_{\mu t} \eta_{\lambda t}}$$

for all $\lambda > \lambda_0$ and a μ. Let ϕ_μ be any accumulation point of $\phi_{\mu,\lambda}$ at $\lambda = +\infty$. Then it is easily seen that the H-spectrum of ϕ_μ is one-sided for $H = \{\eta_t;\ t \in \mathbb{R}\}$ and the restriction of ϕ to $\mathscr{F}^H$ is $(\alpha_t \gamma_{\xi_{\mu t}}, \beta)$-KMS. This remark indicates the physical meaning of N_ϕ: the latter determines the directions where the infinite components of the (vector) chemical potential are concentrated. The finite components of the chemical potential appear in the $\varepsilon_t \zeta_t$ part.

Remark II.6. If G is not assumed to be separable, and if ϕ is assumed to be α_t-invariant, then Theorem II.4 holds with the change that ζ_t will be a one-parameter subgroup of the center of G_ϕ / N_ϕ, instead of being in G_ϕ. The separability of G in Theorems II.2 and II.4 is used only in applying a result quoted in Appendix B for lifting one-parameter subgroups.

(e) Strategy of Proof. For the proof of Theorem II.1 (Section II.2), we first consider the "one-point function" $\phi(\gamma_h a)$, $a \in \mathscr{F}$, as a (continuous) function of $h \in G$. The uniform closure of the set of all one-point functions is shown to be the C^*-algebra of left G_ϕ-invariant continuous functions over G (Lemma II.1). It is shown that the map associating $\phi_2(\gamma_h a)$ to $\phi_1(\gamma_h a)$ is given by a left translation by an element $g \in G$, if ϕ_1 and ϕ_2 have the same restriction to $\mathfrak{A}$. This proves Theorem II.1 (1). Theorem II.2 then follows as a Corollary. Theorem II.1 (2) is more or less known.

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[10] The notation in this paper is adapted to a non-zero β. If $\beta = 0$, $\zeta_{\beta t}$ is to be replaced by a one-parameter family: the conclusion (ii) of Theorem II.4 is that the restriction of ϕ to $\mathscr{F}^{N_\phi}$ is a $(\gamma_{\zeta_t}, 1)$-KMS state

The proof of Theorem II.3 given in Section II.3 is a more or less standard argument.

The proof of Theorem II.4 is split into 4 parts. In Section II.4, the asymmetry subgroup N_ϕ is introduced in connection with the set of "two point functions" $\phi(a\gamma_g b)$ as functions of $g \in G_\phi$. [Because of the definition (II.1.5), $\phi((\gamma_{g_1} a)\gamma_{g_2} b) = \phi(a\gamma_g b))$ with $g = g_1^{-1} g_2$ if $g_1 \in G_\phi$.] In Section II.5, we derive the (α'_t, β)-KMS condition for the restriction of ϕ to $\mathscr{F}^{G_\phi}$ from the (α_t, β)-KMS condition for the restriction of ϕ to $\mathscr{F}^G = \mathfrak{A}$. In Section II.6, we derive the $(\tilde{\alpha}_t, \beta)$-KMS condition for the restriction of ϕ to $\mathscr{F}^{N_\phi}$ from the (α'_t, β)-KMS condition for the restriction of ϕ to $\mathscr{F}^{G_\phi}$. Finally in Section II.7, we prove that the N_ϕ-spectrum is one-sided, the proof depending on the $(\tilde{\alpha}_t, \beta)$-KMS condition for the restriction of ϕ to $\mathscr{F}^{N_\phi}$.

II.2. Uniqueness up to Gauge of a Weakly Clustering Extension

Let left and right translations be defined by

$$[\lambda(h)f](g) = f(h^{-1}g), \quad [\varrho(h)f](g) = f(gh) \tag{II.2.1}$$

for $h, g \in G$ and f belonging to $L^2(G)$ (square integrable functions over G relative to the Haar measure) or $\mathscr{C}(G)$ (continuous functions over G). For a closed subgroup H of G, $\mathscr{C}(H \diagdown G)$ can be identified with the set of all f in $\mathscr{C}(G)$ which are $\lambda(H)$-invariant: $\lambda(h)f = f$ for all $h \in H$.

Lemma II.1. *If ϕ is a weakly τ-clustering state of $\mathscr{F}$, then the norm closure C_ϕ^{1-} of*

$$C_\phi^1(G) = \{g \in G \to f_a^\phi(g) = \phi(\gamma_g a)\,; \quad a \in \mathscr{F}\} \subset \mathscr{C}(G) \tag{II.2.2}$$

coincides with $\mathscr{C}(G_\phi \diagdown G)$.

Proof. Let $a, b \in \mathscr{F}$ and let c be a complex number. Since $f_a^\phi + f_b^\phi = f_{a+b}^\phi$ and $cf_a^\phi = f_{ca}^\phi$, $C_\phi^1(G)$ is a linear subset of $\mathscr{C}(G)$. Since $(f_a^\phi)^* = f_{a^*}^\phi$, $C_\phi^1(G)$ is stable under complex conjugation.

For each $g \in G$,

$$M_n\{f_{a\tau^n b}^\phi(g)\} = f_a^\phi(g) f_b^\phi(g) \tag{II.2.3}$$

by the weak τ-clustering property of ϕ. Since $f_{a\tau^n b}^\phi$, $n \in N$, is an equicontinuous family, the convergence is uniform. Hence $f_a^\phi f_b^\phi \in C_\phi^{1-}(G)$ and $C_\phi^{1-}(G)$ is a C^*-subalgebra of $\mathscr{C}(G)$.

Since $\varrho(h) f_a^\phi = f_{\gamma_h a}^\phi$ for all $a \in \mathscr{F}$ and $h \in G$, $C_\phi^1(G)$ is (globally) right translation invariant: $\varrho(h) C_\phi^1(G) = C_\phi^1(G)$, $h \in G$. By Lemma A.1 of Appendix A, $C_\phi^{1-}(G) = \mathscr{C}(G_\phi \diagdown G)$ where G_ϕ is the set of $h \in G$ such that $f(g) = f(hg)$ for all $f \in C_\phi^1(G)$, namely G_ϕ defined by (II.1.5).

Proof of Theorem II.1(1). Let $G_j = G_{\phi_j}$ and $f_a^j(g) = \phi_j(\gamma_g a)$, $a \in \mathscr{F}$, $g \in G$. Since

$$A(a, \tau^n b) \equiv \int_G \gamma_g(a^* \tau^n b) dg \in \mathfrak{A}, \tag{II.2.4}$$

we have

$$\phi_1(A(a, \tau^n b)) = \phi_2(A(a, \tau^n b)).$$

By the weak τ-clustering property of ϕ_j and by equicontinuity

$$M_n\{\phi_j(A(a,\tau^n b))\} = \int_G f_a^j(g)^* f_b^j(g)dg \equiv \langle f_a^j, f_b^j\rangle_G .$$

Hence

$$\langle f_a^1, f_b^1\rangle_G = \langle f_a^2, f_b^2\rangle_G . \tag{II.2.5}$$

This shows that

$$V_0 f_a^1 = f_a^2 \tag{II.2.6}$$

defines an isometric operator V_0, whose closure V is a unitary map from $L^2(G_1\diagdown G)$ onto $L^2(G_2\diagdown G)$. By (II.2.3),

$$V(f_a^1 f_b^1) = f_a^2 f_b^2 ,$$

which implies for $f_a^j \in C(G_j\diagdown G)$ acting as a multiplication operator on $L^2(G_j\diagdown G)$

$$(\mathrm{Ad}\,V)f_a^1 \equiv V f_a^1 V^* = f_a^2 . \tag{II.2.7}$$

Hence $(\mathrm{Ad}\,V)$ induces a *-isomorphism of $\mathscr{C}(G_1\diagdown G)$ onto $\mathscr{C}(G_2\diagdown G)$. Furthermore V commutes with the right translations:

$$V\varrho(h)f_a^1 = V f_{\gamma_h a}^1 = f_{\gamma_h a}^2 = \varrho(h) f_a^2 = \varrho(h) V f_a^1 .$$

Hence $(\mathrm{Ad}\,V)$ also commutes with the right translations.

By Lemma A.3 of Appendix A, $\mathrm{Ad}\,V = \lambda(g^{-1})$ for some $g \in G$, and $g^{-1}G_1 g = G_2$. In particular

$$\phi_2(a) = f_a^2(1) = \{(\mathrm{Ad}\,V)f_a^1\}(1) = (\lambda(g^{-1})f_a^1)$$
$$(1) = f_a^1(g) = \phi_1(\gamma_g a) .$$

Q.E.D.

Proof of Theorem II.2. Since $\phi_t(a) = \phi(\alpha_t a)$ is a weakly clustering state of $\mathscr{F}$, whose restriction to $\mathfrak{A}$ coincides with ϕ due to the assumed α_t-invariance of $\phi|\mathfrak{A}$, Theorem II.1 implies the existence of $v_t \in G$ such that $\phi_t(a) = \phi(\gamma_{v_t} a)$.

Since α_t commutes with G, $G_{\phi_t} = G_\phi$. Since $G_{\phi_t} = v_t^{-1} G_\phi v_t$, v_t must be in the normalizer $N(G_\phi)$ of G_ϕ in G. Since v_t is unique up to G_ϕ, the cosets $\bar{v}_t = v_t G_\phi$ in $N(G_\phi)/G_\phi$ form a continuous one-parameter subgroup.

By Lemma B of Appendix B, there exist a continuous one-parameter subgroup $\varepsilon_t \in Z(G)$ such that $\varepsilon_t N_\phi = \bar{v}_{-t}$. Then $\phi(\gamma_{\varepsilon_{-t}} a) = \phi_t(a) = \phi(\alpha_t a)$, $a \in \mathscr{F}$, and hence ϕ is invariant under $\alpha_t' = \alpha_t \gamma_{\varepsilon_t}$.

Proof of Theorem II.1.(2). Let ω be a weakly τ-clustering state of $\mathfrak{A}$. The weak τ-clustering property of ω implies that ω is an extremal τ-invariant state of $\mathfrak{A}$ ([10] Theorem 4). The τ-invariant state ω of $\mathfrak{A}$ has an extention to a τ-invariant state. (For any extension ϕ_0 of ω, the mean of $\phi_0 \circ \tau^n$ over $n \in Z$ will give a τ-invariant extension.) By the Krein-Milman Theorem, there exists an extremal element ϕ in the non-empty convex compact set of all τ-invariant states of $\mathscr{F}$, whose restriction to $\mathfrak{A}$ is the given state ω. If $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$, $0 < \lambda < 1$, and if ϕ_1 and ϕ_2 are τ-invariant states of $\mathscr{F}$, the restrictions ω_1 and ω_2 of ϕ_1 and ϕ_2 are τ-invariant states of $\mathfrak{A}$. By the extremality of ω, $\omega_1 = \omega_2 = \omega$, and hence by the extremality of ϕ, $\phi_1 = \phi_2 = \phi$. This shows that ϕ is an extremal τ-invariant state of $\mathscr{F}$. Since $\mathscr{F}$ is asymptotically τ-Abelian, ϕ is weakly τ-clustering [11].

II.3. Semi-Group Property of Spectrum

Lemma II.2. *If a continuous unitary representation U of H on a finite dimensional space V is contained in U_ϕ^H, then there exists a linear map j^U from V into $\mathscr{F}$ such that*

(i) *the map $v \in V \to \pi_\phi(j^U(v))\Omega_\phi$ is isometric and*

(ii) *j^U is intertwining: $j^U(U(h)v) = \gamma_h(j^U(v))$, $h \in H$.*

Proof. Let p be any irreducible representation of H with $\chi_p(h)$ the corresponding character and define

$$\varepsilon_p^H(a) = \int_H \chi_p(h)^* \gamma_h(a) dh, \qquad E_p^H = \int_H \chi_p(h)^* U_\phi^H(h) dh. \tag{II.3.1}$$

Then the E_p^H for different p are orthogonal and $\varepsilon_p^H(\mathscr{F})\Omega_\phi$ is dense in $E_p^H \mathfrak{H}_\phi$. The latter implies that $E_p^H \mathfrak{H}_\phi = \varepsilon_p^H(\mathscr{F})\Omega_\phi$ whenever the dimension of $\varepsilon_p^H(\mathscr{F})\Omega_\phi$ is finite. Hence the Lemma follows from a counting of multiplicities of irreducible representations.

Proof of Theorem II.3. Let the continuous representations U_k of H on finite-dimensional spaces $V_k (k=1,\ 2)$ be contained in U^H. Let $j^{U_k} = j_k$. Since $\pi_\phi(j_1(v_1)\tau^n j_2(v_2))\Omega_\phi$ is bilinear in $v_1 \in V_1$ and $v_2 \in V_2$, and has a bound $\|j_1(v_1)\|\ \|j_2(v_2)\|$ independent of n, we conclude that there is a linear map Ω_n from $V_1 \otimes V_2$ to $\mathfrak{H}_\phi$, such that

$$\Omega_n\left(\sum_m c_m v_m^1 \otimes v_m^2\right) = \sum_N c_m \pi_\phi(j_1(v_m^1)\tau^n j_2(v_m^2))\Omega_\phi, \tag{II.3.2}$$

$$\sup_n \|\Omega_n\| < \infty. \tag{II.3.3}$$

If $\ker \Omega_n = \{0\}$ for some n, Ω_n^{-1} exists on $\Omega_n(V_1 \otimes V_2)$, and $\Omega_n^{-1} U_\phi^H(h) \Omega_n = U_1(h) \otimes U_2(h)$, showing that U_ϕ^H contains $U_1 \otimes U_2$.

Hence it is enough to show that the assumption $\ker \Omega_n \neq \{0\}$ for all n leads to a contradiction. Let $v_n \in \ker \Omega_n$, $\|v_n\| = 1$. Then there is a subsequence $n(p)$ with a limit v in the sense that $\lim_p \|v - v_{n(p)}\| = 0$. This implies

$$\|v\| = \lim \|v_{n(p)}\| = 1, \tag{II.3.4}$$

$$\lim_p \|\Omega_{n(p)}(v)\| = \lim_p \|\Omega_{n(p)}(v - v_{n(p)})\| = 0, \tag{II.3.5}$$

where we have used $\Omega_{n(p)}(v_{n(p)}) = 0$ and (II.3.3). However, the asymptotic Abelianness, weak τ-clustering property of ϕ and the isometric property of Lemma II.2 (i) imply

$$\lim_n \|\Omega_n(v)\|^2 = \|v\|^2$$

which contradicts with (II.3.4) and (II.3.5).

Q.E.D.

II.4. Asymmetry Subgroup of a Weakly Clustering State

Let ϕ be a weakly τ-clustering state of $\mathscr{F}$, which is assumed to be asymptotically Abelian with respect to τ. Let

$$C_\phi(G_\phi) = \{g \in G_\phi \to f_{a,b}^\phi(g) = \phi(a\gamma_g b));\ a, b \in \mathscr{F}\}. \tag{II.4.1}$$

Let $C^{-}_{\phi}(G_{\phi})$ be the uniformly closed linear hull of $C_{\phi}(G_{\phi})$ and $C^{-}_{\phi}(G_{\phi})^*$ denote the set of complex conjugates of functions in $C^{-}_{\phi}(G_{\phi})$.

Lemma II.3. *There exists a closed normal subgroup N_{ϕ} of G_{ϕ} such that*

$$C^{-}_{\phi}(G_{\phi})\cap C^{-}_{\phi}(G_{\phi})^* = \mathscr{C}(G_{\phi}/N_{\phi}). \tag{II.4.2}$$

Proof. By the asymptotic τ-Abelianness of $\mathscr{F}$ and weakly τ-clustering property of ϕ, we obtain

$$M_n\{f^{\phi}_{a\tau^n a', b\tau^n b'}\} = f^{\phi}_{a,b} f^{\phi}_{a',b'} \tag{II.4.3}$$

$$M_n\{f^{\phi}_{a+\tau^n a'+\lambda A, b+\tau^n b'+\mu A}\} = f^{\phi}_{a,b} + f^{\phi}_{a',b'} \tag{II.4.4}$$

where A is an element of $\mathfrak{A}$ with non vanishing $\phi(A^2)$ and complex numbers λ and μ are chosen to satisfy

$$\begin{aligned}\lambda\mu\phi(A^2) + \lambda\{\phi(Ab)+\phi(A)\phi(b')\} + \mu\{\phi(aA)+\phi(a')\phi(A)\}\\ +\phi(a')\phi(b)+\phi(a)\phi(b')=0.\end{aligned}$$

We also have

$$\lambda(g) f^{\phi}_{a,b} = f^{\phi}_{\gamma_g a, b}, \qquad \varrho(g) f^{\phi}_{a,b} = f^{\phi}_{a,\gamma_g b}. \tag{II.4.5}$$

Hence $C^{-}_{\phi}(G_{\phi})$ is a norm closed subalgebra of $\mathscr{C}(G_{\phi})$, globally invariant under left and right translations, which implies that $C^{-}_{\phi}(G_{\phi})\cap C^{-}_{\phi}(G_{\phi})^*$ is a C^*-subalgebra of $\mathscr{C}(G_{\phi})$, globally invariant under left and right translations. By Lemma A.1 of Appendix A, we obtain (II.4.2) for some closed normal subgroup N_{ϕ} of G_{ϕ}. Q.E.D.

We now consider $\mathscr{F}^{N_{\phi}}$ defined by (II.1.10). For a state ϕ_1 of $\mathscr{F}^{N_{\phi}}$ we define $C_{\phi_1}(G_{\phi})$ in a way similar to (II.4.1):

$$C_{\phi_1}(G_{\phi}) = \{g\in G_{\phi} \to f^{\phi}_{a,b}(g) = \phi(a\gamma_g b));\, a, b \in \mathscr{F}^{N_{\phi}}\}.$$

Lemma II.4. *Let ϕ_1 be the restriction of ϕ to $\mathscr{F}^{N_{\phi}}$ defined by* (II.1.10.). *Then*

$$C^{-}_{\phi_1}(G_{\phi}) = C^{-}_{\phi_1}(G_{\phi})^* = \mathscr{C}(G_{\phi}/N_{\phi}). \tag{II.4.6}$$

Proof. If $b\in\mathscr{F}^{N_{\phi}}$, then $\gamma_g b = b$ for all $g\in N_{\phi}$ and hence

$$\varrho(g) f^{\phi}_{a,b} = f^{\phi}_{a,\gamma_g b} = f^{\phi}_{a,b}.$$

Hence

$$C^{-}_{\phi_1}(G_{\phi}) \subset \mathscr{C}(G_{\phi}/N_{\phi}). \tag{II.4.7}$$

Let $p\in\hat{G}_{\phi}$ and

$$e^{N_{\phi}} = \int_{N_{\phi}} \varrho(g)dg = \int_{N_{\phi}} \lambda(g)dg, \tag{II.4.8}$$

$$\varepsilon^{N_{\phi}}(a) = \int_{N_{\phi}} \gamma_g(a)dg. \tag{II.4.9}$$

Then $e^{N_{\phi}}$ is an orthogonal projection in $L^2(G_{\phi})$ and

$$e^{N_{\phi}} f^{\phi}_{a,b} = f^{\phi}_{a_0,b_0}, \qquad a_0 = \varepsilon^{N_{\phi}}(a), \qquad b_0 = \varepsilon^{N_{\phi}}(b).$$

Since $\varepsilon^{N_\phi}(\mathscr{F})=\mathscr{F}^{N_\phi}$, we have

$$e^{N_\phi}C^{\sim}_{\phi}(G_\phi)=C^{\sim}_{\phi_1}(G_\phi). \tag{II.4.10}$$

By (II.4.2) we have

$$e^{N_\phi}C^{\sim}_{\phi}(G_\phi)\supset e^{N_\phi}\mathscr{C}(G_\phi/N_\phi)=\mathscr{C}(G_\phi/N_\phi). \tag{II.4.11}$$

Equations (II.4.7), (II.4.10), and (II.4.11) prove

$$C^{\sim}_{\phi_1}(G_\phi)=\mathscr{C}(G_\phi/N_\phi).$$

Q.E.D.

II.5. Extension from $\mathfrak{A}$ to $\mathscr{F}^{G_\phi}$

Lemma II.5. *Let α'_t be given by Theorem 2. If the restriction of a weakly τ-clustering state ϕ of $\mathscr{F}$ to $\mathfrak{A}$ is an (α_t,β)-KMS state, then the restriction of ϕ to*

$$\mathscr{F}^{G_\phi}=\{a\in\mathscr{F}\,;\,\gamma_g a=a \quad \textit{for all} \quad g\in G_\phi\} \tag{II.5.1}$$

is an (α'_t,β)-KMS state.

Proof. Let $a,a',b,b'\in\mathscr{F}$. Let $k(E)$ be any C^∞*-function of* $E\in\mathbb{R}$ with a compact support and

$$k^{\sim}_{\gamma}(t)=\int_{-\infty}^{\infty}k(E)\exp\{iEt+\gamma E\}\,dE. \tag{II.5.2}$$

Since $\alpha'_t A=\alpha_t A$ for $A\in\mathfrak{A}$, the assumed KMS condition implies

$$\begin{aligned}&\int\phi(A(a,\tau^n a')\alpha'_t A(\tau^m b,\tau^n b'))k^{\sim}_{\beta}(t)dt\\&\quad=\int\phi(\{\alpha'_t A(\tau^m b,\tau^n b')\}A(a,\tau^n a'))k^{\sim}_0(t)dt\end{aligned} \tag{II.5.3}$$

where we have used the notation (II.2.4).

By applying the average M_n first and the average M_m next, we obtain

$$\begin{aligned}&\langle f^\phi_a\otimes f^\phi_b, F(a',b';\,k^{\sim}_\beta)\rangle_{G\times G}\\&\quad=\langle f^\phi_a\otimes f^\phi_b, G(a',b'\,;k_0)\rangle_{G\times G}\end{aligned} \tag{II.5.4}$$

where f^ϕ_a and f^ϕ_b are defined as in (II.2.2),

$F(a',b'\,;k^{\sim}_\beta)(g_1,g_2)=\int\phi((\gamma_{g_1}a')\alpha'_t\gamma_{g_2}b')k^{\sim}_\beta(t)dt\in\mathscr{C}(G\times G)$,

$G(a',b'\,;k^{\sim}_0)(g_1,g_2)=\int\phi(\{\alpha'^t\gamma_{g_2}b'\}\gamma_{g_1}a')k^{\sim}_0(t)dt\in\mathscr{C}(G\times G)$,

and the inner product in (II.5.4) is in $L^2(G\times G)$.

By Lemma 1, $f^\phi_a\otimes f^\phi_b$ for a,b is dense in

$$L^2(G_\phi\times G_\phi\setminus G\times G)=(e^{G_\phi}\otimes e^{G_\phi})L^2(G\times G)$$

where $e^{G_\phi}=\int_{G_\phi}\lambda(g)dg$ is an orthogonal projection. Hence (II.5.4) implies

$$(e^{G_\phi}\otimes e^{G_\phi})F(a',b'\,;k^{\sim}_\beta)=(e^{G_\phi}\otimes e^{G_\phi})G(a',b'\,;k^{\sim}_0). \tag{II.5.5}$$

Since

$$(e^{G_\phi}\otimes e^{G_\phi})F(a',b';k^{\sim}_\beta)(1,1)=\int\phi(\varepsilon^{G_\phi}(a')\alpha'_t\varepsilon^{G_\phi}(b'))k^{\sim}_\beta(t)dt$$
$$(e^{G_\phi}\otimes e^{G_\phi})G(a',b';k^{\sim}_0)(1,1)=\int\phi(\{\alpha'_t\varepsilon^{G_\phi}(b')\}\varepsilon^{G_\phi}(a'))k^{\sim}_0(t)dt\,,$$

the equality (II.5.5) for $a',b'\in\mathscr{F}^{G_\phi}$ implies

$$\int\phi(a'\alpha'_t b')k^{\sim}_\beta(t)dt=\int\phi(\{\alpha'_t b'\}a')k^{\sim}_0(t)dt$$

which is the (α'_t,β)-KMS condition for the restriction of ϕ to $\mathscr{F}^{G_\phi}$. Q.E.D.

II.6. Extension from $\mathscr{F}^{G_\phi}$ to $\mathscr{F}^{N_\phi}$

Proof of Theorem II.4 (ii) and (iii)

By Theorem II.2 and Lemma II.5, ϕ is α'_t-invariant, α'_t commutes with every $g\in G_\phi$ and the restriction of ϕ to $\mathscr{F}^{G_\phi}$ is an (α'_t,β)-KMS state.

Let a, b, a', and b' be α'_t-entire elements of $\mathscr{F}$. (It is known that the set of entire elements is dense.) By the α'_t-invariance of ϕ, we have

$$\phi(\varepsilon^{G_\phi}(a^*\tau^n a')\alpha'_t\varepsilon^{G_\phi}(b^*\tau^n b'))=\phi(a^*\tau^n a'\alpha'_t\varepsilon^{G_\phi}(b^*\tau^n b'))\,.$$

Hence the (α'_t,β)-KMS condition implies

$$\phi(a^*\tau^n a'\alpha'_{i\beta}\varepsilon^{G_\phi}(b^*\tau^n b'))=\phi(\varepsilon^{G_\phi}(b^*\tau^n b')a^*\tau^n a')\,.$$

By applying the average M_n we obtain

$$\langle sf^\phi_{\alpha'_{-i\beta}b,a}, f^\phi_{a',\alpha'_{i\beta}b'}\rangle_{G_\phi}=\langle f^\phi_{a,b}, sf^\phi_{a',b'}\rangle_{G_\phi} \tag{II.6.1}$$

where $f^\phi_{a,b}$ is as in (II.4.1) and

$$(sf)(g)=f(g^{-1})\,.$$

We have used following relations:

$$\alpha'_{i\beta}(b^*)=(\alpha'_{-i\beta}b)^*\,,$$
$$\phi((\gamma_g b)a)=\phi(b\gamma_{g^{-1}}a)=sf^\phi_{b,a}(g)\,,\qquad (g\in G_\phi)\,.$$

By Lemma II.4, the set of $f^\phi_{a,b}$ as well as the set of $sf^\phi_{a,b}=(f^\phi_{b^*,a^*})^*$ for $a,b\in\mathscr{F}^{N_\phi}$ are dense in $\mathscr{C}(G_\phi/N_\phi)$ and hence in $L^2(G_\phi/N_\phi)$.

Let S be temporarily defined by

$$Sf^\phi_{a,b}=sf^\phi_{\alpha'_{-i\beta}b,a}\,.$$

Then (II.6.1), with b' replaced by $\alpha'_{i\beta}b'$, becomes

$$\langle Sf^\phi_{a,b}, f^\phi_{a',b'}\rangle=\langle f^\phi_{a,b}, Sf^\phi_{a',b'}\rangle\,. \tag{II.6.2}$$

If $\Sigma c_i f^\phi_{a_i,b_i}=0$, then (II.6.2) implies that

$$\langle\Sigma c_i Sf^\phi_{a_i,b_i}, f^\phi_{a',b'}\rangle=0$$

for all $a',b'\in\mathscr{F}^{N_\phi}$ and hence $\Sigma c_i Sf^\phi_{a_i,b_i}=0$. Therefore

$$S\Sigma c_i f^\phi_{a_i,b_i}=\Sigma c_i f^\phi_{\alpha'_{-i\beta}b_i,a_i} \tag{II.6.3}$$

defines a symmetric linear operator on a dense linear subset of $L^2(G_\phi/N_\phi)$.

Because of

$$\varrho(g)f_{a,b}=f_{a,\gamma_g b},\ \lambda(g)f_{a,b}=f_{\gamma_g a,b}, \tag{II.6.4}$$

the definition (II.6.3) implies that the domain of S is invariant under right and left translations and that S commutes with them.

Let p be a continuous irreducible unitary representation of G_ϕ/N_ϕ and let

$$e(p)\equiv \int_{G_\phi/N_\phi} \varrho(g)\chi_p(g)^* dg \tag{II.6.5}$$

where χ_p is the character for p. Then $\bar{S}$ commutes with $e(p)$. Since left and right translations together are irreducible on the range of $e(p)$ (which is of finite dimension), $\bar{S}e(p)=S(p)e(p)$ for a function $S(p)$. (The range of $e(p)$ is actually in the domain of S.)

Since the $f^\phi_{a,b}$ span $\mathscr{C}(G_\phi/N)$, there exists $f^\phi_{a,b}$ with $e(p)f^\phi_{a,b}\neq 0$. Then $b_p = \int_{G_\phi/N_\phi} \gamma_g(b)\chi_p(g)^* dg$ satisfies $e(p)f_{a,b}=f_{a,b_p}$ and hence $\phi(b_p^* b_p)\neq 0$. Since α' commutes with G_ϕ, we can find an appropriate f for which the α'_t-entire element $a_p\equiv\int \alpha'_t(b_p)f(t)dt$ fulfils $\phi(a_p^* a_p)\neq 0$. Then $f_{a_{\bar{p}}^*,a_p}$ is in $e(p)\mathscr{C}(G_\phi/N_\phi)$ and hence

$$\begin{aligned}\phi((\alpha'_{-i\beta}a_p)a_p^*)&=(Sf_{a_{\bar{p}}^*,a_p})(1)=S(p)f_{a_{\bar{p}}^*,a_p}\\&=S(p)\phi(a_p^* a_p).\end{aligned} \tag{1}$$

Since

$$\begin{aligned}&\phi((\alpha'_{-i\beta}a_p)a_p^*=\phi((\alpha'_{-i\beta/2}a_p)\alpha'_{i\beta/2}a_p^*)\\&=\phi((\alpha'_{-i\beta/2}a_p)(\alpha'_{-i\beta/2}a_p)^*)\geqq 0,\end{aligned}$$

we obtain $S(p)\geqq 0$.

If we make the same argument with $\bar{p}$, we obtain an α'_t-entire $a_{\bar{p}}$ with $\phi(a_{\bar{p}}^* a_{\bar{p}})\neq 0$. Then $a=\alpha_{i\beta/2}a^-{}_p$ satisfies $f_{a^*,a}\in e(p)\mathscr{C}(G_\phi/N_\phi)$ and hence

$$0<\phi(a_{\bar{p}}^* a_{\bar{p}})=(Sf_{a^*,a})(1)=S(p)\phi(a^*,a)$$

This proves $S(p)\neq 0$.

Therefore $\bar{S}$ is a positive self-adjoint operator and hence $\bar{S}^{it}=\Gamma(t)$ is a continuous one-parameter group of unitary operators commuting with left and right translations.

By the weak τ-clustering, we obtain

$$S(f_{a,b}f_{a',b'})=(Sf_{a,b})(Sf_{a',b'}). \tag{II.6.6}$$

This implies $S(p)S(q)=S(r)$ whenever r is contained in $p\otimes q$. Hence

$$\Gamma(t)(f_{a,b}f_{a',b'})=(\Gamma(t)f_{a,b})(\Gamma(t)f_{a',b'})$$

which implies that

$$\Gamma(t)T_{f_{a,b}}\Gamma(t)^*=T_{\Gamma(t)f_{a,b}}$$

where T_f denotes the multiplication operator by f on $L^2(G_\phi/N_\phi)$.

Hence $\mathrm{Ad}\,\Gamma(t)$ is a continuous one-parameter group of automorphisms of $\mathscr{C}(G_\phi/N_\phi)$ commuting with right and left translations. Hence there exists a continuous one-parameter subgroup ξ_t such that $\mathrm{Ad}\,\Gamma(t)=\lambda(\xi_{\beta t})$ $(=\varrho(\xi_{-\beta t})$ due to Lemma A.2 of Appendix A; and ξ_t must lie in the center of G_ϕ/N_ϕ.

By Lemma B of Appendix B, ξ_t can be lifted to a continuous one-parameter subgroup ζ_t of G. Q.E.D.

II.7. Implications of KMS-Condition for Asymmetry Subgroup

Proof of Theorem II.4 (i)

The following technical Lemma is due to Kishimoto [9].

Lemma II.6. *Let ϕ be a weakly τ-clustering state of $\mathscr{F}$. Assume that the restriction of ϕ to $\mathscr{F}^{N_\phi}$ is an $(\alpha^\sim_t, \beta)$-KMS state for a continuous one-parameter subgroup $\alpha^\sim_t$ of* Aut$\mathscr{F}$ *commuting with τ and satisfying $\alpha^\sim_t N_\phi \alpha^\sim_{-t}=N_\phi$. Let p be an irreducible continuous unitary representation of N_ϕ such that p and $\bar{p}$ are both contained in U_ϕ. Let ε_p be defined by ε_p^H of* (II.3.1) *with $H=N_\phi$. Then for any $a\in\varepsilon_p(\mathscr{F})$, $\phi(a^*a)=0$ is equivalent to $\phi(aa^*)=0$.*

Proof. Let a, a', b, b' be in $\mathscr{F}$. Since

$$a_n=\varepsilon^{N_\phi}(a^*\tau^n a')\in\mathscr{F}^{N_\phi}, \qquad b_n=\varepsilon^{N_\phi}(b^*\tau^n b')\in\mathscr{F}^{N_\phi}, \tag{II.7.1}$$

we have [11] an $F_n(z)$, holomorphic for $0<\mathrm{Im}\,z/\beta<1$, continuous and bounded for $0<\mathrm{Im}\,z/\beta<1$ and satisfying

$$F_n(t)=\phi(a_n\alpha^\sim_t b_n), \qquad F_n(t+i\beta)=\phi((\alpha^\sim_t b_n)a_n)$$

for real t. Both $F_n(t)$ and $F_n(t+i\beta)$ are equicontinuous families of uniformly continuous functions of t and hence there exists a uniform limit $F(z)=M_n\{F_n(z)\}$ which is holomorphic for $0<\mathrm{Im}\,t/\beta<1$, bounded and continuous for $0<\mathrm{Im}\,t/\beta\leqq 1$ and satisfies for real t

$$\begin{aligned}F(t)&=M_n\{\phi(a_n\alpha^\sim_t b_n)\}\\&=\int_{N_\phi}\int_{N_\phi}\phi((\alpha^\sim_t\gamma_{g_2}b)\gamma_{g_1}a)^*\phi((\gamma_{g_1}a')\alpha^\sim_t\gamma_{g_2}b')dg_1dg_2,\end{aligned}$$

$$\begin{aligned}F(t+i\beta)&=M_n\{\phi((\alpha^\sim_t b_n)a_n)\}\\&=\int_{N_\phi}\int_{N_\phi}\phi((\gamma_{g_1}a)\alpha^\sim_t\gamma_{g_2}b)^*\phi((\alpha^\sim_t\gamma_{g_2}b')\gamma_{g_1}a')dg_1dg_2.\end{aligned}$$

In the following we use the implication that $F(t+i\beta)=0$ for all t implies $F(0)=0$.

From the assumption $E_{\bar{p}}\neq 0$, there exists $a_{\bar{p}}\in\varepsilon_{\bar{p}}(\mathscr{F})$ such that $\phi(a_{\bar{p}}^*a_{\bar{p}})\neq 0$, as in the proof of Lemma 5. For $a=\gamma_h a_{\bar{p}}$ and $b=a^*(h\in N_\phi)$, we have

$$\begin{aligned}f_1(h^{-1}g_1^{-1}g_2h)&\equiv\phi((\gamma_{g_2}b)\gamma_{g_1}a)\\&=(U_\phi(h)\pi_\phi(a_{\bar{p}})\Omega_\phi, U_\phi(g_1^{-1}g_2)U_\phi(h)\pi_\phi(a_{\bar{p}})\Omega_\phi)^*.\end{aligned}$$

[11] If $\beta=0$, then $F_n(z)$ is defined for $\mathrm{Im}\,z=0$

If f, f_1, and f_2 are positive type functions in $e_p\mathscr{C}(N_\phi)$ with $e_p = \int_{N_\phi} \varrho(g)\chi_p(g)^* dg$, then the vanishing of $\int f_1(h^{-1}gh)^* f_2(g)dg$ for all $h \in N_\phi$ implies either $f_1 = 0$ or $f_2 = 0$. Hence

$$\int_{N_\phi} \int_{N_\phi} \phi((\gamma_{g_2} b)\gamma_{g_1} a)^* f_2(g_1^{-1} g_2) dg_1 dg_2 = 0$$

for all $h \in N_\phi$ implies $f_2 = 0$ for all positive type f_2 in $e_p\mathscr{C}(N_\phi)$.

If $a_p \in \varepsilon_p(\mathscr{F})$ and $\phi(a_p^* a_p) \neq 0$, then $f_2(g_1^{-1} g_2) \equiv \phi((\gamma_{g_1} a')\gamma_{g_2} b')$ with $b' = a_p$, $a' = a_p^*$ is a non-zero positive function of $g_1^{-1} g_2$ in $e_p\mathscr{C}(N_\phi)$. Hence $F(0) \neq 0$ for some h if $a = \gamma_h a_{\bar{p}}$, $b = a^*$, $b' = a_p$, and $a' = a_p^*$. On the other hand, in $\phi(a_p a_p^*) = 0$, then $F(t + i\beta)$ with $a' = a_p^*$ vanishes for all t. This would imply $F(0) = 0$ in contradiction with the above conclusion. Hence $\phi(a_p a_p^*) \neq 0$.

Since $\varepsilon_p(\mathscr{F})^* = \varepsilon_{\bar{p}}(\mathscr{F})$, $\phi(a_p a_p^*) \neq 0$ implies $\phi(a_p^* a_p) \neq 0$ by the same argument with p and $\bar{p}$ interchanged. Q.E.D.

Lemma II.7. *N_ϕ-spectrum of ϕ is one-sided under the assumption of Lemma II.6.*

Proof. The semi-group property has already been proved in Theorem 3.

Suppose p and $\bar{p}$ are contained in $U_\phi^{N_\phi}$. Then there exists $a_p \in \varepsilon_p(\mathscr{F})$ such that $\phi(a_p^* a_p) \neq 0$ and $\phi(a_p a_p^*) \neq 0$ by Lemma II.6. Among continuous irreducible unitary representations q of G, there exists at least one q such that $E_q^{G_\phi} \pi_\phi(a_p)\Omega_\phi = \pi_\phi(\varepsilon_q^{G_\phi} a_p)\Omega_\phi \neq 0$.

Since

$$(\pi_\phi(a_p)\Omega_\phi, \pi_\phi(\varepsilon_q^{G_\phi} a_p)\Omega_\phi) = \|\pi_\phi(\varepsilon_q^{G_\phi} a_p)\Omega_\phi\|^2 \neq 0, \quad \text{(II.7.2)}$$

the representation q of G_ϕ, when restricted to N_ϕ, must contain p.

The equation (II.7.2) also shows that

$$\pi_\phi(\varepsilon_p^{N_\phi}(\varepsilon_q^{G_\phi} a_p))\Omega_\phi \neq 0$$

and hence by Lemma II.7

$$E_p^{N_\phi} \pi_\phi(\varepsilon_q^{G_\phi} a_p)^* \Omega_\phi = \pi_\phi(\varepsilon_p^{N_\phi}(\varepsilon_q^{G_\phi} a_p))\Omega_\phi \neq 0$$

and hence

$$\pi_\phi(\varepsilon_q^{G_\phi} a_p)^* \Omega_\phi \neq 0 .$$

(The same conclusion follows also from Lemma II.6 in which N_ϕ is replaced by G_ϕ.) This proves that

$$e_q^{G_\phi}\mathscr{C}(G_\phi) \subset C_\phi^-(G_\phi) \cap C_\phi^-(G_\phi)^* .$$

By (II.4.2), N_ϕ must be trivially represented in q. Hence $p = 1$. Q.E.D.

Because of Theorem II.4 (ii) and (iii) already proved, Lemma II.7 with $\alpha_t^\sim = \alpha_t \varepsilon_t \zeta_t$ implies Theorem II.4 (i).

II.8. The Case Where Fermion Operators Are Present

In the foregoing discussion, we assumed the asymptotic Abelianness (II.1.1). We shall now show that all the foregoing arguments go through if we make a modification of (II.1.1) appropriate for the presence of Fermion operators.

In addition to assumptions II.1.(a), we consider an involutive automorphism θ of $\mathscr{F}(\theta^2=1)$ commuting with α_t, every γ_g, $g\in G$, and τ. $\mathscr{F}$ is split into a sum

$$\mathscr{F}=\mathscr{F}_+ + \mathscr{F}_- \tag{II.8.1}$$

where $a\in\mathscr{F}_\pm$ satisfies $\theta(a)=\pm a$. Operators in $\mathscr{F}_-$ will be called *Fermion operators.* The assumption (II.1.1) is modified to the following: we consider 4 different cases (i) $a\in\mathscr{F}_+$, $b\in\mathscr{F}_+$, (ii) $a\in\mathscr{F}_+$, $b\in\mathscr{F}_-$, (iii) $a\in\mathscr{F}_-$, $b\in\mathscr{F}_+$, (iv) $a\in\mathscr{F}_-$, $b\in\mathscr{F}_-$. We assume

$$\lim_n \|(\tau^n a)b-\varepsilon_{b,a}b\tau^n a\|=0 \tag{II.8.2}$$

where $\varepsilon_{b,a}=+1$ for the cases (i), (ii), and (iii) and $\varepsilon_{b,a}=-1$ for the case (iv).

We can now show that all results in II.1 through II.7 hold even if we change the assumption (II.1.1) to (II.8.2). We have only to check those part of proof where the asymptotic abelianness is used. We do this item by item in the following:

(a) An extremal τ-invariant state is weakly τ-clustering
[This is relevant to the proof of Theorem II.1.(2)]

Let ϕ be an extremal τ-invariant state, π_ϕ on $\mathfrak{H}_\phi$ the associated cyclic representation, Φ the associated vector, U_τ the unitary operator on $\mathfrak{H}_\phi$ associated with τ and P_τ the projection operator on U_τ-invariant vectors. Then

$$M_n\{\phi(a^*\tau^n a)\}=(\Phi,\pi_\phi(a)^*P_\tau\pi_\phi(a)\Phi)\geqq 0\,, \tag{II.8.3}$$

$$M_n\{\phi((\tau^n a)a^*)\}=(\Phi,\pi_\phi(a)P_\tau\pi_\phi(a)^*\Phi)\geqq 0\,. \tag{II.8.4}$$

If a is a Fermion operator, (II.8.2) implies

$$M_n\{\phi(a^*\tau^n a)\}=-M_n\{\phi((\tau^n a)a^*)\}\,,$$

which must vanish due to (II.8.3) and (II.8.4). Hence

$$P_\tau\pi_\phi(a)\Phi=0$$

for any $a\in\mathscr{F}_-$. In particular, $\phi(\mathscr{F}_-)=0$ and hence ϕ is θ-invariant. Let U_θ be the unitary operator on $\mathfrak{H}_\phi$ associated with θ.

For any $a\in\mathscr{F}$, we have

$$M_n\{\pi_\phi(\tau^n a)\Omega_\phi\}=M_n\{U_\tau^n\}\pi_\phi(a)\Omega_\phi=P_\tau\pi_\phi(a)\Omega_\phi\,.$$

Hence

$$M_n\{\pi_\phi(\tau^n a)\pi_\phi(b)\Omega_\phi\}=\varepsilon_{a,b}\pi_\phi(b)M_n\{\pi_\phi(\tau^n a)\Omega_\phi\}$$

exists for the 4 cases considered in (II.8.2). Hence the uniformly bounded sequence $N^{-1}\sum_{n=1}^{N}\pi_\phi(\tau^n a)$ is weakly convergent:

$$M_n\{\pi_\phi(\tau^n a)\}\equiv M_\tau(a)\,.$$

If $a\in\mathscr{F}_+$, $M_\tau(a)$ commutes with $\mathscr{F}$ by (II.8.2) and with τ by the definition. The ergodicity then implies that it is a number, which can be computed by

$$M_\tau(a)=(\Phi,M_\tau(a)\Phi)1=\phi(a)1\,.$$

Hence

$$M_n\{\phi(b\tau^n a)\} = \phi(b)\phi(a). \tag{II.8.5}$$

If $a \in \mathscr{F}_-$, $M_\tau(a)U_\theta$ commutes with $\mathscr{F}$ by (II.8.2) and by the definition of U_θ. It also commutes with τ. Hence

$$M_\tau(a)U_\theta = (\Phi, M_\tau(a)U_\theta\Phi)1 = \phi(a)1 = 0.$$

Hence

$$M_\tau(a) = 0 = \phi(a)1.$$

Therefore (II.8.5) holds also for this case and hence in general. This proves the weakly τ-clustering property of an arbitrary τ-ergodic state ϕ, in the presence of Fermion operators.

(b) Proof of Theorem II.3

Since U_θ commutes with $g \in G$, each $E_p^H \mathfrak{H}_\phi = (\varepsilon_p^H(\mathscr{F})\Omega_\phi)^-$ in the proof of Lemma 2 can be split into a direct sum of two representations $U_\pm$ according to the eigenvalue ± 1 of U_θ. Accordingly, we have

$$j^U(v) = j^{U_+}(v_+) + j^{U_-}(v_-),\ j^{U_\pm}(v_\pm) \in \mathscr{F}_\pm. \tag{II.8.6}$$

when $v = v_+ + v_-$. As far as the transformation property under $g \in G$ and the action on Ω_ϕ are concerned, the operator

$$j_0^U(v) = j^{U_+}(v_+) + j^{U_-}(v_-)U_\theta \tag{II.8.7}$$

also transforms unitarily equivalently to U and produces the same vactor when applied on Ω_ϕ. We then consider

$$\Omega_n(v_1; v_2) \equiv \pi_\phi(j^{U_1}(v_1)\tau^n j_0^{U_2}(v_2))\Omega_\phi$$

which carries the representation $U_1 \otimes U_2$ provided that $\ker \Omega_n = \{0\}$. The condition $\ker \Omega_n = \{0\}$ holds by the same reason as before. Namely the equation

$$\lim_n \|\Omega_n(v)\|^2 = \|v\|^2,$$

follows from the asymptotic commutativity of $j^{U_1}(v_1)$ and $j_0^{U_2}(v_2)$ and from the weak τ-clustering property, which is not affected by the presence of U_θ.

(c) Proof of Lemma II.3

Since (II.4.4) holds without change due to the weak τ-clustering property, we have only to prove $f_{a,b}^\phi, f_{a',b'}^\phi \in C^-_\phi(G_\phi)$ when a, b, a', b' are in $\mathscr{F}_+ \cup \mathscr{F}_-$. Since (II.4.3) holds when the right hand side is multiplied by $\varepsilon_{a',b}$ and since $-f_{a,b}^\phi = f_{-a,b}^\phi$, this is the case.

(d) Proof of Lemma II.5

The only place where care should be taken is the passage from (II.5.3) to (II.5.4). If a, b, a', b' are in $\mathscr{F}_+ \cup \mathscr{F}_-$, then either both sides of (II.5.4) get the same additional sign $\varepsilon_{a'b} = \varepsilon_{b'a}$ or else they both vanish due to $\phi(\mathscr{F}) = 0$. This implies that (II.5.4) holds for either case, and hence for a, b, a', b' in $\mathscr{F}$.

(e) Proof of Theorem II.4 (ii) and (iii)

The only place where care should be taken is the derivations of (II.6.1) and (II.6.6). These can be done exactly in the same manner as (d) above.

(f) Proof of Theorem II.4 (i)

The only place where care should be taken is the computation of $F(t)$ and $F(t+i\beta)$, where the same argument as (d) above can be applied.

III. Analysis of Representations and the Associated von Neumann Algebras

III.1

In the last section, we saw that any weakly τ-clustering state ω of the observable algebra $\mathfrak{A}$ extends uniquely, up to a translation by the gauge group G, to a weakly clustering state ϕ of the field algebra $\mathscr{F}$. Since the stabilizer G_ϕ of ϕ need not be the whole gauge group G, the action of G may not be implementable in the *GNS*-representation $\{\pi_\phi, \mathfrak{H}_\phi, \Omega_\phi\}$ of $\mathscr{F}$ (in other words, the gauge symmetry may be broken in the state ϕ, with only the smaller symmetry, G_ϕ, still present). In Subsection III.2, we shall study how one can restore the whole symmetry from the smaller one. It turns out that the induced covariant representation $\operatorname{Ind}_{G_\phi \uparrow G} \{\pi_\phi, U_\phi\}$ from the smaller covariant system $\{\mathscr{F}, G_\phi\}$ is unitarily equivalent to the *GNS*-covariant representation $\{\pi_{\bar{\omega}}, U_{\bar{\omega}}\}$ given by the unique G-invariant extension $\bar{\omega}$. From this analysis, we shall derive that $\pi_\phi(\mathscr{F}^{G_\phi})'' = \pi_\phi(\mathfrak{A})''$.

Subsection III.3 is devoted to the analysis of the von Neumann algebras $\pi_\phi(\mathscr{F})''$, $\pi_\phi(\mathfrak{A})''$ and $\pi_\omega(\mathfrak{A})''$ in the absence of the asymmetry group N_ϕ. We shall show that (i) any automorphism α on $\mathfrak{M} = \pi_\phi(\mathscr{F})''$, which leaves $\mathfrak{N} = \pi_\phi(\mathfrak{A})''$ pointwise invariant and commutes with τ, must be of the form $\alpha = \gamma_{g_\alpha}$ for some $g_\alpha G_\phi$; (ii) $\mathfrak{M}$ is the dual crossed product of $\mathfrak{N}$ by some dual action $\gamma\hat{}$ of G_ϕ on $\mathfrak{N}$ and the action γ of G_ϕ on $\mathfrak{M}$ is realized as the dual action to $\gamma\hat{}$. As a byproduct, we shall give an alternative proof of a part of Theorem II.4. A basic tool for this analysis is a von Neumann algebra version of the Tannaka duality theorem adapted to the formalism developed by Roberts in [11].

III.2. Broken Symmetry Versus Induced Covariant Representation

Let $\mathscr{F}$, $\{\gamma_g : g \in G\}$, $\mathfrak{A} = \mathscr{F}^G$ and τ be as before. We showed that any weakly τ-clustering state ω of $\mathfrak{A}$ extends uniquely, up to a gauge transformation, to a τ-clustering state ϕ of $\mathscr{F}$. However, this extended state ϕ need not be invariant under G, in which case the action γ is not unitarily implementable in the *GNS*-representation $\{\pi_\phi, \mathfrak{H}_\phi, \Omega_\phi\}$ of $\mathscr{F}$. On the other hand, any state ω of $\mathfrak{A}$ has a unique G-invariant extension $\bar{\omega}$ to $\mathscr{F}$ given by the following simple procedure:

$$\bar{\omega}(a) = \omega\left(\int_G \gamma_g(a)\,dg\right), \quad a \in \mathscr{F}. \tag{III.2.1}$$

The action γ of G on $\mathscr{F}$ is canonically implemented in the *GNS*-representation $\{\pi_{\bar{\omega}}, \mathfrak{H}_{\bar{\omega}}, \Omega_{\bar{\omega}}\}$ of $\mathscr{F}$ by the unitary representation $U_{\bar{\omega}}$ of G on $\mathfrak{H}_{\bar{\omega}}$ given by

$$U_{\bar{\omega}}(g)\pi_{\bar{\omega}}(a)\Omega_{\bar{\omega}} = \pi_\omega(\gamma_g(a))\Omega_\omega, \quad a \in \mathscr{F}. \tag{III.2.2}$$

Contrasting with this, the GNS-representation $\{\pi_\phi, \mathfrak{H}_\phi, \Omega_\phi\}$ of $\mathscr{F}$ given by the weakly τ-clustering extension ϕ carries only the unitary representation U_ϕ of the stabilizer of ϕ in G which implements the restricted action $\gamma|G_\phi$ on $\mathscr{F}$. The relation between $\{\bar{\omega}, \pi_{\bar{\omega}}, U_{\bar{\omega}}\}$ and $\{\phi, \pi_\phi, U_\phi\}$ is described as follows:

Theorem III.2.1. *Let $\mathscr{F}$ be a C^*-algebra equipped with a continuous action γ of a compact group G, and let $\mathfrak{A} = \mathscr{F}^G$ be the fixed point algebra. Suppose that $\mathscr{F}$ admits an automorphism τ which is asymptotically abelian and commutes with γ. Let ω be a weakly τ-clustering state on $\mathfrak{A}$, ϕ a weakly τ-clustering extension of ω to $\mathscr{F}$ and $\bar{\omega}$ the G-invariant extension of ω to $\mathscr{F}$. The following then holds:*

(i) *the state $\bar{\omega}$ is given by the integral:*

$$\bar{\omega} = \int_{G_\phi \setminus G} \phi \circ \gamma_g dg \tag{III.2.3}$$

where dg is the cononical image of the normalized Haar measure dg of G on the right coset space $G_\phi \setminus G = \{G_\phi g : g \in G\}$ of G modulo the stabilizer G_ϕ of ϕ in G. Furthermore, this integral is orthogonal and subcentral in the sense of [12];

(ii) $\{\pi_{\bar{\omega}}, U_{\bar{\omega}}\} \cong \operatorname*{Ind}_{G_\phi \uparrow G} \{\pi_\phi, U_\phi\}$

in the sense of [13, 14].

Proof. Equality (III.2.3) trivially follows from the uniqueness of the G-invariant extension. Denote

$$\{\pi^\sim, U^\sim, \mathfrak{H}^\sim\} = \operatorname*{Ind}_{G_\phi \uparrow G} \{\pi_\phi, U_\phi, \mathfrak{H}_\phi\}. \tag{III.2.4}$$

The Hilbert space $\mathfrak{H}^\sim$ consists of all $\mathfrak{H}_\phi$-valued square integrable functions ξ on G such that

$$\xi(hg) = U_\phi(h)\xi(g), \quad h \in G, \quad g \in G; \tag{III.2.5}$$

and the inner product in $\mathfrak{H}^\sim$ is given by:

$$\begin{aligned}(\xi|\eta) &= \int_G (\xi(g)|\eta(g))dg \\ &= \int_{G_\phi \setminus G} (\xi(g)|\eta(g))dg.\end{aligned} \tag{III.2.6}$$

The representation $\{\pi^\sim, U^\sim\}$ is defined by:

$$\begin{aligned}&\{\pi^\sim(a)\xi\}(g) = {}_\phi(\gamma_g(a))\xi(g), \quad a \in \mathscr{F}, \quad g \in G; \\ &\{U^\sim(g)\xi\}(h) = \xi(hg), \quad g, \quad h \in G.\end{aligned} \tag{III.2.7}$$

We define a vector $\Omega^\sim$ in $\mathfrak{H}^\sim$ by

$$\Omega^\sim(g) = \Omega_\phi, \quad g \in G. \tag{III.2.8}$$

It then follows that

$$(\Omega^{\sim}|\pi^{\sim}(a)\Omega^{\sim}) = \int_G (\Omega_\phi|\pi_\phi\circ\gamma_g(a)\Omega_\phi)dg$$
$$= \int_G \phi\circ\gamma_g(a)dg = \bar{\omega}(a), \qquad a\in\mathscr{F}$$

Hence the map: $\pi_{\bar{\omega}}(a)\Omega_{\bar{\omega}} \to \pi^{\sim}(a)\Omega^{\sim}$ extends to an isometry of $\mathfrak{H}_{\bar{\omega}}$ into $\mathfrak{H}^{\sim}$, which we shall denote by W. We want to show that W is onto. To this end, we further introduce operators $\theta(f)$, $f\in L^\infty(G_\phi\backslash G)$, as follows:

$$\theta(f)\xi(g) = f(\dot{g})\xi(g), \qquad g\in G, \qquad f\in L^\infty(G_\phi\backslash G), \tag{III.2.9}$$

where $\dot{g} = G_\phi g\in G_\phi\backslash G$. It then follows that θ is a σ-weakly continuous faithful representation of $L^\infty(G_\phi\backslash G)$ on $\mathfrak{H}^{\sim}$. The range $\mathscr{A}$ of θ is called the *imprimitivity system of the induction* (III.2.4).

Lemma III.2.2. *The imprimitivity system $\mathscr{A}$ is contained in the center of $\pi^{\sim}(\mathscr{F})''$.*

Assuming this lemma, we continue the proof of the theorem. For each operator $A\in U_\phi(G_\phi)'$, we put

$$A^{\sim}\xi(g) = A\xi(g), \qquad \xi\in\mathfrak{H}^{\sim}, \qquad g\in G. \tag{III.2.10}$$

It is then known, [15], that the mapping $A\in U_\phi(G_\phi)' \to A\in\mathscr{B}(\mathfrak{H}^{\sim})$ gives rise to an isomorphism of $U_\phi(G_\phi)'$ onto $U^{\sim}(G)'\cap\mathscr{A}'$. Since the range of W is invariant under $U^{\sim}(G)$ and $\pi^{\sim}(\mathscr{F})$, it is also invariant under $\mathscr{A}$ by Lemma III.2.2, so that the projection of $\mathfrak{H}^{\sim}$ onto $W\mathfrak{H}_{\bar{\omega}}$ must be of the form $P^{\sim}$ for some projection $P\in U_\phi(G_\phi)'$. But $\{(W\xi)(g): \xi\in\mathfrak{H}_{\bar{\omega}}\}$ contains $\pi_\phi\circ\gamma_g(\mathscr{F})\Omega_\phi$ for all $g\in G$, and is thus dense in $\mathfrak{H}_\phi$, so that P must be the identity 1. Hence we conclude that W is an isometry of $\mathfrak{H}_{\bar{\omega}}$ *onto* $\mathfrak{H}^{\sim}$, which intertwines $\{\pi_{\bar{\omega}}, U_{\bar{\omega}}\}$ and $\{\pi^{\sim}, U^{\sim}\}$. This completes the proof.

Proof of Lemma 2.2. For each $n=1, 2, \ldots$, we put

$$T_n(a) = (2n+1)^{-1}\sum_{k=-n}^{n}\tau^k(a), \qquad a\in\mathscr{F}. \tag{III.2.11}$$

As in Section II, we have for all $a, b, c\in\mathscr{F}$,

$$\lim_{n\to\infty}(\pi_\phi(c)\Omega_\phi|\pi_\phi(T_n(a))\pi_\phi(b)\Omega_\phi)$$
$$= \phi(a)(\pi_\phi(c)\Omega_\phi|\pi_\phi(b)\Omega_\phi).$$

Since $\{\pi_\phi(T_n(a)): n=1, 2, \ldots\}$ is bounded, $\pi_\phi(T_n(a))$ converges weakly to $\phi(a)1$. Therefore, we get, for each $\xi, \eta\in\mathfrak{H}^{\sim}$,

$$\lim_{n\to\infty}(\eta|\pi^{\sim}\circ T^n(a)\xi) = \lim_{n\to\infty}\int_G(\eta(g)|\pi_\phi\circ\gamma_g\circ T_n(a)\xi(g))dg$$
$$= \lim_{n\to\infty}\int_G(\eta(g)|\pi_\phi\circ T_n\circ\gamma_g(a)\xi(g))dg$$
$$= \int_G \phi\circ\gamma_g(a)(\eta(g)|\xi(g))dg$$
$$= \int_G f_a^\phi(g)(\eta(g)|\xi(g))dg = (\eta|\theta(f_a^\phi)\xi),$$

where $f_a^\phi(g)=\phi\circ\gamma_g(a)$ as in (II.2.2.). Hence $\theta(f_a^\phi)$ belongs to $\pi^\sim(\mathscr{F})''$. Since $\{f_a^\phi : a\in\mathscr{F}\}$ $=C_\phi^1(G)$ is dense in $\mathscr{C}(G_\phi\setminus G)$ by Lemma II.2.1, θ maps $\mathscr{C}(G_\phi\setminus G)$ into $\pi^\sim(\mathscr{F})''$. The σ-weak continuity of θ yields that $\mathscr{A}\subset\pi^\sim(\mathscr{F})''$. By its construction, $\mathscr{A}\subset\pi^\sim(\mathscr{F})'$. Hence $\mathscr{A}\subset\pi^\sim(\mathscr{F})''\cap\pi^\sim(\mathscr{F})'$. Q.E.D.

We will now identify the covariant representations $\{\pi_{\bar\omega}, U_{\bar\omega}, \mathfrak{H}_{\bar\omega}, \Omega_{\bar\omega}\}$ with the induced covariant representation $\{\pi^\sim, U^\sim, \mathfrak{H}^\sim, \Omega^\sim\}$. We define

$$\begin{aligned}
&\mathfrak{M}^\sim=\pi^\sim(\mathscr{F})'', \qquad \mathfrak{M}=\pi_\phi(\mathscr{F})'',\\
&\gamma^\sim_g(A)=U^\sim(g)AU^\sim(g)^*, \qquad A\in\mathfrak{M}, \qquad g\in G;\\
&\gamma_g^\phi(A)=U_\phi(g)AU_\phi(g)^*, \qquad A\in\mathfrak{M}, \qquad g\in G_\phi.
\end{aligned}\tag{III.2.12}$$

Corollary III.2.3. *With the notation and the assumptions of Theorem III.2.1.*

(i) $\{\mathfrak{M}^\sim, G, \gamma^\sim\}\cong \operatorname{Ind}_{G_\phi\uparrow G}\{\mathfrak{M}, G_\phi, \gamma^\phi\}$, *in the sense of* [16];
(ii) $\mathfrak{M}^{\sim G}\cong\mathfrak{M}^{G_\phi}$ *under the correspondence given by* (*III.2.10*).
(iii) $\pi_\phi(\mathscr{F}^{G_\phi})''=\pi_\phi(\mathfrak{A})''$.

Proof. Let $\{\mathfrak{M}^\sim, G, \gamma^\sim\}=\operatorname{Ind}_{G_\phi\uparrow G}\{\mathfrak{M}, G_\phi, \gamma^\phi\}$. By definition, $\mathfrak{M}^-$ is the subalgebra of $\mathfrak{M}^-\otimes L^\infty(G)=L^\infty(G, dg, \mathfrak{M})$ consisting of all elements such that

$$x(hg)=\gamma^\phi(h)x(g), \qquad h\in G_\phi, \qquad g\in G. \tag{III.2.13}$$

The action γ^- of G on $\mathfrak{M}^-$ is defined by

$$\gamma^-_g(x)(k)=x(kg), \qquad k, g\in G. \tag{III.2.14}$$

For each $A=\pi(a)$, $a\in\mathscr{F}$, let

$$A(g)=\{\pi_\phi\circ\gamma_g\}(a), \qquad g\in G. \tag{III.2.15}$$

It follows that $A(\cdot)$ belongs to $\mathfrak{M}^-$ and the correspondence: $A\leftrightarrow A(\cdot)$ is a normal isomorphism. Identifying A and $A(\cdot)$, $\mathfrak{M}^\sim$ is regarded as a von Neumann subalgebra of $\mathfrak{M}^-$, globally invariant under γ^-_G and containing $\mathscr{A}$ by Lemma III.2.2, so that [16; Proposition 10.4] yields that $\mathfrak{M}^\sim$ must be obtained from a von Neumann subalgebra $\mathfrak{N}$ of $\mathfrak{M}$ as $\mathfrak{M}^\sim=\{x\in\mathfrak{M}^- : x(g)\in\mathfrak{N}, g\in G\}$. This means, however, that $\mathfrak{N}$ contains $\pi_\phi(\mathscr{F})$. Thus $\mathfrak{N}=\mathfrak{M}$; and so $\mathfrak{M}^\sim=\mathfrak{M}^-$. This completes the proof of (i). We now prove (ii). By definition

$$\begin{aligned}
&\mathfrak{M}^{\sim G}=\mathfrak{M}^\sim\cap U^\sim(G)'\\
&\mathfrak{M}^{G_\phi}=\mathfrak{M}\cap U_\phi(G_\phi)'.
\end{aligned}$$

Since, by Lemma III.2.2, $\mathfrak{M}^\sim$ contains the system $\mathscr{A}$ of imprimitivity in its center, $\mathfrak{M}^{\sim G}$ is contained in $\mathscr{A}'\cap U^\sim(G)'$. Hence $\mathfrak{M}^{\sim G}$ corresponds to a von Neumann subalgebra of $U_\phi(G_\phi)'$ under the correspondence (III.2.10). Under the identification of $\mathfrak{M}^\sim$ with $\mathfrak{M}^-$, the correspondence (III.2.10) means that

$$A\in U_\phi(G_\phi)'\leftrightarrow A^\sim=A\otimes 1. \tag{III.2.16}$$

Thus the subalgebra of $U_\phi(G_\phi)'$ corresponding to $\mathfrak{M}^{\sim G}$ must be $\mathfrak{M}^{G_\phi}$.

In order to prove that $\pi_\phi(\mathscr{F}^{G_\phi})'' = \pi_\phi(\mathfrak{A})''$ it suffices to check that the images of these von Neumann algebras under the isomorphism (III.2.10) coincide. We just showed that the image of $\pi_\phi(\mathscr{F}^{G_\phi})'' = \mathfrak{M}^{G_\phi}$ is $\mathfrak{M}^{\sim G}$. On the other hand, since the image of $\pi_\phi(\mathfrak{A})$ is $\pi^\sim(\mathfrak{A})$, the image of $\pi_\phi(\mathfrak{A})''$ is $\pi^\sim(\mathfrak{A})'' = \mathfrak{M}^{\sim G}$. Q.E.D.

Remark. Corollary III.2.3 affords an alternative to the first stage (cf. II.5) of the proof of Theorem II.4.

We therefore come to the following situation:

$$\mathfrak{M}^\sim \cong \mathfrak{M} \bar{\otimes} L^\infty(G_\phi \backslash G);$$
$$\mathfrak{M}^{\sim G} \cong \mathfrak{M}^{G_\phi}.$$

This means that the W^*-covariant system $\{\mathfrak{M}, G_\phi, \gamma^\phi\}$ describes entirely the whole W^*-covariant system $\{\mathfrak{M}^\sim, G, \gamma^\sim\}$. Thus we shall consider only the case where $G = G_\phi$ for the analysis of the associated von Neumann algebras in the next section.

III.3. The von Neumann Algebras Associated with Weakly Clustering Extended States

In this section, we study the von Neumann algebras $\pi_\phi(\mathscr{F})''$, $\pi_\phi(\mathfrak{A})''$, and $\pi_\omega(\mathfrak{A})''$ together with the gauge group G. Let $\mathscr{F}$, $\{\gamma_g : g \in G\}$, $\mathfrak{A} = \mathscr{F}^G$, τ and ω be as before. Take a weakly τ-clustering extension ϕ of ω to $\mathfrak{A}$. We further assume that ω and ϕ are both separating in the following sense:

Definition III.3.1. A state Ψ on a C^*-algebra $\mathfrak{B}$ is said to be *separating* if the vector Ω_Ψ associated with the GNS-representation $\{\pi_\Psi, \mathfrak{H}_\Psi, \Omega_\Psi\}$ of $\mathfrak{B}$ is separating for $\pi_\Psi(\mathfrak{B})''$.

When a one-parameter automorphism group $\{\alpha_t\}$ of $\mathscr{F}$, commuting with $\{\gamma_g : g \in G\}$ and τ, for which ω satisfies the KMS-condition at $\beta > 0$, is given, the separating assumption on ϕ is equivalent to the triviality of the asymmetry group N_ϕ[12].

Proposition III.3.2. *Let $\mathscr{F}$ be a C^*-algebra and $\mathfrak{A}$ a C^*-subalgebra of $\mathscr{F}$. Let ϕ be a state of $\mathscr{F}$ with $\omega = \phi|_{\mathfrak{A}}$. If ϕ is separating, then the restriction $\pi_\phi|_{\mathfrak{A}}$ of the GNS-representation $\{\pi_\phi, \mathfrak{H}_\phi, \Omega_\phi\}$ of $\mathscr{F}$ to $\mathfrak{A}$ is quasi-equivalent to the GNS-representation $\{\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega\}$ of $\mathfrak{A}$.*

Proof. Let $\mathfrak{K} = [\pi_\phi(\mathfrak{A})\Omega_\phi]$. Trivially the representation $\{\pi_\phi, \mathfrak{K}, \Omega_\phi\}$ of $\mathfrak{A}$ is unitarily equivalent to the GNS-representation $\{\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega\}$ of $\mathfrak{A}$ under the isometry U defined by $U\pi_\omega(x)\Omega_\omega = \pi_\phi(x)\Omega_\phi$, $x \in \mathfrak{A}$. Let E be the projection of $\mathfrak{H}_\phi$ onto $\mathfrak{K}$: E belongs to $\pi_\phi(\mathfrak{A})'$. The central support F of E in $\pi_\phi(\mathfrak{A})'$ is the projection of $\mathfrak{H}_\phi$ onto $[\pi_\phi(\mathfrak{A})'\mathfrak{K}]$. But we have

$$[\pi_\phi(\mathfrak{A})'\mathfrak{K}] \supset [\pi_\phi(\mathfrak{A})'\Omega_\phi] \supset [\pi_\phi(\mathscr{F})'\Omega_\phi].$$

Since Ω_ϕ is separating for $\pi_\phi(\mathscr{F})''$, it is cyclic for $\pi_\phi(\mathscr{F})'$:

12 Cf. Remark 4 of Section II.1

$[\pi_\phi(\mathscr{F})'\Omega_\phi]=\mathfrak{H}_\phi$, thus $F=1$; this means that the mapping: $A\in\pi_\phi(\mathfrak{A})''\to A_E\in U\pi_\omega(\mathfrak{A})''U^*$ is an isomorphism. Q.E.D.

Thus, if ϕ is separating, we know that $\pi_\phi(\mathfrak{A})''$ is isomorphic to $\pi_\phi(\mathfrak{A})''$.

The situation of interest for us is that of an extremal (α_t, β) KMS state ω of $\mathfrak{A}$: it is known that in this case ω is primary, that is $\pi_\omega(\mathfrak{A})''$ is a factor [6]. We further have that the automorphism τ of $\pi_\omega(\mathfrak{A})''$ extending τ on $\mathfrak{A}$ is *ergodic* in the sense that its fixed point algebra in $\pi_\omega(\mathfrak{A})''$ reduces to the scalars[13]. A result of Hugenholtz [17] then radily tells us that (unless α_t reduces to the identity automorphism group) $\pi_\omega(\mathfrak{A})''$ is a factor of type III. Setting $\mathfrak{M}=\pi_\phi(\mathscr{F})''$, and $\mathscr{G}=G_\phi$ (the stabilizer of the state ϕ) we thus reach a situation corresponding to the assumptions of the following Theorem, which is the main result of this subsection[14]:

Theorem III.3.3. *Let $\mathfrak{M}$ be a von Neumann algebra equipped with a continuous action γ of a compact group $\mathscr{G}$*[15] *such that the fixed point algebra $\mathfrak{M}^\gamma=\mathfrak{N}$ of γ in $\mathfrak{M}$ is a factor of type III. Assume further that the group*

$$\mathrm{Aut}_\gamma\mathfrak{M}=\{\varrho\in\mathrm{Aut}\,\mathfrak{M}:\varrho\gamma_g=\gamma_g\varrho, g\in\mathscr{G}\} \tag{III.3.1}$$

has a subgroup $\mathscr{S}$ ergodic in $\mathfrak{M}$ in the sense

$$\{x\in\mathfrak{M}:\tau(x)=x \text{ for all } \tau\in\mathscr{S}\}=\mathbb{C}. \tag{III.3.2}$$

Then

(i) *If α is an automorphism of $\mathfrak{M}$ commuting with $\mathscr{S}$ and such that $\alpha(x)=x$ for every $x\in\mathfrak{M}^\gamma$ there is an element $g_\alpha\in\mathscr{G}$ such that $\alpha=\gamma_{g_\alpha}$ [in particular α normalizes $\gamma(\mathscr{G})$, that is $\alpha\gamma(\mathscr{G})\alpha^{-1}=\gamma(\mathscr{G})$].*

(ii) *If the action γ of $\mathscr{G}$ on $\mathfrak{M}$ is faithful the crossed product $W^*(\mathfrak{M},\mathscr{G},\gamma)$ is isomorphic to $\mathfrak{M}^\gamma\otimes\mathfrak{B}(L^2(\mathscr{G}))\cong\mathfrak{M}^\gamma$.*

We shall prove this theorem in several steps. The basic tool for proving (i) will be the Roberts-Tannaka duality theorem discussed in Appendix C. To show that the latter applies we have to prove the two following facts:

(a) One has $\alpha\mathfrak{H}\subset\mathfrak{H}$ for each $\mathfrak{H}\in\mathscr{H}_\gamma(\mathfrak{M})$.

(b) Each irreducible subrepresentation of γ on $\mathfrak{M}$ belongs to $\mathscr{R}_\gamma(\mathscr{G})$.

(We denote by $\mathscr{H}_\gamma(\mathfrak{M})$ the collection of finite dimensional γ-invariant Hilbert spaces in $\mathfrak{M}$, and by $\mathscr{R}_\gamma(\mathscr{G})$ the set of equivalence classes of representations of $\mathscr{G}$ associated with these Hilbert spaces—see Appendix C.)

For the proof of both (a) and (b) Proposition 3.6 in [11] plays an essential role. We need in fact, for the proof of (a), the background machinery of this proposition which we now describe.

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[13] The weak τ-clustering property of ω together with the asymptotic abelianness of τ imply that the fixed points of $\pi_\omega(\mathfrak{A})'$ under the unitary implementaring τ in π_ω reduce to the scalars [10]. Since this unitary commutes with the modular conjugation J of ω (due to the fact that ω is τ-invariant), $\pi_\omega(\mathfrak{A})''=J\pi_\omega(\mathfrak{A})'J$ is ergodic for τ

[14] This theorem will later be applied to the case $\alpha=\sigma_t\alpha_t^{-1}$, with σ_t^ϕ the modular automorphism group of ϕ and α_t the extension to $\mathfrak{M}$ of the dynamical group

[15] I.e. $g\in\mathscr{G}\to\gamma_g(x)$ is continuous from $\mathscr{G}$ to $\mathfrak{M}$ equipped with its σ-weak topology. We use the notation $\mathrm{Aut}(\mathfrak{M})$ for the set of all automorphisms of the von Neumann algebra $\mathfrak{M}$

Let $\mathfrak{H}\in\mathscr{H}_\gamma(\mathfrak{M})$: with n the dimension of $\mathfrak{H}$, one first builds a Hilbert space $\mathfrak{H}_0$ in $\mathfrak{M}^\gamma$ (hence in $\mathfrak{M}$) of the same dimension by taking as one of its orthonormal bases a set of n isometries $u_i\in\mathfrak{M}^\gamma$ with mutually orthonormal ranges adding up to 1. (This is possible since $\mathfrak{M}^\gamma$ is properly infinite.) Now consider an isometry v from $\mathfrak{H}_0$ onto $\mathfrak{H}$ and the $U(g)$, $g\in\mathscr{G}$, where $U=\gamma|_{\mathfrak{H}}$ is the representation of $\mathscr{G}$ on $\mathfrak{H}$ determined by $\mathscr{G}$: Lemma 2.3 in [11] allowed us to consider v and the $U(g)$ as elements of $\mathfrak{M}$. If $x_0\in\mathfrak{H}_0$, we then have

$$\gamma_g(v)x_0=\gamma_g(vx_0)=U(g)vx_0\,,\qquad g\in\mathscr{G}\,, \tag{III.3.3}$$

whence, using the separating property of Hilbert spaces in $\mathfrak{M}$,

$$\gamma_g(v)=vV(g)\,,\qquad g\in\mathscr{G}\,. \tag{III.3.4}$$

where $V(g)=v^*U(g)v$ is an element of $(\mathfrak{H}_0,\mathfrak{H}_0)$, the subspace consisting of those elements of $\mathfrak{M}$ mapping $\mathfrak{H}_0$ into $\mathfrak{H}_0$. We now use the fact [11] that

$$\mathfrak{M}=\phi_0(\mathfrak{M})\bar{\otimes}(\mathfrak{H}_0,\mathfrak{H}_0) \tag{III.3.5}$$

with ϕ_0 the morphism attached to the Hilbert space $\mathfrak{H}_0$:

$$\phi_0(x)=\sum_{i=1}^{n}u_ixu_i^*\,,\qquad x\in\mathfrak{M}\,. \tag{III.3.6}$$

Using this set up we now prove (a). Since $(\mathfrak{H}_0,\mathfrak{H}_0)\in\mathfrak{M}^\gamma$ and $\alpha\circ\phi_0=\phi_0\circ\alpha$, $\gamma_g\circ\phi_0=\phi_0\circ\gamma_g$ (following from the fact that $\mathfrak{M}^\gamma$ is stable under α and the γ_g), the action of α resp. γ_g on $\mathfrak{M}$ in (III.3.5) is as follows

$$\begin{cases}\alpha=\alpha\otimes i\\ \gamma_g=\gamma_g\otimes i\end{cases},\qquad i=\text{the identity automorphism}\,. \tag{III.3.7}$$

If we define, for $\tau\in\mathscr{S}$

$$\tau^-=\tau^\sim\otimes i\,,\qquad \tau^\sim \text{ defined by } \tau^\sim\circ\phi_0(a)=\phi_0\circ\tau(a), a\in\mathfrak{M}\,, \tag{III.3.8}$$

the assumed commutativity of τ with γ_g and α entails that τ^- commutes with γ_g and α. The proof of (a) is now obtained as follows: since $\mathfrak{H}=v\mathfrak{H}_0$ the inclusion $\alpha\mathfrak{H}\subset\mathfrak{H}$ is synonimous with the fact that $v^*\alpha(v)$ belongs to $(\mathfrak{H}_0,\mathfrak{H}_0)$. But since we assumed that τ acts ergodically on $\mathfrak{M}$ [consequently $\tau^\sim$ on $\phi_0(\mathfrak{M})$], this will follow from (III.3.5) if we show that

$$\tau^-(v^*\alpha(v))=v^*\alpha(v)\,. \tag{III.3.9}$$

To check this, set

$$a_\tau=\tau^-(v)v^*\,. \tag{III.3.10}$$

Since τ^- and α commute, the l.h.s. of (III.3.9) reads

$$v^*a_\tau^*\alpha(a_\tau)\alpha(v)\,, \tag{III.3.11}$$

we thus need to show that the unitary a_τ fulfills $\alpha(a_\tau)=a_\tau$. But we assumed $\alpha=i$ on $\mathfrak{M}^\gamma$: hence our proof boils down to checking that

$$\gamma_g(a_\tau)=a_\tau, \quad g\in\mathcal{G}. \tag{III.3.12}$$

Now, since τ^- commutes with γ_g, using (III.3.4), we have

$$\gamma_g(a_\tau)=\tau^-(\gamma_g(v))\gamma_g(v^*)=\tau^-(v)\tau^-(V(g))V(g)^*v, \tag{III.3.13}$$

yielding our result since $V(g)$ is unitary and $\tau^-(V(g))=V(g)$, [remember that $V(g)\in(\mathfrak{H}_0,\mathfrak{H}_0)$]. We completed the proof of statement (a).

We now turn to the proof of (b): for this we shall now need the statement

Proposition 3.6 in Roberts [11]. *Let U be an n-dimensional unitary representation of $\mathcal{G}$. For the equivalence class of U to belong to $\mathcal{R}_\gamma(\mathcal{G})$ it is necessary and sufficient that there be a unitary V in $\mathfrak{M}\otimes M_n$ (M_n the $n\times n$ complex matrices) such that*

$$i\otimes U(g)=V^*\{\gamma_g\otimes i\}(V). \tag{III.3.14}$$

[We note that (III.3.14) is obtained by transporting (III.3.4) to $\mathfrak{M}\otimes M_n$ via the isomorphism $\phi_0^{-1}\otimes\Psi$, Ψ an isomorphism of $(\mathfrak{H}_0,\mathfrak{H}_0)$ onto M_n—whereby v goes to V and $V(g)$ to $U(g)$—and using (III.3.7).]

For shortness we need the following notation: we set

$$\mathcal{P}_n=\mathfrak{M}\otimes M_n \tag{III.3.15}$$

on which we define γ^-_g, $g\in\mathcal{G}$, as

$$\gamma^-_g=\gamma_g\otimes i, \quad g\in\mathcal{G}. \tag{III.3.16}$$

With U a nonzero representation of $\mathcal{G}$ on M_n we set

$$\mathcal{P}_n^{\gamma^-}(U)=\{x\in\mathcal{P}_n;\ \gamma^-_g(x)=x(i\otimes U(g)),\ g\in\mathcal{G}\}. \tag{III.3.17}$$

It is easily seen that we have the equivalence

$$\mathcal{P}_n^{\gamma^-}(U)\neq\{0\}\Leftrightarrow \text{there is a subrepresentation of } \gamma \text{ on } \mathfrak{M} \text{ equivalent to the representation } U. \tag{III.3.18}$$

(If the a_j span a non vanishing subspace of $\mathfrak{M}$ carrying the representation U,

$$\gamma_g(a_j)=\sum_k a_k U_{kj}(g), \tag{III.3.19}$$

then $(a_{ij})=(\delta_{i1}a_j)\neq 0$ belongs to $\mathcal{P}_n^{\gamma^-}(U)$. Conversely if $(a_{ij})\in\mathcal{P}_n^{\gamma^-}(U)$ does not vanish then $a_{ij}\neq 0$ for some i, j, and

$$\gamma_g(a_{ij})=\sum_k a_{ik}U_{kj}(g). \tag{III.3.20}$$

In particular $\mathcal{P}_n^{\gamma^-}(U)\neq 0$ if U is a non-zero subrepresentation of γ acting on $\mathfrak{M}$. On the other hand (III.3.14) states the existence of unitary V in $\mathcal{P}_n^{\gamma^-}(U)$. Thus Proposition 3.6 of Roberts stated above implies statement (*b*) if we establish

Lemma III.3.4. *If U is an irreducible representation of $\mathscr{G}$ on M_n with $\mathscr{P}_n^{\gamma^-}(U) \neq \{0\}$, then $\mathscr{P}_n^{\gamma^-}(U)$ contains a unitary v and*

$$\mathscr{P}_n^{\gamma^-}(U) = \mathscr{P}_n^{\gamma^-} v. \tag{III.3.21}$$

Proof. In addition to γ^-_g defined in (III.3.16) we shall need

$$\gamma^\sim_g = \gamma_g \otimes \operatorname{Ad} U(g), \quad g \in \mathscr{G}. \tag{III.3.22}$$

with corresponding fixed-point subalgebra $\mathscr{P}_n^{\gamma^\sim}$. Note that

$$\begin{aligned} &\mathscr{P}_n^{\gamma^-}(U)^* \mathscr{P}_n^{\gamma^-}(U) \subset \mathscr{P}_n^{\gamma^\sim}, \\ &\mathscr{P}_n^{\gamma^-}(U) \mathscr{P}_n^{\gamma^-}(U)^* \subset \mathscr{P}_n^{\gamma^-}. \end{aligned} \tag{III.3.23}$$

Consider now the 2×2 matrix algebra, say, $\mathscr{P}$, over $\mathscr{P}_n$ and a new action, say, σ, of G on $\mathscr{P}$ given by

$$\sigma_g\left(\begin{bmatrix} x_{11}, & x_{12} \\ x_{21}, & x_{22} \end{bmatrix}\right) = \begin{bmatrix} \gamma^-_g(x_{11}), & U(g)\gamma^-_g(x_{12}) \\ \gamma^-_g(x_{21})U(g)^*, & U(g)\gamma^-_g(x_{22})U(g)^* \end{bmatrix}. \tag{III.3.24}$$

We claim that of the projections

$$e_1 = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0, & 0 \\ 0, & 1 \end{bmatrix}$$

are equivalent in the fixed point algebra $\mathscr{P}^\sigma$ in $\mathscr{P}$ under σ.

First, we note that

$$\begin{bmatrix} 0, & 0 \\ x, & 0 \end{bmatrix} \in \mathscr{P}^\sigma \Leftrightarrow x \in \mathscr{P}_n^{\gamma^-}(U). \tag{III.3.25}$$

Since $e_1 \mathscr{P}^\sigma e_1 \cong \mathscr{P}_n^{\gamma^-}$ and $e_2 \mathscr{P}^\sigma e_2 \cong \mathscr{P}_n^{\gamma}$ both contain $\mathfrak{M}^\gamma \otimes \mathbb{C}$, e_1 and e_2 are both properly infinite projections in $\mathscr{P}^\sigma$. In order to prove $e_1 \sim e_2$ in $\mathscr{P}^\sigma$, it suffices to prove that the central supports of e_1 and e_2 in $\mathscr{P}^\sigma$ are both the identity 1. To do this, we must show that $e_2 \mathscr{P}^\sigma e_1$ has left and right supports equal to e_2 and e_1 respectively. But we know by (III.3.25) that $e_2 \mathscr{P}^\sigma e_1 = \mathscr{P}_n^{\gamma^-}(U) \otimes e_{21}$, where $e_{21} = \begin{bmatrix} 0, & 0 \\ 1, & 0 \end{bmatrix}$. Thus, we have only to prove that the left and the right supports of $\mathscr{P}_n^{\gamma^-}(U)$ are both one in $\mathscr{P}_n$.

For each $\varrho \in \operatorname{Aut}_\gamma \mathfrak{M}$, we consider $\varrho^- = \varrho \otimes i \in \operatorname{Aut}_{\gamma^-}(\mathscr{P}_n)$. Since $\varrho^-(U(g)) = U(g)$, $g \in \mathscr{G}$, we have

$$\varrho^-(\mathscr{P}_n^{\gamma^-}(U)) = \mathscr{P}_n^{\gamma^-}(U), \quad \varrho \in \operatorname{Aut}_\gamma \mathfrak{M}. \tag{III.3.26}$$

Therefore, the left support z_2 and the right support z_1 of $\mathscr{P}_n^{\gamma^-}(U)$ are both invariant under ϱ^- for every $\varrho \in \operatorname{Aut}_\gamma \mathfrak{M}$. Hence it follows from the ergodicity of $\operatorname{Aut}_\gamma \mathfrak{M}$ that z_1 and z_2 both fall into M_n. Furthermore, the inclusion:

$$\mathscr{P}_n^{\gamma^-} \mathscr{P}_n^{\gamma^-}(U) \mathscr{P}_n^{\gamma^\sim} \subset \mathscr{P}_n^{\gamma^-}(U) \tag{III.3.27}$$

yields that z_1 (resp. z_2) are central in $\mathscr{P}_n^{\gamma^\sim}$ (resp. $\mathscr{P}_n^{\gamma}$). Since $\mathscr{P}_n^{\gamma^-} \subset M_n$, we get $z_1 = 1$. Since $\mathscr{P}_n^{\gamma}(v)$ contains $U(g)$, the commutativity $z_2 U(g) = U(g) z_2$, $g \in \mathscr{G}$, and the irreducibility imply that $z_2 = 1$. Thus, we have proved the claim: $e_1 \sim e_2$.

Therefore, there exists $v^{\sim}\in\mathscr{P}^{\sigma}$ such that $v^{\sim *}v^{\sim}=e_1$ and $v^{\sim}v^{\sim *}=e_2$. Since $v^{\sim}=e_2v^{\sim}e_1$, $v^{\sim}$ must be of the form:

$$v^{\sim}=\begin{bmatrix}0, & 0\\ v, & 0\end{bmatrix}$$

for some $v\in\mathscr{P}_n^{\gamma^{\sim}}(U)$. The equalities $v^{\sim *}v^{\sim}=e_1$ and $v^{\sim}v^{*}=e_2$ mean precisely that v is unitary. This proves the first conclusion of Lemma III.3.4. Furthermore (III.3.25) and $e_1\mathscr{P}^{\sigma}e_1=\mathscr{P}_n^{\gamma^{\sim}}$ proves (III.3.21).

This concludes the proof of Lemma III.3.4, and thus of Theorem III.3.3(i). A result of Roberts [11; Proposition 6.9] concludes assertion (ii) of this theorem, since $\mathscr{R}_{\gamma}(\mathscr{G})$ contains all irreducible representations of $\mathscr{G}$ if the action γ is faithful (cf. the last assertion of the theorem in Appendix C).

Corollary III.3.5. *With the same assumptions as in Theorem III.3.3 (ii), there exists a dual action $\gamma^{\wedge}$ of $\mathscr{G}$ on $\mathfrak{M}^{\gamma}=\mathfrak{M}$ in the sense of Nakagami,* [20], *such that*[16] $\{\mathfrak{M},\gamma\}\cong\{W^*(\mathfrak{M}^{\gamma},\gamma^{\wedge}),\gamma^{\wedge\wedge}\}$, *where $W^*(\mathfrak{M}^{\gamma},\gamma^{\wedge})$ means the crossed product of $\mathfrak{M}^{\gamma}$ by the dual action $\gamma^{\wedge}$ and $\gamma^{\wedge\wedge}$ means the action on $W^*(\mathfrak{M}^{\gamma},\gamma^{\wedge})$ dual to $\gamma^{\wedge}$.*

Proof. Let $\mathscr{P}=\mathfrak{M}\bar{\otimes}\mathfrak{B}(L^2(\mathscr{G}))$ and $\gamma^{\sim}_g=\gamma_g\otimes\mathrm{Ad}(\varrho(g))$, $g\in\mathscr{G}$, where ϱ is the right regular representation of $\mathscr{G}$. Put $\gamma^{-}_g=\gamma_g\otimes i$. Since $\mathfrak{M}$ is properly infinite being a factor of type III, $\{\mathfrak{M},\gamma^{-}\}\cong\{\mathscr{P},\gamma^{-}\}$. It is known from Digernes [18] that $\mathscr{P}^{\gamma}$ is isomorphic to the crossed product $W^*(\mathfrak{M},\gamma)$ of $\mathfrak{M}$ by $\mathscr{G}$. A duality theorem due to Nakagami [19], and Roberts [11], says that $\{\mathscr{P};\gamma^{\sim}\}=\{W^*(W^*(\mathfrak{M},\gamma),\gamma^{\wedge});\gamma^{\wedge\wedge}\}$. Thus we have only to prove that $\{\mathscr{P};\gamma^{-}\}\cong\{\mathscr{P};\gamma^{\sim}\}$, which is equivalent to the fact that $\mathscr{P}^{\gamma^{-}}(\varrho)$ contains a unitary. But it follows from Theorem III.3.3.ii that $\mathscr{P}^{\gamma^{\sim}}=W^*(\mathfrak{M},\mathscr{G})\cong\mathfrak{M}^{\gamma^{\sim}}$ is a factor of type III. We know also that $\mathscr{P}^{\gamma^{-}}=\mathfrak{M}^{\gamma}\bar{\otimes}\mathfrak{B}(L^2(\mathscr{G}))$ is a factor of type III. Thus $\mathscr{P}^{\gamma}(\varrho)\neq\{0\}$ entails the existence of a unitary in $\mathscr{P}^{\gamma^{-}}(\varrho)$.

The proof of the next corollary provides an alternative way of establishing the second part of Theorem II.4 (ii), (iii) (extension from $\mathscr{F}^{G_{\phi}}$ to $\mathscr{F}^{N_{\phi}}$—cf. II.6), since one knows that a weakly clustering extension to $\mathscr{F}$ of a τ-weakly clustering, (α_t,β) KMS state of $\mathfrak{A}$ separates $\mathscr{F}$.

Corollary III.3.6. *In addition to the assumptions of Theorem III.3.3 relative to the system $\{\mathfrak{M},\gamma\}$, let $\{\alpha'_t\}$ be a continuous one parameter group of automorphisms of $\mathfrak{M}$ such that $\alpha'_t\circ\gamma_g=\gamma_g\circ\alpha'_t$, $g\in\mathscr{G}$, $t\in\mathbb{R}$, and $H=\{\sigma\ \mathrm{Aut}_{\gamma}\mathfrak{M};\tau\circ\alpha'_t=\alpha'_t\circ\tau, t\in\mathbb{R}\}$ is ergodic on $\mathfrak{M}$. If ϕ is a faithful normal state of $\mathfrak{M}$ such that the restriction of ϕ to $\mathfrak{M}^{\gamma}$ satisfies the* KMS-*condition for $\{\alpha'_t\}$ at $\beta>0$ and $\phi\circ\sigma=\phi$, $\tau\in H$, then we conclude the following*

(i) *ϕ is invariant for both $\gamma(\mathscr{G})$ and $\{\alpha'_t\}$;*

(ii) *there exists a one parameter subgroup $\{g_t\}$ of $Z(\mathscr{G})$ such that ϕ satisfies the* KMS *condition for the one-parameter automorphism group $\{\alpha'_t\gamma_{g_t}\}$ at $\beta>0$.*

Proof. Suppose that $\alpha\in\mathrm{Aut}\mathfrak{M}$ commutes with H. Then $\phi\circ\alpha$ and ϕ are both invariant under H, so that the cocycle Radon-Nicodym derivative $(D\phi\circ\alpha:D\phi)_t$ belongs to the fixed point algebra $\mathfrak{M}^H$ under H (which is $\mathbb{C}1$ by assumption). Hence $\phi\circ\alpha$ is a scalar multiple of ϕ. But being a state, $\phi\circ\alpha=\phi$. Thus statement (i) follows.

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[16] In what follows we omit for shortness mentioning the group $\mathscr{G}$ in the notation for cross products

The modular automorphism group $\{\sigma_t^\phi\}$ of ϕ commutes with $\gamma(\mathscr{G})$ and H. Hence it leaves $\mathfrak{M}^\gamma$ globally invariant and $\sigma_{\beta t}^\phi(x)=\alpha_t(x)$ for every $x\in\mathfrak{M}^\gamma$. It follows then that $\{\alpha_{-t}\sigma_{\beta t}^\phi\}$ is a one parameter automorphism group commuting with $\gamma(\mathscr{G})$ and H, which leaves $\mathfrak{M}^\gamma$ pointwise invariant. Hence it follows from Theorem III.3.3. (i) that there exists a one parameter group $\{g_t\}$ in $\mathscr{G}$ such that $\gamma_{g_t}=\alpha_{-t}\sigma_{\beta t}^\phi$. The commutativity of $\{\gamma_{g_t}\}$ and $\gamma(\mathscr{G})$ yields that $\{g_t\}$ is a subgroup of the center $Z(\mathscr{G})$ of $\mathscr{G}$. Q.E.D.

IV. Intrinsic Characterization of the Chemical Potential

Since expectation values of the non-observables elements in $\mathscr{F}$ are without physical significance, the role of the chemical potential for the equilibrium state $\omega_{\beta,\mu}$ should be seen directly without considering the extension problem of the state from the observable algebra $\mathfrak{A}$ to the field algebra $\mathscr{F}$.

If one considers $\mathfrak{A}$ as the given fundamental object then, following [7] the physically relevant part of the structure involving $\mathscr{F}$ and G may be seen to arise from the fact that $\mathfrak{A}$ possesses localized automorphisms which are not inner (we treat in this section the special case of an abelian gauge group which we take for simplicity to be the one-dimensional torus). The following two properties are discussed in [7]:

a) Space or time translations do not change the equivalence class of a localized morphism modulo inner automorphisms. More precisely, if ϱ is a localized morphism giving rise to an α_t-covariant representation of $\mathfrak{A}$ we have[17]

$$\varrho^{-1}\circ\alpha_t\circ\varrho=\mathrm{Ad}\,v_t\circ\alpha_t\,,\qquad t\in\mathbb{R}\,,\tag{IV.1}$$

with v_t a continuous one-parameter family of unitaries of $\mathfrak{A}$ fulfilling the "cocycle condition"

$$v_{t+s}=v_t\alpha_t(v_s)\,,\qquad s,t\in\mathbb{R}\,.\tag{IV.2}$$

b) The equivalence classes of localized automorphisms form an Abelian group the "charge group". In the case of our (simplest) example the charge group is the additive group of integers.

The field algebra $\mathscr{F}$ arises then as a covariance algebra in which the mentioned automorphisms are realized by unitaries $u\in\mathscr{F}$[18]:

$$\varrho(A)=uAu^*\,,\qquad A\in\mathfrak{A}\,.\tag{IV.3}$$

The (Abelian) gauge group is the dual of the "charge group".

Unlike the situation in [7] where irreducible representation of $\mathfrak{A}$ were considered and the (non inner) localized automorphisms led to inequivalent representations (superselection rules) we have now

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[17] By definition $\mathrm{Ad}\,v_t(A)=v_tAv_t^*$, $A\in\mathfrak{A}$

[18] ϱ is then of class n if $\gamma_\theta(u)=e^{in\theta}u$, $\theta\in\mathbb{R}$

Proposition IV.1. *If ω is an extremal (α_t, β)-KMS state of $\mathfrak{A}$ and ϱ a localized automorphism, then $\omega\circ\varrho$ is equivalent to ω. Consequently ω and $\omega\circ\varrho$ have normal faithful extensions to the same von Neumann algebra $\mathfrak{N}=\pi_\omega(\mathfrak{A})''$.*

Proof. Since $\varrho=\mathrm{Ad}\,u$ with $u\in\mathscr{F}$ the state $\omega\circ\varrho$ generates a representation of $\mathfrak{A}$ contained in the restriction to $\mathfrak{A}$ of the representation π_ϕ of $\mathscr{F}$ where ϕ is a clustering extension of ω to $\mathscr{F}$. The proposition then results from Proposition III.2.2. Note that the assumption of unitariness of u entails that the spectrum of the gauge group is not one-sided (N_ϕ is trivial). Theorem II.4 then assures us that ϕ is β-KMS for a one parameter group $t\to\alpha_{-\beta t}\gamma_{-\beta t}$: ϕ is thus separating and Proposition IV.1 applies.

According to known results in the theory of modular automorphisms [21], Proposition IV.1 tells us that there is a unitary cocycle $W_t\in\pi_\omega(\mathfrak{A})''$ (for which one has a unique definition procedure) called the *cocycle Radon-Nikodym derivative* of $\omega\circ\varrho$ with respect to ω:

$$W_t=(D(\omega\circ\varrho):D\omega)_t\,,$$

such that

$$\pi_\omega(\varrho^{-1}\circ\alpha_{-\beta t}\circ\varrho(A))=W_t\pi_\omega(\alpha_{-\beta t}(A))W_t^*\,,\qquad A\in\mathfrak{A}\,,\qquad t\in R \tag{IV.4}$$

(we used the fact that the modular automorphism group of the continuous extension of ω to $\pi_\omega(\mathfrak{A})''$ coincides with $\alpha_{-\beta t}$ on $\mathfrak{A}$, so that the modular automorphism group of $\omega\circ\varrho$ is $\varrho^{-1}\circ\alpha_{-\beta t}\circ\varrho$ on $\mathfrak{A}$ cf. [21] Lemma 1.2.10). The relation (IV.1) gives on the other hand

$$\begin{aligned}&\pi_\omega(\varrho^{-1}\circ\alpha_{-\beta t}\circ\varrho(A))\\&\quad=\pi_\omega(v_{-\beta t})\pi_\omega(\alpha_{-\beta t}(A))\pi_\omega(v_{-\beta t})^*\,,\qquad A\in\mathfrak{A}\,,\qquad t\in R\,.\end{aligned} \tag{IV.5}$$

Comparison of (IV.4) and (IV.5), together with the fact that $\pi_\omega(\mathfrak{A})''$ is a factor, entails that $W_t\pi_\omega(v^*_{-\beta t})$ is a scalar. Furthermore the cocycle properties of W_t and v_t imply that this scalar is of the form $e^{i\lambda t}$ for some real λ:

$$W_t=(D(\omega\circ\varrho)_t:D\omega)_t=e^{i\lambda t}\pi_\omega(v_{-\beta t})\,. \tag{IV.6}$$

We now relate this constant λ to the chemical potential μ of the state ω. We have, for the separating states ϕ and ϕ_u of $\mathscr{F}$, where $\phi_u(a)=\phi(uau^*)$, $a\in\mathscr{F}$, (or rather for their continuous extensions to $\mathfrak{M}=\pi_\phi(\mathscr{F})''$, see [21] Lemma 1.2.3 (e)):

$$(D\phi_u:D\phi)_t=\pi_\phi(u^*)\sigma_t^\phi(\pi_\phi(u))\,,\qquad t\in\mathbb{R}\,,$$

with σ_t^ϕ the modular automorphism of ϕ such that $\sigma_t^\phi\circ\pi_\phi=\pi_\phi\circ\alpha_{-\beta t}\circ\gamma_{-\beta\mu t}$. Thus, if the automorphism ϱ carries the charge n [i.e. $\gamma_\theta(u)=e^{in\theta}u$, $\theta\in\mathbb{R}$] one has $\sigma_t^\phi(u)=e^{in\beta\mu t}u$, and thus

$$(D\phi_u:D\phi)_t=e^{in\beta\mu t}\pi_\phi(u^*\alpha_{-\beta t}(u))\,. \tag{IV.7}$$

Taking the restriction of both sides to the cyclic component for $\pi_\phi(\mathfrak{A})$ of the vector corresponding to ϕ in its GNS construction we see that

$$(D(\omega\circ\varrho):D\omega)_t = e^{in\beta\mu t}\pi_\omega(u^*\alpha_{-\beta t}(u)). \tag{IV.8}$$

Now, since $v'_t = u^*\alpha_t(u)$ satisfies (IV.1) as well as v_t, we have that $u^*\alpha_t(u) = e^{-inct}v_t$, which c a real constant which is chosen independantly of ω. We proved

Proposition II.2. *If ϱ is a localized automorphism in class n, ω an extremal (α_t, β)-KMS state of $\mathfrak{A}$ and v_t a continuous cocycle in $\mathfrak{A}$ satisfying (IV.1) then*

$$(D(\omega\circ\varrho):D\omega)_t = e^{in\beta\mu\hat{}t}\pi_\omega(v_{-\beta t}) \tag{IV.9}$$

where $\mu\hat{} = \mu + c$ with μ the chemical potential as treated in the previous sections and c a real constant independant of ω. (We recall that $(D(\omega\circ\varrho):D\omega)_t$ denotes the cocycle Radon-Nicodym derivative of the continuous extension of the states $\omega\circ\varrho$ and ω to $\pi_\omega(\mathfrak{A})''$ (see Proposition II.1).)

Comments. (i) By (IV.9) the chemical potential of an equilibrium state of $\mathfrak{A}$ is intrinsically defined, in terms of objects related to $\mathfrak{A}$ alone, up to a conventional constant (zero point of μ) which is fixed by the convention adopted in choosing v_t. This freedom corresponds to the freedom in changing the definition of time translation on $\mathscr{F}$ by a factor γ_{ct}.

(ii) (IV.9) shows that the Radon-Nikodym derivative lies actually not only in $\pi_\omega(\mathfrak{A})''$ but in $\pi_\omega(\mathfrak{A})$.

(iii) Let ϱ be an automorphism of $\mathfrak{A}$ such that: (A) ϱ fulfills (IV.1) with a continuous unitary cocycle v_t [satisfying (IV.2)] with values in $\mathfrak{A}$; (B) $\omega\circ\varrho$ and ω are quasi equivalent for all (β, α_t)-KMS states of $\mathfrak{A}$, $\beta\in\mathbb{R}$. Then we have (II.6) (so ϱ defines a kind of a chemical potential).

(iv) If is replaced by an automorphism ϱ' of the same class modulo inner automorphisms ($\varrho' = \varrho\circ\mathrm{Ad}\,U$, U unitary in $\mathfrak{A}$), ϱ' satisfies (IV.1) with the continuous unitary cocycle $v'_t = U^{-1}v_t\alpha_t(U)$. On the other hand, using (IV.1) and the composition property of cocycle Radon-Nikodym derivatives, one sees that

$$(D(\omega\circ\varrho'):D\omega)_t = U^{-1}v_{-\beta t}\alpha_{-\beta t}(U)v^*_{-\beta t}(D(\omega\circ\varrho):D\omega)_t$$

thus (IV.9) holds with the same $\mu\hat{}$ and with ϱ', $v'_{-\beta t}$ instead of ϱ, $v_{-\beta t}$.

Appendix A. Right Translation Invariant C*-Subalgebras of Continuous Functions on a Compact Group

Let G be a compact group. We shall consider $\mathscr{C}(G)$, the set of complex continuous functions on G, as a C^*-algebra with pointwise multiplication, complex conjugation as the *-operation and the sup norm $\|\ \|_\infty$. For $g\in G$ we recall that we defined the right, resp. left translation acting on functions f on G by $(\varrho(h)f)(g) = f(gh)$, $(\lambda(h)f)(g) = f(h^{-1}g)$, $g\in G$. (cf. II.2.1).

Lemma A.1. *The following correspondences establish a bijection between the closed subgroups K of G and the C*-subalgebras $\mathscr{A}$ of $\mathscr{C}(G)$ globally invariant under all right* *translations (i.e. with $\varrho(g)f\in\mathscr{A}$ for all $f\in\mathscr{A}$ and $g\in G$):*

$$K\to\mathscr{A}(=\mathscr{C}(K\diagdown G))^{19}=\{f\in\mathscr{C}(G);\lambda(k)f=f \quad \text{for all} \quad k\in K\} \tag{A.1}$$

$$\mathscr{A}\to K=\{k\in G;f(kg)=f(g) \quad \text{for all} \quad f\in\mathscr{A} \quad \text{and} \quad g\in G\} \tag{A.2}$$

This bijection associates to the normal subgroups of G those globally right-translation invariant C^*-subalgebras of $\mathscr{C}(G)$ which are also globally left translation invariant (or, equivalently, invariant under the symmetry s, where $(sf)(g)=f(g^{-1})$, $g\in G$).

Note that owing to the right translation invariance of $\mathscr{A}$, K in (A.2) can also be defined as

$$K=\{k\in G;f(kg_0)=f(g_0) \quad \text{for all} \quad f\in\mathscr{A}\} \tag{A.3}$$

with g_0 any fixed element of G (for instance the unit e of G).

Proof. $\mathscr{A}=\mathscr{A}_K$ defined in (A.1) is obviously (for each subset K of G) a C^*-subalgebra of $\mathscr{C}(G)$ globally invariant by all $\varrho(g)$, $g\in G$. On the other hand $K=K_{\mathfrak{A}}$ defined in (A.2) is clearly [for each subset $\mathscr{A}$ of $\mathscr{C}(G)$] a subgroup of G, closed as an intersection of closed sets.

Now, with $\mathscr{A}$ a globally right translation invariant C^*-subalgebra of $\mathscr{C}(G)$, one has clearly $\mathscr{A}\subset\mathscr{A}_{K_\mathscr{A}}=\mathscr{C}(K_\mathscr{A}\diagdown G)$: the Stone-Weierstrass theorem will therefore imply that $\mathscr{A}=\mathscr{A}_{K_\mathscr{A}}$ if we show that $\mathscr{A}$ separates the spectrum of $\mathscr{A}_{K_\mathscr{A}}$; i.e. that $f(g_1)=f(g_2)$, $f\in\mathscr{A}$, entails $g_1g_2^{-1}\in K_\mathscr{A}$. But this follows from (A.3) with $g_1=g_2$ and $k=g_1g_2^{-1}$. Since, on the other hand, the map $K\to K_\mathscr{A}$ is obviously injective, we proved that (A.1), (A.2) establish a bijection.

Suppose now that $\mathscr{A}$ is globally left translation invariant and let $k\in K$, $g\in G$: (A.2) with f replaced by $\lambda(g)f$ and $x=gx_0$, $x_0\in G$, shows that $f(g^{-1}kgg_0)=f(g_0)$, hence $g^{-1}kg\in K$ by (A.3): we proved that K is a normal subgroup. Conversely, if this is the case, $K\diagdown G=G/K$, thus $\mathscr{A}$ consists of the $f\in\mathscr{C}(K)$ invariant under the $\varrho(k)$, $k\in K$, and is thus globally left-translation invariant. Further, the fact that $s(\mathscr{A})=\mathscr{A}$ evidently entails global left-translation invariance; and conversely normality of K entails that $s(\mathscr{A})=\mathscr{A}$.

Lemma A.2. *Let K_i, $i=1,2$, be closed subgroups of the compact group G: every C^*-isomorphism of $\mathscr{C}(K_1\diagdown G)$ onto $\mathscr{C}(K_2\diagdown G)$ commuting with all right translations $\varrho(g)$, $g\in G$, is of the form $\lambda(h)$ for some $h\in G$ such that $K_2=hK_1h^{-1}$.*

Proof. Let σ be such an automorphism: the transposed σ^t of σ maps (homeomorphically) $K_2\diagdown G$ onto $K_1\diagdown G$ whilst commuting with all $\varrho(g)$, $g\in G$. Therefore

$$\begin{aligned}\sigma^t(K_2g)&=\sigma^t(K_2e\cdot g)=\sigma^t(K_2e)g=(K_1h)g\\&=K_1(hg), \quad g\in G,\end{aligned}$$

[19] This notation presupposes that we may identify the functions $f\in\mathscr{C}(G)$ invariant under all left translations $\lambda(k)$, $k\in K$, with the continuous functions $\not{f}$ on the space $K\diagdown G$ of the right cosets Kg, $g\in G$, endowed with the usual quotient topology. The identification if by means of the bijection $\not{f}\leftrightarrow f=\not{f}\circ\phi$, with ϕ the canonical map $G\to K\diagdown G$

with h any element of G such that $\sigma^t(K_2)=K_1h$. By transposition we obtain that $\sigma=\lambda(h)$. Since σf, $f\in\mathscr{C}(K_1\diagdown G)$, has to be invariant under all $\lambda(k_2)$, $k_2\in K_2$, we must have $f(h^{-1}k_2g)=f(h^{-1}g)$, $g\in G$, $k_2\in K_2$, $f\in\mathscr{C}(K_1\diagdown G)$, whence $h^{-1}k_2h\in K_1$, thus $K_2=hK_1h^{-1}$.

Lemma A.3. *Let $\mathscr{A}^i_0$, $i=1,2$, with norm closures $\mathscr{A}^i$, be globally right translations invariant *-subalgebras of $\mathscr{C}(G)$ and let K_i be the closed subgroup associated to $\mathscr{A}^i$. Every linear, multiplicative U_0; $\mathscr{A}^1_0\to\mathscr{A}^2_0$ which commutes with all right-translations $\varrho(g)$, $g\in G$ and is isometric for the $L^2(K_i\diagdown G)$ norms is given by $\lambda(h)$ for some $h\in G$ such that $K_2=hK_1h^{-1}$.*

Proof. Let U be the linear isometry from $L^2(K_1\diagdown G)$ onto $L^2(K_2\diagdown G)$ obtained by continuous extension of U_0. Let T^i be the *-representation of $\mathscr{A}^i$ on $L^2(K_i\diagdown G)$ defined by pointwise multiplication:

$$\{T^i_{f_i}\phi_i\}(g)=f_i(g)\phi_i(g)\,,\qquad f_i\in\mathscr{A}^i\,,$$
$$\phi_i\in L^2(K_i\diagdown G)\,,\qquad g\in G\,,$$

and define σ on $\mathscr{A}^i$ by

$$T^2_{\sigma_f}=UT^1_fU^*\,,\qquad f\in\mathscr{A}^1\,. \tag{A.4}$$

Applying the r.h.s. of (A.4) to U_0f^1, $f^1\in\mathscr{A}^1_0\subset L^2(K_1\diagdown G)$ with $f\in\mathscr{A}^1_0$ we have that

$$UT^1_ff^1=U(ff^1)=(U_0f)(U_0f^1)=T_{U_0f}U_0f^1$$

whence $\sigma f=U_0f$ by comparison with the l.h.s. of (A.4) since $U_0\mathscr{A}^1_0$ is dense in $L^2(K_2\diagdown G)$: thus (A.4) defines a continuous extension of U_0 to $\mathscr{A}^1$. Since T^1 and T^2 are faithful representations this extension fulfills the assumptions of Lemma A.3, which yields the result.

Appendix B. Lifting of a One-Parameter Subgroup

For the convenience of the reader, we shall state a Lemma, which is a trivial combination of results of Iwazawa [23], §1, and Montgomery and Zippin, [24], §4.1.5.

Lemma B. *Let G be a locally compact group with H a compact normal subgroup, and π the natural homomorphism of G onto G/H. If $\{\dot g\,(t)\}$ is a one parameter subgroup of G/H, then there exists a one parameter subgroup $\{g(t)\}$ in G such that* (i) $\pi(g(t))=\dot g\,(t)$ *and* (ii) *$g(t)$ commutes with H for any $t\in\mathbb{R}$.*

Appendix C. Robert's Version of the Tannaka Duality Theorem in von Neumann Algebras

Let $\mathfrak{M}$ be a properly infinite von Neumann algebra equipped with a continuous action γ of a compact group $\mathscr{G}$ such that the fixed point algebra $\mathfrak{M}^\gamma$ is also properly infinite. By definition (see Roberts in [11]), a (non-degenerate) *Hilbert space in* $\mathfrak{M}$ is a closed subspace $\mathfrak{H}$ of $\mathfrak{M}$ such that (i) y^*x is a scalar multiple of the identity for every $x, y \in \mathfrak{H}$ and (ii) $a\mathfrak{H} = \{0\}$ implies $a = 0$ for any $a \in \mathfrak{M}$. The inner product $(y|x)$ in $\mathfrak{H}$ is accordingly given by y^*x. We note also that any element of $\mathfrak{H}$ with norm one is an isometry, and that a complete orthogonal basis $\{u_i\}$ of $\mathfrak{H}$ is a collection of isometries with orthogonal ranges and $\sum u_i u_i^* = 1$. If $\mathfrak{H}$ is globally invariant under γ, then we have, for any x, y

$$(\gamma_g(y)|\gamma_g(x)) = \gamma_g(y)^*\gamma_g(x) = \gamma_g(y^*x) = y^*x = (y|x).$$

Hence the restriction of γ to $\mathfrak{H}$ gives rise to a unitary representation of $\mathscr{G}$. Let $\mathscr{H}_\gamma(\mathfrak{M})$ denote the collection of all finite dimensional Hilbert spaces in $\mathfrak{M}$ globally invariant under γ. Let $\mathscr{R}_\gamma(\mathscr{G})$ denote the collection of unitary representations of $\mathscr{G}$ obtained by restricting γ to all members of $\mathscr{H}_\gamma(\mathfrak{M})$. We quote the Tannaka duality theorem in the cast of the formalism given by Roberts [11] as follows:

Tannaka Duality Theorem. *With the assumptions and notation above suppose that each irreducible equivalence class of subrepresentations of the action γ of $\mathscr{G}$ on $\mathfrak{M}$ occurs in $\mathscr{R}_\gamma(\mathfrak{M})$*[20]. *If α is an automorphism of $\mathfrak{M}$ such that* (i) $\alpha(\mathfrak{H}) \subset \mathfrak{H}$ *for each* $\mathfrak{H} \in \mathscr{H}_\gamma(\mathfrak{M})$, (ii) $\alpha(x) = x$ *for each* $x \in \mathfrak{M}^\gamma$, *then there exists an element $g_\alpha \in \mathscr{G}$ such that $\alpha = \gamma_{g_\alpha}$. If the action of G on $\mathfrak{M}$ is faithful,* Msp(γ) *contains (up to equivalence) all irreducible representations of $\mathscr{G}$.*

Proof. For $x, y \in \mathfrak{H}$, $\mathfrak{H} \in \mathscr{H}_\gamma(\mathfrak{M})$, let $f_{x,y}$ be the function on $\mathscr{G}$ given by

$$f_{x,y}(g) = (x|\gamma_g(y)) = x^*\gamma_g(y), \qquad g \in \mathscr{G}, \tag{C.1}$$

and let $\mathscr{C}_\gamma(\mathscr{G})$ be the set of all such functions. We claim that $\mathscr{C}_\gamma(\mathscr{G})$ is a *-subalgebra of $\mathscr{C}(\mathscr{G})$. Let $x_1, y_1 \in \mathfrak{H}_1$, $x_2, y_2 \in \mathfrak{H}_2$, $\mathfrak{H}_1, \mathfrak{H}_2 \in \mathscr{H}_\gamma(\mathfrak{M})$. If we choose isometric u, v in $\mathfrak{M}^\gamma$ with complementary orthogonal ranges (possible since $\mathfrak{M}^\gamma$ is properly infinite) the set $\mathfrak{H} = u\mathfrak{H}_1 + v\mathfrak{H}_2$ is a member of $\mathscr{H}_\gamma(\mathfrak{M})$: one has namely, with $x = ux_1 + vx_2$, $y = uy_1 + vy_2$,

$$x^*y = x_1^*y_1 + x_2^*y_2 \in \mathbb{C}.$$

More generally

$$x^*\gamma_g(y) = x_1^*\gamma_g(y_1) + x_2^*\gamma_g(y_2), \qquad g \in \mathscr{G},$$

i.e. $f_{x_1,y_1} + f_{x_2,y_2} = f_{x,y}$: $\mathscr{C}_\gamma(\mathscr{G})$ is thus additive (moreover obviously linear). To show that $\mathscr{C}_\gamma(\mathscr{G})$ is multiplicative we observe that

$$x_1^*\gamma_g(y_1)x_2^*\gamma_g(y_2) = x_2^*x_1^*\gamma_g(y_1)\gamma_g(y_2),$$

in other words

$$f_{x_1,y_1}f_{x_2,y_2} = f_{x_1y_1,x_2y_2} \tag{C.2}$$

with x_1y_1, x_2y_2 elements of the tensor product Hilbert space $H_1 \otimes H_2$ (see [11]). It remains us to show that $\mathscr{C}_\gamma(\mathscr{G})$ is closed for complex conjugation. For this we

[20] In other terms the *spectrum* Sp(γ) *of* γ (irreducible equivalence classes of subrepresentations of γ) coincides with its *monoidal spectrum* Msp(γ) (irreducible equivalence classes of restrictions of γ to Hilbert spaces in $\mathfrak{M}$)

observe that the $f_{x,y}(g)$, $x, y \in \mathfrak{H}$, are the matrix elements of the representation $U = \gamma|_{\mathfrak{H}}$ of $\mathscr{G}$: therefore it suffices to check that if U belongs to $\mathscr{R}_\gamma(\mathscr{G})$, the same holds for the conjugate representation U^-. Since $\mathscr{R}_\gamma(\mathscr{G})$ is closed for direct sums (observe that $\gamma|_{\mathfrak{H}} = \gamma|_{\mathfrak{H}_1} \oplus \gamma|_{\mathfrak{H}_2}$ for $\mathfrak{H} = u\mathfrak{H}_1 + v\mathfrak{H}_2$ as constructed above), it is enough to examine an irreducible $U \in \mathscr{R}_\gamma(\mathscr{G})$: now, if $U = \gamma|_{\mathfrak{H}}$, $\gamma|_{\mathfrak{H}^*}$ is equivalent to U^-: U^- thus appears as a subrepresentation of γ: but it then by assumption also occurs in $\mathscr{R}_\gamma(\mathscr{G})$. We completed the proof that $\mathscr{C}_\gamma(\mathscr{G})$ is a *-subalgebra of $\mathscr{C}(\mathscr{G})$.

We now set, for $f_{x,y} \in \mathscr{C}_\gamma(\mathscr{G})$

$$U f_{x,y} = f_{\alpha(x)y}. \tag{C.3}$$

The fact that this defines a map U on $\mathscr{C}_\gamma(\mathscr{G})$ results from the calculation

$$\begin{aligned}(Uf_{x,y}|Uf_{x',y'})_{L^2(\mathscr{G})} &= \int f_{\alpha(x'),y'}(g)^* f_{\alpha(x),y}(g)dg \\ &= \int \alpha(x)^*\gamma_g(y)\{\alpha(x')^*\gamma_g(y')\}^* dg \\ &= \alpha(x)^* \int \gamma_g(yy'^*)dg\,\alpha(x') \\ &= \alpha\{x^* \int \gamma_g(yy'^*)dg x'\} \\ &= (f_{x,y}|f_{x',y'})_{L^2(\mathscr{G})}\end{aligned}$$

[we used the fact that $\int \gamma_g(y'y^*)dg$ belongs to $\mathfrak{M}^\gamma$, thus is left invariant by α]. The last relation shows that (C.2) defines an isometry of $\mathscr{C}_\gamma(\mathscr{G})$ for the $L^2(\mathscr{G})$ norm. As, on the other hand, by (C.1),

$$\begin{aligned}U(f_{x_1,y_1}\cdot f_{x_2,y_2}) &= U(f_{x_1x_2,y_1y_2}) = f_{\alpha(x_1x_2),y_1y_2} \\ &= f_{\alpha(x_1)\alpha(x_2),y_1y_2} = f_{\alpha(x_1)y_1} f_{\alpha(x_2)y_2},\end{aligned}$$

U is multiplicative. Since moreover U obviously commutes with right translations, Lemma A.3 of Appendix A tells us that we have, for all $f_{x,y} \in \mathscr{C}_\gamma(\mathscr{G})$,

$$(Uf_{x,y})(g) = f_{x,y}(g_\alpha^{-1}g),$$

with some $g_\alpha \in \mathscr{G}$. This means that $\alpha(x)^*\gamma_g(y) = x^*\gamma_{g_\alpha^{-1}g}(y) = \gamma_{g_\alpha}(x)^*\gamma_g(y)$ for all $x, y \in \mathfrak{H}$, $\mathfrak{H} \in \mathscr{H}_\gamma(\mathfrak{M})$. It follows that $\alpha = \gamma_{g_\alpha}$ in restriction to all $\mathfrak{H} \in \mathscr{H}_\gamma(\mathfrak{M})$.
In order to show that $\alpha = \gamma_{g_\alpha}$ on all of $\mathfrak{M}$, it now suffices to check that $\alpha = \alpha_{g_\alpha}$ in restriction to each subspace $\mathscr{V}$ on which γ induces an irreducible representation of $\mathscr{G}$ (since $\mathscr{G}$ is compact, the collection of these subspaces is total in $\mathfrak{M}$). Let thus $\mathscr{V}$ be one of these subspaces and let $\gamma|_{\mathscr{V}} \sim U$, U an irreducible representation of $\mathscr{G}$. By assumption, there exists an $\mathfrak{H} \in \mathscr{H}_\gamma(\mathfrak{M})$ such that $\gamma|_{\mathfrak{H}} \sim U$. We can, therefore, choose a basis $\{e_1, \ldots, e_n\}$ of $\mathscr{V}$ and an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathfrak{H}$ such that

$$\gamma_g(e_i) = \sum_{j=1}^{n} (U_g)_{ji} e_j,$$

$$\gamma_g(f_i) = \sum_{j=1}^{n} (U_g)_{ji} f_j,$$

with the same unitary matrices $\{(U_g)_{ji}\}$ realizing U. As a result

$$A=\sum_{i=1}^{n} e_i f_i^*$$

belongs to $\mathfrak{M}^\gamma$, whilst $\mathscr{V}=A\mathfrak{H}$ since $Af_i=e_i$, $i=1,\ldots,n$. Thus, for each $Ax\in\mathscr{V}$, $x\in\mathfrak{H}$, one has

$$\alpha(Ax)=A\alpha(x)=A\gamma_{g_\alpha}(x)=\gamma_{g_\alpha}(Ax).$$

This proves the first conclusion of the Theorem.

Assume now that the action γ of $\mathscr{G}$ on $\mathfrak{M}$ is faithful i.e. that $\gamma_g=i$ entails $g=e$. Since $\mathscr{C}_\gamma(\mathscr{G})$ is a *-subalgebra of $\mathscr{C}(\mathscr{G})$ it follows (cf. Lemma A.1 of Appendix A) that $\overline{\mathscr{C}_\gamma(\mathscr{G})}=\mathscr{C}(K\diagdown\mathscr{G})$, with

$$K=\{g\in\mathscr{G}\,;f_{x,y}(g)=f_{x,y}(e)\quad\text{for all}\quad f_{x,y}\in\mathscr{C}_\gamma(\mathscr{G})\}.$$

From this we infer that if $g\in K$, $U(g)=1$ for all $U\in\mathscr{R}_\gamma(\mathfrak{M})$, and thus, according to our assumption, for all irreducible subrepresentations of γ: therefore $\gamma_g=i$ and $g=e$, if γ is faithful. We proved that $K=\{e\}$, thus $\overline{\mathscr{C}_\gamma(\mathscr{G})}=\mathscr{C}(\mathscr{G})$. The Peter-Weyl theorem then implies that $\mathscr{R}_\gamma(\mathscr{G})$ contains all irreducible representations of $\mathscr{G}$.

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