Extension of Plurisubharmonic Functions with Growth Control

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EXTENSION OF PLURISUBHARMONIC FUNCTIONS WITH GROWTH CONTROL

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Abstract. Suppose that X is an analytic subvariety of a Stein manifold M and that φ is a plurisubharmonic (psh) function on X which is dominated by a continuous psh exhaustion function u of M. Given any number $c>1$, we show that φ admits a psh extension to M which is dominated by cu on M.

We use this result to prove that any ω-psh function on a subvariety of the complex projective space is the restriction of a global ω-psh function, where ω is the Fubini-Study Kähler form.

Introduction

Let $X\subset\mathbb{C}^n$ be a (closed) analytic subvariety. In the case when X is smooth it is well known that a plurisubharmonic (psh) function on X extends to a psh function on $\mathbb{C}^n$ [Sa] (see also [BL, Theorem 3.2]). Using different methods, Coltoiu generalized this result to the case when X is singular [Co, Proposition 2].

In this article we follow Coltoiu's approach and show that it is possible to obtain extensions with global growth control:

Theorem A. *Let X be an analytic subvariety of a Stein manifold M and let φ be a psh function on X. Assume that u is a continuous psh exhaustion function on M so that $\varphi(z)<u(z)$ for all $z\in X$. Then for every $c>1$ there exists a psh function $\psi=\psi_c$ on M so that $\psi|_X=\varphi$ and $\psi(z)<c\max\{u(z),0\}$ for all $z\in M$.*

We recall that a function $\varphi:X\to[-\infty,+\infty)$ is called psh if $\varphi\not\equiv-\infty$ on X and if every point $z\in X$ has a neighborhood U in $\mathbb{C}^n$ so that $\varphi=u|_U$ for some psh function u on U. We refer to [FN] and [D2, section 1] for a detailed discussion of this notion. We note here that if φ is not identically $-\infty$ on an irreducible component Y of X then φ is locally integrable on Y with respect to the area measure of Y. Let us stress that the more general notion of *weakly psh* function is not appropriate for the extension problem (see section 3).

We then look at a similar problem on a compact Kähler manifold V. Here psh functions have to be replaced by quasiplurisubharmonic (qpsh) ones. Given a Kähler form ω, we let

$$PSH(V,\omega)=\left\{\varphi\in L^1(V,[-\infty,+\infty)):\ \varphi\text{ upper semicontinuous, }dd^c\varphi\geq-\omega\right\}$$

denote the set of ω-plurisubharmonic (ω-psh) functions. If $X\subset V$ is an analytic subvariety, we define similarly the class $PSH(X,\omega|_X)$ of ω-psh functions on X (see section 2 for precise definitions).

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By restriction, ω-psh functions on V yield $\omega|_X$-psh functions on X. Assuming that ω is a *Hodge form*, i.e. a Kähler form with integer cohomology class, our second result is that every $\omega|_X$-psh function on X arises in this way.

Theorem B. *Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. Then any $\omega|_X$-psh function on X is the restriction of an ω-psh function on V.*

Note that in the assumptions of Theorem B there exists a positive holomorphic line bundle L on V whose first Chern class $c_1(L)$ is represented by ω. In this case the ω-psh functions are in one-to-one correspondence with the set of (singular) positive metrics of L (see [GZ]). Thus an alternate formulation of Theorem B is the following:

Theorem B'. *Let X be a subvariety of a projective manifold V and L be an ample line bundle on V. Then any (singular) positive metric of $L|_X$ is the restriction of a (singular) positive metric of L on V.*

Recall that it is possible to regularize qpsh functions on $\mathbb{P}^n$, since it is a homogeneous manifold. Hence Theorem B has the following immediate corollary:

Corollary C. *Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. If $\varphi \in PSH(X, \omega|_X)$ then there exists a sequence of smooth functions $\varphi_j \in PSH(V, \omega)$ which decrease pointwise on V so that $\lim \varphi_j = \varphi$ on X.*

When X is smooth this regularization result is well known to hold even when the cohomology class of ω is not integral (see [D3], [BK]).

Corollary C allows to show that the singular Kähler-Einstein currents constructed in [EGZ1] have *continuous* potentials, a result that has been obtained recently in [EGZ2] by completely different methods (see also [DZ] for partial results in this direction).

We prove Theorem A in section 1. The compact setting is considered in section 2, where Theorem B is derived from Theorem A. In section 3 we discuss the special situation when X is an algebraic subvariety of $\mathbb{C}^n$. As an application of Theorem B, we give a characterization of those psh functions in the Lelong class $\mathcal{L}(X)$ which admit an extension in the Lelong class $\mathcal{L}(\mathbb{C}^n)$ (see section 3 for the necessary definitions). In particular, we give simple examples of algebraic curves $X \subset \mathbb{C}^2$ and of functions $\eta \in \mathcal{L}(X)$ which do not have extensions in $\mathcal{L}(\mathbb{C}^2)$.

1. Proof of Theorem A

The following proposition will allow us to reduce the proof of Theorem A to the case $M = \mathbb{C}^n$. We include its short proof for the convenience of the reader.

Proposition 1.1. *Let V be a complex submanifold of $\mathbb{C}^N$ and u be a continuous psh exhaustion function on V. Then there exists a continuous psh exhaustion function $\widetilde{u}$ on $\mathbb{C}^N$ so that $\widetilde{u}|_V = u$.*

Proof. The argument is very similar to the one of Sadullaev ([Sa],[BL, Theorem 3.2]). By [Si], there exists an open neighborhood W of V in $\mathbb{C}^N$ and a holomorphic retraction $r : W \to V$. We can find an open neighborhood U of V so that $U \subset W$ and $\|r(z) - z\| < 2$ for every $z \in U$. Indeed, if $B(p,r)$ denotes the open ball in $\mathbb{C}^N$ centered at p and of radius r, then $U_p = r^{-1}(B(p,1)) \cap B(p,1)$ is an open neighborhood of $p \in V$, and we let $U = \bigcup_{p\in V} U_p$. Since u is a continuous psh exhaustion function on V, it follows that the function $u(r(z))$ is continuous psh on U and $\lim_{z\in U,\|z\|\to+\infty} u(r(z)) = +\infty$.

It is well known that there exist entire functions $f_0, \dots, f_N$, so that $V = \{z \in \mathbb{C}^N : f_k(z) = 0,\ 0 \le k \le N\}$ (see [Ch, p.63]). The function $\rho = \log(\sum |f_k|^2)$ is psh on $\mathbb{C}^N$ and $V = \{\rho = -\infty\}$.

Let D be an open set so that $V \subset D \subset \overline{D} \subset U$. Since ρ is continuous on $\mathbb{C}^N \setminus V$, we can find a convex increasing function χ on $[0, +\infty)$ which verifies for every $R \ge 0$ the following two properties:

(i) $\chi(R) > R - \rho(z)$ for all $z \in \mathbb{C}^N \setminus D$ with $\|z\| = R$.

(ii) $\chi(R) > u(r(z)) - \rho(z)$ for all $z \in \partial D$ with $\|z\| = R$.

Then

$$\widetilde{u}(z) = \begin{cases} \max\{u(r(z)), \chi(\|z\|) + \rho(z)\}, \text{ if } z \in D, \\ \chi(\|z\|) + \rho(z), \text{ if } z \in \mathbb{C}^N \setminus D, \end{cases}$$

is a continuous psh exhaustion function on $\mathbb{C}^N$ and $\widetilde{u} = u$ on V. □

Employing the methods of Coltoiu [Co] we now construct psh extensions with growth control over bounded sets in $\mathbb{C}^n$.

Proposition 1.2. *Let χ be a psh function on a subvariety $X \subset \mathbb{C}^n$ and let v be a continuous psh function on $\mathbb{C}^n$ with $\chi < v$ on X. If $R > 0$, there exists a psh function $\widetilde{\chi} = \widetilde{\chi}_R$ on $\mathbb{C}^n$ so that $\widetilde{\chi}|_X = \chi$ and $\widetilde{\chi}(z) < v(z)$ for all $z \in \mathbb{C}^n$ with $\|z\| \le R$.*

Proof. We use a similar argument to the one in the proof of Proposition 2 in [Co]. Consider the subvariety $A = (X \times \mathbb{C}) \cup (\mathbb{C}^n \times \{0\}) \subset \mathbb{C}^{n+1}$, and let

$$D = \{(z,w) \in X \times \mathbb{C} : \log|w| + \chi(z) < 0\} \cup (\mathbb{C}^n \times \{0\}) \subset A.$$

Since $D \cap (X \times \mathbb{C})$ is Runge in $X \times \mathbb{C}$, it follows that D is Runge in A. Let

$$K = \{(z,w) \in \mathbb{C}^{n+1} : \rho(z,w) = \max\{\log^+(\|z\|/R), \log|w| + v(z)\} \le 0\}.$$

Since v is continuous, ρ is a continuous psh exhaustion function on $\mathbb{C}^{n+1}$, so K is a polynomially convex compact set. As $\chi < v$ on X, we have $K \cap A \subset D$. By [Co, Theorem 3] there exists a Runge domain $\widetilde{D} \subset \mathbb{C}^{n+1}$, with $\widetilde{D} \cap A = D$ and $K \subset \widetilde{D}$. Let $\delta(z,w)$ denote the distance from $(z,w) \in \widetilde{D}$ to $\partial\widetilde{D}$ in the w-direction. Since $\widetilde{D}$ is pseudoconvex, $-\log\delta$ is psh on $\widetilde{D}$ (see e.g. [FS, Proposition 9.2]). Hence $\widetilde{\chi}(z) = -\log\delta(z,0)$ is psh on $\mathbb{C}^n$, as $\mathbb{C}^n \times \{0\} \subset \widetilde{D}$. Since $\widetilde{D} \cap A = D$, it follows that $\widetilde{\chi}|_X = \chi$. Moreover, $K \subset \widetilde{D}$ implies that $\widetilde{\chi}(z) < v(z)$ for all $z \in \mathbb{C}^n$ with $\|z\| \le R$. □

The proof of Theorem A proceeds like this. Given a partition

$$\mathbb{C}^n = \bigcup\{m_{j-1} < u \le m_j\},$$

where $m_j \nearrow +\infty$, we apply Proposition 1.2 inductively to construct an extension dominated in each "annulus" $\{m_{j-1} < u \le m_j\}$ by $\gamma_j u$, where $\gamma_j > 1$ is an increasing sequence defined in terms of the m_j's. Theorem A will follow by showing that it is possible to choose $\{m_j\}$ rapidly increasing so that $\lim\gamma_j$ is arbitrarily close to 1.

We fix next an increasing sequence $\{m_j\}_{j\ge-1}$ so that

$$m_{-1} = m_0 = 0 < m_1 < m_2 < \dots,\ \{u < m_1\} \neq \emptyset,\ m_j \nearrow +\infty.$$

Define inductively a sequence $\{\gamma_j\}_{j\geq 0}$, as follows:

$$\gamma_0 = 1,\ \ \gamma_j(m_j - m_{j-1}) = \gamma_{j-1}(m_j - m_{j-2}) + 1 \text{ for } j \geq 1. \tag{1}$$

Clearly, $\gamma_j > \gamma_{j-1} > 1$ for all $j > 1$.

Proposition 1.3. *Let X, φ, u be as in Theorem A with $M = \mathbb{C}^n$, and let $\{m_j\}$, $\{\gamma_j\}$ be as above. There exists a psh function ψ on $\mathbb{C}^n$ so that $\psi|_X = \varphi$ and for all $z \in \mathbb{C}^n$ we have*

$$\psi(z) < \begin{cases} \gamma_j u(z), \text{ if } m_{j-1} < u(z) \leq m_j,\ j \geq 2, \\ \gamma_1 \max\{u(z), 0\}, \text{ if } u(z) \leq m_1. \end{cases}$$

Proof. We introduce the sets

$$D_j = \{z \in \mathbb{C}^n : u(z) < m_j\}\ ,\ \ K_j = \{z \in \mathbb{C}^n : u(z) \leq m_j\}.$$

Since u is a continuous psh exhaustion function, K_j is a compact set. Let

$$\rho_j = \gamma_j \max\{u - m_{j-1}, 0\} - j,\ j \geq 0.$$

Then ρ_j is psh on $\mathbb{C}^n$ and (1) implies that

$$\rho_j(z) = \rho_{j-1}(z) \text{ if } u(z) = m_j,\ j \geq 1. \tag{2}$$

We claim that

$$\rho_j(z) \geq u(z) \text{ if } z \in \mathbb{C}^n \setminus D_j,\ j \geq 0. \tag{3}$$

Indeed, since $\gamma_j \geq 1$ and using (1) we obtain

$$\begin{aligned} \rho_j(z) - u(z) &= (\gamma_j - 1)u(z) - \gamma_j m_{j-1} - j \geq (\gamma_j - 1)m_j - \gamma_j m_{j-1} - j \\ &= (\gamma_{j-1} - 1)m_j - \gamma_{j-1}m_{j-2} - j + 1 \\ &\geq (\gamma_{j-1} - 1)m_{j-1} - \gamma_{j-1}m_{j-2} - (j-1). \end{aligned}$$

So $x_j := (\gamma_j - 1)m_j - \gamma_j m_{j-1} - j \geq x_0 = 0$, and (3) is proved.

Let $\varphi_j = \max\{\varphi, -j\}$. We construct by induction on $j \geq 1$ a sequence of continuous psh functions ψ_j on $\mathbb{C}^n$ with the following properties:

$$\psi_j(z) > \varphi_j(z) \text{ for } z \in X\ ,\ \ \int_{X \cap K_{j-1}} (\psi_j - \varphi_j) < 2^{-j}. \tag{4}$$

$$\psi_j(z) \geq \rho_j(z) \text{ for } z \in D_j\ ,\ \ \psi_j(z) = \rho_j(z) \text{ for } z \in \mathbb{C}^n \setminus D_j. \tag{5}$$

$$\psi_j(z) < \psi_{j-1}(z) \text{ for } z \in K_{j-1}, \text{ where } \psi_0 = \rho_0 = \max\{u, 0\}. \tag{6}$$

Here the integral in (4) is with respect to the area measure on each irreducible component, i.e.

$$\int_{X \cap K} f := \sum \int_{Y \cap K} f\, \beta^{\dim Y},$$

where the sum is over all irreducible components Y of X which intersect K and β is the standard Kähler form on $\mathbb{C}^n$. (Note that this is a finite sum.)

Assume that the function ψ_{j-1} is constructed with the desired properties. We construct ψ_j by applying Proposition 1.2 with $\chi = \varphi_j$ and $v = \psi_{j-1}$. (If $j = 1$, ψ_1 is constructed in the same way by applying Proposition 1.2 with $\chi = \varphi_1$ and $v = \psi_0$.) By (4), $\varphi_j \leq \varphi_{j-1} < \psi_{j-1}$ on X (and for $j = 1$, clearly $\varphi_1 < \psi_0$ on X). Therefore Proposition 1.2 yields a psh function $\widetilde{\varphi}_j$ on $\mathbb{C}^n$ so that $\widetilde{\varphi}_j|_X = \varphi_j$ and $\widetilde{\varphi}_j < \psi_{j-1}$ on K_j. Using the standard regularization of $\widetilde{\varphi}_j$ and the dominated convergence theorem (as $\varphi_j \geq -j$) we obtain a continuous psh function $\widetilde{\psi}_j$ on $\mathbb{C}^n$ which verifies

$$\widetilde{\psi}_j(z) > \varphi_j(z) \text{ for } z \in X \ , \ \int_{X\cap K_j} (\widetilde{\psi}_j - \varphi_j) < 2^{-j}.$$

Moreover, since ψ_{j-1} is continuous, we can ensure by the Hartogs lemma that we also have $\widetilde{\psi}_j(z) < \psi_{j-1}(z)$ for $z \in K_j$.

We now define

$$\psi_j(z) = \begin{cases} \max\{\widetilde{\psi}_j(z), \rho_j(z)\}, \text{ if } z \in D_j, \\ \rho_j(z), \text{ if } z \in \mathbb{C}^n \setminus D_j. \end{cases}$$

By (5) and (2) we have $\widetilde{\psi}_j < \psi_{j-1} = \rho_{j-1} = \rho_j$ on ∂D_j (for $j = 1$, recall that $\psi_0 = \rho_0$ by definition). So ψ_j is a continuous psh function on $\mathbb{C}^n$ which verifies (5). On $X \setminus D_j$ we have by (3) that $\psi_j = \rho_j \geq u > \varphi_j$, while on $X \cap D_j$, $\psi_j \geq \widetilde{\psi}_j > \varphi_j$. Since $\rho_j = -j \leq \varphi_j < \widetilde{\psi}_j$ on $X \cap K_{j-1}$, we see that $\psi_j = \widetilde{\psi}_j$ on $X \cap K_{j-1}$ so

$$\int_{X\cap K_{j-1}} (\psi_j - \varphi_j) \leq \int_{X\cap K_j} (\widetilde{\psi}_j - \varphi_j) < 2^{-j}.$$

Hence ψ_j verifies (4). Finally, we have by (5), $\rho_j = -j < \rho_{j-1} \leq \psi_{j-1}$ on K_{j-1} (and for $j = 1$, $\rho_1 = -1 < \psi_0 = 0$ on K_0). Since $\widetilde{\psi}_j < \psi_{j-1}$ on K_j we conclude that $\psi_j < \psi_{j-1}$ on K_{j-1}, so (6) is verified.

So we have constructed a sequence of continuous psh functions ψ_j on $\mathbb{C}^n$ verifying properties (4)-(6). Since $\bigcup_{j\geq 1} D_j = \mathbb{C}^n$, we have by (6) that the function

$$\psi(z) = \lim_{j\to\infty} \psi_j(z)$$

is well defined and psh on $\mathbb{C}^n$. As $\ldots < \psi_{j+2} < \psi_{j+1} < \psi_j$ on K_j, it follows that $\psi < \psi_j$ on K_j.

Suppose now that $z \in K_j \setminus D_{j-1}$, for some $j \geq 2$, so $m_{j-1} \leq u(z) \leq m_j$. By the above construction and property (5), we have

$$\widetilde{\psi}_j(z) < \psi_{j-1}(z) = \rho_{j-1}(z) \Longrightarrow \psi(z) < \psi_j(z) \leq \max\{\rho_{j-1}(z), \rho_j(z)\} \leq \gamma_j u(z).$$

Similarly, for $z \in K_1$ we have

$$\psi(z) < \psi_1(z) \leq \max\{\rho_0(z), \rho_1(z)\} \leq \gamma_1 \max\{u(z), 0\}.$$

Hence ψ satisfies the desired global upper estimates on $\mathbb{C}^n$.

Property (4) implies that $\psi(z) \geq \varphi(z)$ for every $z \in X$. Let K be a compact in $\mathbb{C}^n$ and Y be an irreducible component of X so that $\varphi|_Y \not\equiv -\infty$. By (4) we have that for all j sufficiently large

$$0 \leq \int_{Y\cap K} (\psi_j - \varphi) = \int_{Y\cap K} (\psi_j - \varphi_j) + \int_{Y\cap K} (\varphi_j - \varphi) \leq 2^{-j} + \int_{Y\cap K} (\varphi_j - \varphi).$$

Hence by dominated convergence, $\int_{Y\cap K} (\psi - \varphi) = 0$, which shows that $\psi = \varphi$ on Y.

Assume now that Y is an irreducible component of X so that $\varphi|_Y \equiv -\infty$. Then using (4) and the monotone convergence theorem we conclude that

$$\int_{Y\cap K} \psi = \lim_{j\to\infty} \int_{Y\cap K} \psi_j = \lim_{j\to\infty} \left(\int_{Y\cap K} (\psi_j - \varphi_j) + \int_{Y\cap K} \varphi_j \right) = -\infty,$$

so $\psi|_Y \equiv -\infty$. Therefore $\psi = \varphi$ on X, and the proof is finished. $\square$

Proof of Theorem A. We consider first the case $M = \mathbb{C}^n$. Fix $c > 1$. We define inductively a sequence $\{m_j\}$ with the following properties: $m_{-1} = m_0 = 0 < m_1$, $\{u < m_1\} \neq \emptyset$, and for $j \geq 1$, $m_j > m_{j-1}$ is chosen large enough so that

$$a_j = \frac{m_{j-1} - m_{j-2} + 1}{m_j - m_{j-1}} \leq \frac{\log c}{2^j} .$$

Since $\gamma_j \geq \gamma_0 = 1$ we have by (1),

$$\gamma_j(m_j - m_{j-1}) \leq \gamma_{j-1}(m_j - m_{j-2} + 1) \Longrightarrow \gamma_j \leq \gamma_{j-1}(1 + a_j).$$

Thus

$$\gamma_j < \gamma = \prod_{j=1}^{\infty}(1 + a_j) \ , \ \ \log \gamma \leq \sum_{j=1}^{\infty} a_j \leq \log c.$$

Let $\psi = \psi_c$ be the psh extension of φ provided by Proposition 1.3 for this sequence $\{m_j\}$. Then for every $z \in \mathbb{C}^n$ we have

$$\psi(z) < \gamma \max\{u(z), 0\} \leq c \max\{u(z), 0\}.$$

Assume now that M is a Stein manifold of dimension n. Then M can be properly embedded in $\mathbb{C}^{2n+1}$, hence we may assume that M is a complex submanifold of $\mathbb{C}^{2n+1}$ (see e.g. [Ho, Theorem 5.3.9]). Proposition 1.1 implies the existence of a continuous psh exhaustion function $\widetilde{u}$ on $\mathbb{C}^{2n+1}$ so that $\widetilde{u} = u$ on M. By what we already proved, given $c > 1$ there exists a psh function $\widetilde{\psi}$ on $\mathbb{C}^{2n+1}$ which extends φ and such that $\widetilde{\psi} < c \max\{\widetilde{u}, 0\}$ on $\mathbb{C}^{2n+1}$. We let $\psi = \widetilde{\psi}\,|_M$. □

We end this section by noting that some hypothesis on the growth of u is necessary in Theorem A. Indeed, suppose that X is a submanifold of $\mathbb{C}^n$ for which there exists a non-constant negative psh function φ on X. Then any psh extension of φ to $\mathbb{C}^n$ cannot be bounded above. However, by Theorem A, given any $\varepsilon > 0$ there exists a psh function $\psi = \psi_\varepsilon$ so that $\psi\,|_X = \varphi$ and $\psi(z) < \varepsilon \log^+ \|z\|$ on $\mathbb{C}^n$.

2. Extension of qpsh functions

Let V be a compact Kähler manifold equipped with a Kähler form ω. We let $PSH(V, \omega)$ denote the set of ω-psh functions on V. These are upper semicontinuous functions $\varphi \in L^1(V, [-\infty, +\infty))$ such that $\omega + dd^c\varphi \geq 0$, where $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$. We refer the reader to [GZ] for basic properties of ω-psh functions.

Let X be an analytic subvariety of V. Recall that an upper semicontinuous function $\varphi : X \to [-\infty, +\infty)$ is called $\omega\,|_X$-psh if $\varphi \not\equiv -\infty$ on X and if there exist an open cover $\{U_i\}_{i \in I}$ of X and psh functions φ_i, ρ_i defined on U_i, where ρ_i is smooth and $dd^c\rho_i = \omega$, so that $\rho_i + \varphi = \varphi_i$ holds on $X \cap U_i$, for every $i \in I$. Moreover, φ is called *strictly* $\omega\,|_X$-psh if it is $(1-\varepsilon)\omega\,|_X$-psh for some small $\varepsilon > 0$. The current $\omega\,|_X + dd^c\varphi$ is then called a Kähler current on X (see [EGZ1, section 5.2]). We denote by $PSH(X, \omega\,|_X)$, resp. $PSH^+(X, \omega\,|_X)$, the class of $\omega\,|_X$-psh, resp. strictly $\omega\,|_X$-psh functions on X.

Every ω-psh function φ on V yields, by restriction, an $\omega\,|_X$-psh function $\varphi\,|_X$ on X, as soon as $\varphi\,|_X \not\equiv -\infty$. The question we address here is whether this restriction operator is surjective. In other words, is there equality

$$PSH(X, \omega\,|_X) \stackrel{?}{=} PSH(V, \omega)\,|_X .$$

2.1. **The smooth case.** We start with the elementary observation that smooth strictly ω-psh functions can easily be extended.

Proposition 2.1. *Let V be a compact Kähler manifold equipped with a Kähler form ω, and let X be a complex submanifold of V. Then*

$$PSH^+(X,\omega\,|_X) \cap \mathcal{C}^\infty(X,\mathbb{R}) = \big(PSH^+(V,\omega) \cap \mathcal{C}^\infty(V,\mathbb{R})\big)\,|_X.$$

We include a proof for the convenience of the reader, although this is probably part of the "folklore" (see e.g. [Sch] for the case where ω is a Hodge form).

Proof. Let $\varphi \in \mathcal{C}^\infty(X,\mathbb{R})$ be such that $(1-\varepsilon)\omega\,|_X + dd^c\varphi \geq 0$ on X, for some $\varepsilon > 0$. We first choose $\tilde{\varphi}$ to be any smooth extension of φ to V. Consider

$$\psi := \tilde{\varphi} + A\chi\,\mathrm{dist}(\cdot,X)^2,$$

where χ is a test function supported in a small neighborhood of X and such that $\chi \equiv 1$ near X. Here *dist* is any Riemannian distance on V, for instance the distance associated to the Kähler metric ω. Then ψ is yet another smooth extension of φ to V, which now satisfies $(1-\varepsilon/2)\omega + dd^c\psi \geq 0$ near X, if A is chosen large enough.

The function $\log(\mathrm{dist}(\cdot,X)^2)$ is well defined and qpsh in a neighborhood of X. Let χ be a test function supported in this neighborhood so that $\chi \equiv 1$ near X. The function $u = \chi\log(\mathrm{dist}(\cdot,X)^2)$ is $N\omega$-psh on V for a large integer N. Moreover, $\exp(u)$ is smooth and $X = \{u = -\infty\}$. Replacing ω by $N\omega$, φ by $N\varphi$, and ψ by $N\psi$, we may assume that $N = 1$. Set now

$$\psi_C := \frac{1}{2}\log\left[e^{2\psi} + e^{u+C}\right].$$

This again is a smooth extension of φ, and a straightforward computation yields

$$dd^c\psi_C \geq \frac{2e^{2\psi}dd^c\psi + e^{u+C}dd^c u}{2(e^{2\psi}+e^{u+C})}\,.$$

Hence

$$\left(1-\frac{\varepsilon}{2}\right)\omega + dd^c\psi_C \geq \frac{2e^{2\psi}\left[\left(1-\frac{\varepsilon}{2}\right)\omega + dd^c\psi\right] + (1-\varepsilon)e^{u+C}\omega}{2(e^{2\psi}+e^{u+C})} \geq 0,$$

if C is chosen large enough. □

This proof breaks down when φ is singular and hence a different approach is needed. We consider in the next section the particular case when ω is a Hodge form.

2.2. **Proof of Theorem B.** We assume here that ω is a *Hodge form*, i.e. that the cohomology class $\{\omega\}$ belongs to $H^2(V,\mathbb{Z})$ (more precisely to the image of $H^2(V,\mathbb{Z})$ in $H^2(V,\mathbb{R})$ under the mapping induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$). We prove the following more precise version of Theorem B.

Theorem 2.2. *Let X be a subvariety of a projective manifold V equipped with a Hodge form ω. If $\varphi \in PSH(X,\omega\,|_X)$ then given any constant $a > 0$ there exists $\psi \in PSH(V,\omega)$ so that $\psi\,|_X = \varphi$ and $\max_V \psi < \max_X \varphi + a$.*

In the assumptions of Theorem 2.2 there exists a positive holomorphic line bundle L on V whose first Chern class $c_1(L)$ is represented by ω. By Kodaira's embedding theorem L is ample, hence for large k there exists an embedding $\pi : V \hookrightarrow \mathbb{P}^n$ such that $L^k = \pi^*\mathcal{O}(1)$.

Replacing ω by $k\omega$, φ by $k\varphi$, we can assume that $L = \mathcal{O}(1)$, V is an algebraic submanifold of the complex projective space $\mathbb{P}^n$, and $\omega = \omega_{FS}\,|_V$ is the Fubini-Study Kähler form. Hence X is an algebraic subvariety of $\mathbb{P}^n$, and Theorem 2.2 follows if we show that ω_{FS}-psh functions on X extend to ω_{FS}-psh functions on $\mathbb{P}^n$.

Therefore we assume in the sequel that $X \subset V = \mathbb{P}^n$ and ω is the Fubini-Study Kähler form on $\mathbb{P}^n$. Let $[z_0 : \ldots : z_n]$ denote the homogeneous coordinates. Without loss of generality, we may assume that they are chosen so that no coordinate hyperplane $\{z_j = 0\}$ contains any irreducible component of X.

Let

$$\theta(z) = \log \frac{\max\{|z_0|, \ldots, |z_n|\}}{\sqrt{|z_0|^2 + \ldots + |z_n|^2}}\,,\ z = [z_0 : \ldots : z_n] \in \mathbb{P}^n.$$

This is an ω-psh function and for all $z \in \mathbb{P}^n$,

$$-m \leq \theta(z) \leq 0\,,\ \text{ where } m = \log\sqrt{n+1}.$$

We start by noting that Theorem A yields special subextensions of ω-psh functions on X.

Lemma 2.3. *Let $\varepsilon \geq 0$ and u be a continuous $(1+\varepsilon)\omega$-psh function on $\mathbb{P}^n$ so that $u(z) \leq 0$ for all $z \in \mathbb{P}^n$. If $c > 1$ and φ is an ω-psh function on X so that $\varphi < u$, then there exists a $c\omega$-psh function ψ on $\mathbb{P}^n$ so that*

$$\frac{1}{c}\,\psi(z) \leq \frac{1}{1+\varepsilon}\,u(z),\ \forall z \in \mathbb{P}^n,$$

and

$$\psi(z) = \varphi(z) + (c-1)\theta(z) + (c-1)\min_{\zeta\in\mathbb{P}^n} u(\zeta),\ \forall z \in X.$$

Proof. Let

$$M = -\min_{\zeta\in\mathbb{P}^n} u(\zeta) \geq 0.$$

We work first in an affine chart $\{z_j = 1\} \equiv \mathbb{C}^n$. Let $X_j = X \cap \{z_j = 1\}$ and let $\rho_j \geq 0$ be the potential of ω in this chart with $\rho_j(0) = 0$. Then $\varphi + \rho_j$ is psh on X_j and since $u \leq 0$,

$$\varphi + \rho_j + M < u + \rho_j + M \leq \frac{1}{1+\varepsilon}\,u + \rho_j + M \text{ on } X_j.$$

Note that $(1+\varepsilon)^{-1}u + \rho_j + M \geq 0$ is a continuous psh exhaustion function on $\mathbb{C}^n$. Theorem A yields a psh function $\widetilde{\psi}$ on $\mathbb{C}^n$ so that

$$\widetilde{\psi} < \frac{c}{1+\varepsilon}\,u + c\rho_j + cM \text{ on } \mathbb{C}^n\ ,\ \ \widetilde{\psi} = \varphi + \rho_j + M \text{ on } X_j.$$

The function $\psi_j = \widetilde{\psi} - c\rho_j - cM$ extends uniquely to a $c\omega$-psh function on $\mathbb{P}^n$ which verifies

$$\psi_j \leq \frac{c}{1+\varepsilon}\,u \ \text{ on } \mathbb{P}^n.$$

Moreover on $X \cap \{z_j = 1\}$ we have

$$\psi_j = \varphi - (c-1)\rho_j - (c-1)M = \varphi + (c-1)\theta_j - (c-1)M,$$

where

$$\theta_j(z) = \log \frac{|z_j|}{\sqrt{|z_0|^2 + \ldots + |z_n|^2}} \,.$$

Hence $\psi_j = -\infty$ on $X \cap \{z_j = 0\}$.

We finally let $\psi = \max\{\psi_0, \ldots, \psi_n\}$. This is a $c\omega$-psh function on $\mathbb{P}^n$ which verifies the desired conclusions, since $\theta = \max\{\theta_0, \ldots, \theta_n\}$. □

Proof of Theorem 2.2. Fix $a > 0$. Replacing φ by $\varphi - \max_X \varphi - a$ we may assume that $\max_X \varphi = -a$. We will show that there exists a sequence of smooth ω-psh functions φ_j on $\mathbb{P}^n$ which decrease pointwise on $\mathbb{P}^n$ to a negative ω-psh function ψ so that $\psi = \varphi$ on X.

Let X' be the union of the irreducible components W of X so that $\varphi|_W \not\equiv -\infty$. We first construct by induction on $j \geq 1$ a sequence of numbers $\varepsilon_j \searrow 0$ and a sequence of negative smooth $(1+\varepsilon_j)\omega$-psh functions ψ_j on $\mathbb{P}^n$ so that for all $j \geq 2$

$$\frac{\psi_j}{1+\varepsilon_j} < \frac{\psi_{j-1}}{1+\varepsilon_{j-1}} \text{ on } \mathbb{P}^n \,, \;\; \psi_{j-1} > \varphi \text{ on } X \,, \;\; \int_{X'} (\psi_j - \varphi) < \frac{1}{j} \,, \;\; \int_W \psi_j < -j \,,$$

for every irreducible component W of X where $\varphi|_W \equiv -\infty$. Here the integrals are with respect to the area measure on each irreducible component X_j of X, i.e.

$$\int_X f := \sum_{X_j} \int_{X_j} f\, \omega^{\dim X_j}.$$

Let $\varepsilon_1 = 1$, $\psi_1 = 0$, and assume that ε_{j-1}, ψ_{j-1}, where $j \geq 2$, are constructed with the above properties. Since $\varphi < \psi_{j-1}|_X$ and the latter is continuous on the compact set X, we can find $\delta > 0$ so that $\varphi < \psi_{j-1} - \delta$ on X.

Let $c > 1$. By Lemma 2.3, there exists a $c\omega$-psh function ψ_c so that

$$\frac{\psi_c}{c} \leq \frac{\psi_{j-1} - \delta}{1+\varepsilon_{j-1}} \text{ on } \mathbb{P}^n \,, \;\; \psi_c = \varphi + (c-1)\theta - (c-1)M_{j-1} \text{ on } X,$$

where

$$M_{j-1} = \delta - \min_{\zeta \in \mathbb{P}^n} \psi_{j-1}(\zeta) \geq 0.$$

We can regularize ψ_c on $\mathbb{P}^n$: there exists a sequence of smooth $c\omega$-psh functions decreasing to ψ_c on $\mathbb{P}^n$. Therefore we can find a smooth $c\omega$-psh function ψ'_c on $\mathbb{P}^n$ so that

$$\frac{\psi'_c}{c} < \frac{\psi_{j-1} - \frac{\delta}{2}}{1+\varepsilon_{j-1}} \text{ on } \mathbb{P}^n, \;\; \psi'_c > \varphi + (c-1)\theta - (c-1)M_{j-1} \geq \varphi - (c-1)(m + M_{j-1}) \text{ on } X.$$

By dominated, resp. monotone convergence, we can in addition ensure that

$$\int_{X'} (\psi'_c - \varphi) \leq \int_{X'} (\psi'_c - \varphi - (c-1)\theta + (c-1)M_{j-1}) < c - 1,$$

$$\int_W \psi'_c < -j - (c-1)(m + M_{j-1})|W|,$$

for every irreducible component W of X where $\varphi|_W \equiv -\infty$. Here $|W|$ denotes the (projective) area of W.

Now let $\psi''_c = \psi'_c + (c-1)(m + M_{j-1})$. Then on $\mathbb{P}^n$ we have

$$\frac{\psi''_c}{c} < \frac{\psi_{j-1} - \frac{\delta}{2}}{1+\varepsilon_{j-1}} + \frac{(c-1)(m+M_{j-1})}{c} < \frac{\psi_{j-1}}{1+\varepsilon_{j-1}} - \frac{\delta}{4} + (c-1)(m+M_{j-1}).$$

Moreover, $\psi''_c > \varphi$ on X and

$$\begin{aligned} \int_{X'} (\psi''_c - \varphi) &= \int_{X'} (\psi'_c - \varphi) + (c-1)(m+M_{j-1})|X'| \\ &< (c-1)(1 + m|X'| + M_{j-1}|X'|) , \\ \int_W \psi''_c &= \int_W \psi'_c + (c-1)(m+M_{j-1})|W| < -j , \end{aligned}$$

for every irreducible component W of X where $\varphi|_W \equiv -\infty$.

We take $c = 1 + \varepsilon_j$ and $\psi_j = \psi''_c$, where $\varepsilon_j > 0$ is so that

$$\varepsilon_j < \varepsilon_{j-1}/2 , \ \ \varepsilon_j(m + M_{j-1}) < \frac{\delta}{4} , \ \ \varepsilon_j(1 + m|X'| + M_{j-1}|X'|) < \frac{1}{j} .$$

Then ε_j, ψ_j have the desired properties.

We conclude that $\varphi_j = (1+\varepsilon_j)^{-1}\psi_j$ is a decreasing sequence of smooth negative ω-psh function on $\mathbb{P}^n$, so that $\varphi_j > (1+\varepsilon_j)^{-1}\varphi > \varphi$ on X. Hence $\psi = \lim_{j\to\infty} \varphi_j$ is a negative ω-psh function on $\mathbb{P}^n$ and $\psi \geq \varphi$ on X. Note that

$$\int_{X'} (\varphi_j - \varphi) = \frac{1}{1+\varepsilon_j} \int_{X'} (\psi_j - \varphi) - \frac{\varepsilon_j}{1+\varepsilon_j} \int_{X'} \varphi < \frac{1}{j} - \frac{\varepsilon_j}{1+\varepsilon_j} \int_{X'} \varphi ,$$

$$\int_W \varphi_j = \frac{1}{1+\varepsilon_j} \int_W \psi_j < -\frac{j}{2} ,$$

for every irreducible component W of X where $\varphi|_W \equiv -\infty$. It follows that $\psi = \varphi$ on X and the proof of Theorem 2.2 is finished. □

3. Algebraic subvarieties of $\mathbb{C}^n$

If X is an analytic subvariety of $\mathbb{C}^n$ and γ is a positive number, we denote by $\mathcal{L}_\gamma(X)$ the *Lelong class* of psh functions φ on X which verify $\varphi(z) \leq \gamma \log^+ \|z\| + C$ for all $z \in X$, where C is a constant that depends on φ. We let $\mathcal{L}(X) = \mathcal{L}_1(X)$. By Theorem A, functions $\varphi \in \mathcal{L}(X)$ admit a psh extension in each class $\mathcal{L}_\gamma(\mathbb{C}^n)$, for every $\gamma > 1$. [1]

We assume in the sequel that X is an *algebraic* subvariety of $\mathbb{C}^n$ and address the question whether it is necessary to allow the arbitrarily small additional growth. More precisely, is it true that

$$\mathcal{L}(X) \stackrel{?}{=} \mathcal{L}(\mathbb{C}^n)\,|_X ,$$

i.e. is every psh function with logarithmic growth on X the restriction of a globally defined psh function with logarithmic growth? We will give a criterion for this to hold, but show that in general this is not the case.

[1] If X is algebraic this result is claimed in [BL, Proposition 3.3], but there is a gap in their proof.

3.1. **Extension preserving the Lelong class.** Consider the standard embedding

$$z \in \mathbb{C}^n \hookrightarrow [1:z] \in \mathbb{P}^n,$$

where $[t:z]$ denote the homogeneous coordinates on $\mathbb{P}^n$. Let ω be the Fubini-Study Kähler form and let

$$\rho(t,z) = \log \sqrt{|t|^2 + \|z\|^2}$$

be its logarithmically homogeneous potential on $\mathbb{C}^{n+1}$.

We denote by $\overline{X}$ the closure of X in $\mathbb{P}^n$, so $\overline{X}$ is an algebraic subvariety of $\mathbb{P}^n$. It is well known that the class $PSH(\mathbb{P}^n, \omega)$ is in one-to-one correspondence with the Lelong class $\mathcal{L}(\mathbb{C}^n)$ (see [GZ]). Let us look at the connection between ω-psh functions on $\overline{X}$ and the class $\mathcal{L}(X)$.

The mapping

$$F_X : PSH(\overline{X}, \omega\,|_{\overline{X}}) \longmapsto \mathcal{L}(X),\ (F_X\varphi)(z) = \rho(1,z) + \varphi([1:z]),$$

is well defined and injective. However, it is in general not surjective, as shown by Examples 3.2 and 3.3 that follow.

Conversely, a function $\eta \in \mathcal{L}(X)$ induces an upper semicontinuous function $\widetilde{\eta}$ on $\overline{X}$ defined in the obvious way:

$$\widetilde{\eta}([t:z]) = \begin{cases} \eta(z) - \rho(1,z), & \text{if } t = 1,\ z \in X, \\ \limsup\limits_{[1:\zeta]\to[0:z],\zeta\in X} (\eta(\zeta) - \rho(1,\zeta)), & \text{if } t = 0,\ [0:z] \in \overline{X} \setminus X. \end{cases}$$

The function $\widetilde{\eta}$ is in general only *weakly ω-psh on $\overline{X}$*, i.e. it is bounded above on $\overline{X}$ and it is $\omega\,|_{\overline{X}_r}$-psh on the set $\overline{X}_r$ of regular points of $\overline{X}$. This notion is in direct analogy to that of *weakly psh* function on an analytic variety (see [D2, section 1]). We do not pursue it any further here.

Note that $\eta \in F_X\left(PSH(\overline{X}, \omega\,|_{\overline{X}})\right)$ if and only if $\widetilde{\eta} \in PSH(\overline{X}, \omega\,|_{\overline{X}})$. The following simple characterization is a consequence of Theorem B.

Proposition 3.1. *Let $\eta \in \mathcal{L}(X)$. The following are equivalent:*

(i) There exists $\psi \in \mathcal{L}(\mathbb{C}^n)$ so that $\psi = \eta$ on X.

(ii) $\widetilde{\eta} \in PSH(\overline{X}, \omega\,|_{\overline{X}})$.

(iii) For every point $a \in \overline{X} \setminus X$ the following holds: if (X_j, a) are the irreducible components of the germ $(\overline{X}, a)$ then the value

$$\limsup_{X_j \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta))$$

is independent of j.

In particular, if the germs $(\overline{X}, a)$ are irreducible for all points $a \in \overline{X} \setminus X$ then $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)\,|_X$.

Proof. Assume that (i) holds. It follows that $\widetilde{\eta} = \varphi\,|_{\overline{X}}$, where

$$\varphi([t:z]) := \begin{cases} \psi(z) - \rho(1,z), & \text{if } t = 1, \\ \limsup_{[1:\zeta]\to[0:z]} (\psi(\zeta) - \rho(1,\zeta)), & \text{if } t = 0, \end{cases}$$

is an ω-psh function on $\mathbb{P}^n$. Hence $\widetilde{\eta} \in PSH(\overline{X}, \omega\,|_{\overline{X}})$.

Conversely, if (ii) holds then by Theorem B there exists an ω-psh function φ on $\mathbb{P}^n$ which extends $\widetilde{\eta}$. Hence $\psi(z) = \rho(1,z) + \varphi([1:z])$ is an extension of η and $\psi \in \mathcal{L}(\mathbb{C}^n)$.

The equivalence of (ii) and (iii) follows easily from [D2, Theorem 1.10]. $\square$

3.2. **Explicit examples.** In view of section 3.1, it is easy to construct examples of algebraic curves $X \subset \mathbb{C}^2$ and functions in $\mathcal{L}(X)$ which do not admit an extension in $\mathcal{L}(\mathbb{C}^2)$. We write $z = (x,y) \in \mathbb{C}^2$.

Example 3.2. *Let $X = \{y = 0\} \cup \{y = 1\} \subset \mathbb{C}^2$ and $\eta \in \mathcal{L}(X)$, where*

$$\eta(z) = \begin{cases} \rho(1,z), & \text{if } z = (x,0), \\ \rho(1,z) + 1, & \text{if } z = (x,1). \end{cases}$$

The function $\widetilde{\eta}$ is not ω-psh on $\overline{X} = \{y = 0\} \cup \{y = t\}$, hence η does not have an extension in $\mathcal{L}(\mathbb{C}^2)$. Indeed, the maximum principle is violated along $\{y = 0\}$ near the point $a = [0:1:0]$, since $\widetilde{\eta}([t:1:0]) = 0$ for $t \neq 0$, while $\widetilde{\eta}([t:1:t]) = 1$.

With a little more effort we can give an example as above where X is an irreducible curve. Let $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$.

Example 3.3. *Let $X \subset \mathbb{C}^2$ be the irreducible cubic with equation $xy = x^3 + 1$. Then*

$$\overline{X} = \{[t:x:y] \in \mathbb{P}^2 : xyt = x^3 + t^3\},\ \overline{X} = X \cup \{a\},\ a = [0:0:1].$$

The germ $(\overline{X}, a)$ has two irreducible components X_1, X_2, both are smooth at a, X_1 being tangent to the line $\{x = 0\}$, and X_2 to the line $\{t = 0\}$.

Note that in fact $X \subset \mathbb{C}^\star \times \mathbb{C}$ is the graph of the rational function $y = x^2 + x^{-1}$, $x \in \mathbb{C}^\star$. If $(x,y) \in X$ and $x \to 0$ then $(x,y) \to a$ along X_1, while as $x \to \infty$ then $(x,y) \to a$ along X_2. The function

$$u(x,y) = \max\{-\log|x|, 2\log|x| + 1\}$$

is psh in $\mathbb{C}^\star \times \mathbb{C}$. It is easy to check that $\eta := u|_X \in \mathcal{L}(X)$ and

$$\limsup_{X_1 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = 0\ ,\quad \limsup_{X_2 \ni [1:\zeta] \to a} (\eta(\zeta) - \rho(1,\zeta)) = 1.$$

Hence η does not admit an extension in $\mathcal{L}(\mathbb{C}^2)$.

We conclude this section with an example of a cubic X in $\mathbb{C}^2$ and a psh function on X of the form $\eta = \log|P|$, where P is a polynomial, so that η admits a "transcendental" extension with exactly the same growth, but small additional growth is necessary if we look for an "algebraic" extension.

Proposition 3.4. *Let $X = \{x = y^3\}$ and $\eta(x,y) = \log|1+y|$, so $\eta|_X \in \mathcal{L}_{1/3}(X)$.*

Given $k \geq 1$, there is a polynomial $Q_k(x,y)$ of degree $k+1$ so that $Q_k(y^3,y) = (y+1)^{3k}$. In particular, $\psi_k = \frac{1}{3k}\log|Q_k| \in \mathcal{L}_{(k+1)/3k}(\mathbb{C}^2)$ is an extension of $\eta|_X$.

There exists no polynomial $Q(x,y)$ of degree k so that $Q(y^3,y) = (y+1)^{3k}$. However, $\eta|_X$ has an extension in $\mathcal{L}_{1/3}(\mathbb{C}^2)$.

Proof. We construct Q_k by replacing y^3 by x in the polynomial

$$(y+1)^{3k} = \sum_{j=0}^{3k} \binom{3k}{j} y^j.$$

Since $j = 3[j/3] + r_j$, $r_j \in \{0, 1, 2\}$, it follows that

$$Q_k(x, y) = \sum_{j=0}^{3k} \binom{3k}{j} x^{[j/3]} y^{r_j} = 3kx^{k-1}y^2 + l.d.t.\,.$$

We now check that there is no polynomial $Q(x, y)$ of degree k so that $Q(y^3, y) = (y + 1)^{3k}$. Indeed, if $Q(x, y) = \sum_{j+l\leq k} c_{jl} x^j y^l$ then

$$Q(y^3, y) = c_{k0} y^{3k} + c_{k-1,1} y^{3k-2} + l.d.t.$$

does not contain the monomial y^{3k-1}.

Note that $\overline{X} = \{xt^2 = y^3\} = X \cup \{a\}$, where $a = [0 : 1 : 0]$, so the germ $(\overline{X}, a)$ is irreducible. Proposition 3.1 implies that $\eta\,|_X$ has an extension in $\mathcal{L}_{1/3}(\mathbb{C}^2)$. $\square$

We conclude with some remarks regarding our last example. If X is an algebraic subvariety of $\mathbb{C}^n$ and f is a holomorphic function on X, f is said to have polynomial growth if there is an integer $N(f)$ and a constant A so that

$$|f(z)| \leq A(1 + \|z\|)^{N(f)}, \ \ \forall\, z \in X.$$

Then it is well known that there exists a polynomial P of degree at most $N(f)+\varepsilon(X)$ so that $P\,|_X = f$, where $\varepsilon(X) > 0$ is a constant depending only on X (see e.g. [Bj] and references therein). However, if $\overline{X} \subset \mathbb{P}^N$ is irreducible at each of its points at infinity then by Proposition 3.1 the psh function $\eta = N(f)^{-1} \log |f| \in \mathcal{L}(X)$ has a psh extension in the Lelong class $\mathcal{L}(\mathbb{C}^n)$.

On the other hand, Demailly [D1] has shown that in the case of the transcendental curve $X = \{e^x + e^y = 1\}$ any holomorphic function f on X, of polynomial growth, has a polynomial extension of the same degree to $\mathbb{C}^n$. Hence it is natural to ask if for this curve one has that $\mathcal{L}(X) = \mathcal{L}(\mathbb{C}^n)\,|_X$.

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