Extension of the axiomatic analyticity domain of scattering amplitudes by unitarity-I

Alain J. Martin

Institutions: CERN

Published on: 21 Aug 1966 - Nuovo Cimento Della Societa Italiana Di Fisica A-nuclei Particles and Fields (Springer Berlin Heidelberg)

Topics: Scattering amplitude

EXTENSION OF THE AXIOMATIC ANALYTICITY DOMAIN OF SCATTERING AMPLITUDES BY UNITARITY - I

A. Martin
CERN - Geneva

ABSTRACT

It is shown that any scattering amplitude satisfying fixed transfer dispersion relations for $-t_0 < t \leqslant 0$ is in fact analytic in the topological product $|t| \leqslant R \times s$ in cut plane with cuts $s = (M_A+M_B)^2 + \lambda$, $s = (M_A-M_B)^2 - \nu - t$, λ, $\nu \geqslant 0$. R is some fixed number. For the pion-pion case, $R = 4\mu^2$ where μ is the pion mass. For pion-pion scattering the domain is extended further by using elastic unitarity. The region of validity of dispersion relations is a certain domain in the complex t plane, which contains in particular the real segment $t = -28\mu^2$ to $t = 4\mu^2$. The fixed energy sections of the amplitude for not too high energies contain part of the Mandelstam cuts. In particular in the elastic region the analyticity domain of the absorptive part contains part of the Mandelstam double spectral function. The domain thus obtained can be extended further both by analytic completion and by repeated use of the unitarity condition.

65/1857/5-TH.637
13 December 1965

This work is dedicated to V. Glaser whose courage and tenacity gave the necessary starting point.

I. INTRODUCTION

Soon after the discovery of dispersion relations for fixed transfer[1] ($-t_0 < t \leq 0$) it was realized by Mandelstam[2] that if one really wants to get physical consequences from the analyticity properties of scattering amplitudes (except of course merely testing dispersion relations, where you insert a quantity given from experiment and you obtain another physical quantity), one needs a larger analyticity domain. Mandelstam proposed that we shall call the Mandelstam domain, which so far has never been disproved in the lowest mass case (pion-pion scattering), nor proved to be correct even in perturbation theory. The merit of Mandelstam has been to postulate a domain which escapes criticism because it possesses the most obvious singularities induced by unitarity.

Later, it was realized that some of the physical consequences of Mandelstam representation could be obtained by postulating a smaller analyticity domain for the scattering amplitude[3]. Let me describe this domain. Let $F(s,t,u)$ be the scattering amplitude for $A+B \rightarrow A+B$ with

$$s+t+u = 2M_A^2 + 2M_B^2$$

s square of the c.m. energy for $s = (M_A+M_B)^2 + \lambda + i\varepsilon$, $\lambda > 0$, $t = -2k^2(1-\cos\theta)$ where k and θ are respectively the c.m. momentum and the c.m. scatteing angle. u, for $u = \mu + i\varepsilon + (M_A+M_B)^2$ for $\mu > 0$, $\varepsilon \geqslant 0$ is the square of the c.m. energy for reaction $A+\bar{B} \rightarrow A+\bar{B}$. (1)

Then it is postulated that $F(s,t,u)$ is analytic (except for possible fixed t poles) in the topological product

$|t| < 4\mu^2$, μ pion mass, $\times$ the s cut plane with cuts $s = (M_A+M_B)^2 + \lambda$, $\lambda \geqslant 0$, $s=(M_A-M_B)^2-t-\mu$ $\mu \geqslant 0$. (2)

This domain is not completely ad hoc. It contains somehow the postulate that the range of forces between elementary particles does not grow with energy and is also based on an analogy with potential scattering : any potential decreasing faster than $e^{-2\mu r}$ at large distances will admit such a domain of analyticity (with no left-hand cut of course) while only a very restricted class of potentials gives rise to an amplitude satisfying the Mandelstam representation.

What we want to prove in this paper is that in all cases where dispersion relations have been proved for $-t_0 < t \leqslant 0$ a domain of the type (2) can be obtained by combining the unitarity condition with the results of axiomatic field theory. The domain we derive is

$$|t| < R \quad \times \quad s \text{ in cut plane .}$$

So far we have only been able to prove that $R = 4\mu^2$, where μ is the pion mass, for the pion-pion amplitude. We think that it will be possible later to prove that it is generally true.

The pion-pion case is especially nice because then these results hold for any permutation of s, t, and u, giving rise therefore to a non-natural holomorphy domain which can be further enlarged. Also the use of the unitarity condition allows some further enlargement.

The essential unitarity ingredient of the whole proof is the following : let $A_s(s,\cos\theta)$ be the absorptive part of the amplitude in the s channel. Then

$$\left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\Big|_{\cos\theta=1} \geqslant 0$$

$$\left|\left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\right|_{-1\leqslant\cos\theta\leqslant 1} \leqslant \left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\Big|_{\cos\theta=1} \qquad (3)$$

The analyticity ingredients are the following :

1) existence of dispersion relations for $-t_M < t \leqslant 0$;

2) for any fixed physical energy s the amplitude and the absorptive part are analytic in the Lehmann ellipse [4];

3) from the results of Bros, Epstein and Glaser [5),6)] we know that in the neighbourhood of any point s_o, t_o, $-t_M < t_o \leqslant 0$, s_o outside the cuts, there is analyticity in both s and t in

$$|s - s_o| < \eta(s_o, t_o) \quad , \quad |t - t_o| < \eta(s_o, t_o)$$

A priori the size of this neighbourhood can vary with s_o, t_o.

Section II is devoted to a simple pedagogical case : the case when dispersion relations for $-t_o < t \leqslant 0$ have no left-hand cut. First, unsubtracted dispersion relations are assumed. Then subtractions are included.

In Section III the general case is presented, first without subtractions, then with subtractions.

Section IV deals with the pion-pion case. In Section V the analyticity region is extended by using unitarity in the elastic region for the pion-pion case.

Section VI presents a tentative survey of the physical consequences of the existence of the analyticity domain we obtained.

Section VIII contains concluding remarks and is especially concerned with the possibility of extending further the domain.

II. SIMPLIFIED CASE : NO LEFT-HAND CUT

We assume here that the scattering amplitude is analytic for $-t_M < t \leqslant 0$ in the cut s plane with a cut starting at $s = (M_A+M_B)^2$. This is essentially the case of potential scattering. Now, following the results of field theory, we assume that at any point s_o, t_o, $-t_M < t_o$ real $\leqslant 0$, s_o <u>in</u> the cut plane, there is a neighbourhood $|s-s_o| < \eta(s_o,t_o)$, $|t-t_o| < \eta(s_o,t_o)$ in which $F(s,t)$ is analytic in both s and t. A priori this neighbourhood can shrink as $s_o \to \infty$ for instance. What we shall manage to prove is that it does not.

First, let us restate property (3) :

$$\left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\Big|_{\cos\theta=1} \geqslant 0$$

$$\left|\left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\right|_{-1\leqslant\cos\theta\leqslant 1} \leqslant \left(\frac{d}{d\cos\theta}\right)^n A_s(s,\cos\theta)\Big|_{\cos\theta=1}$$

This property follows from the convergence of the partial wave expansion in the physical region,

$$A_s(s,\cos\theta) = \frac{\sqrt{s}}{k}\sum(2\ell+1)\,\mathrm{Im}\, f_\ell\, P_\ell(\cos\theta)$$

and also of any derivative of A, which is guaranteed for instance by this existence of the Lehmann ellipse [4], of the unitarity condition $\mathrm{Im}\, f_\ell(s) \geqslant 0$, and of the property of Legendre polynomials

$$\left|\left(\frac{d}{d\cos\theta}\right)^n P_\ell(\cos\theta)\right|_{-1\leqslant\cos\theta\leqslant +1} \leqslant \left(\frac{d}{d\cos\theta}\right)^n P_\ell(1) \quad . \qquad (4),$$

Now consider in the s cut plane a fixed point s_1 real $<(M_A+M_B)^2$. At this point $F(s_1,t)$ is analytic in some neighbourhood in t : $|t| < R$. Therefore the derivatives $(d/dt)^n F(s_1,0)$ exist and are bounded by the Cauchy inequalities :

$$\left|\left(\frac{d}{dt}\right)^n F(s_1,0)\right| \leqslant \frac{M\,n!}{R^n} \tag{5}$$

where M is the maximum of $F(s_1,t)$ on the circle $|t| = R$ (if F becomes infinite take any smaller R).

From now on we assume (provisionally) unsubtracted dispersion relations for fixed transfer $-t_0 < t \leqslant 0$. Then

$$F(s,t) = \frac{1}{\pi}\int \frac{A_s(s',t)\,ds'}{s'-s} \tag{6}$$

In particular

$$F(s_1,t) = \frac{1}{\pi}\int \frac{A_s(s',t)\,ds'}{s'-s_1} \tag{7}$$

Now, the first question to answer is this : can one, in calculating the successive derivatives of $F(s_1,t)$ at $t=0$ differentiate under the integral ?

Since the first derivative exists, we are free to define it as :

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \frac{F(s_1,0)-F(s_1,-\tau)}{\tau} = \frac{1}{\pi}\lim \int_{(M_A+M_B)^2}^{\infty} \frac{\frac{A_s(s',0)-A_s(s',-\tau)}{\tau}\,ds'}{s'-s_1} \tag{8}$$

Now for $s' > (M_A+M_B)^2 + \tau$ we can apply inequality (3) and we see that the integrand is positive. Hence :

$$\lim_{\tau \to 0} \frac{F(s_1,0)-F(s_1,-\tau)}{\tau} \geqslant \lim \frac{1}{\pi}\int_{(M_A+M_B)^2}^{x} \frac{\frac{A_s(s',0)-A_s(s',-\tau)}{\tau}}{s'-s_1}\, ds' \qquad (9)$$

and the right-hand integral is an integral over a compact. If the integral is taken in the Lebesgue sense, it has a limit for $\tau \to 0$ if the integrand has a limit [7] and is bounded by some function $F(s')$. This is the case because given x one can always manage to take τ small enough so that for all $(M_A+M_B)^2 \leqslant s' \leqslant x$. The segment $-\tau \leqslant t \leqslant 0$ lies entirely in the Lehmann ellipse for $A_s(s',t)$. The second condition is fulfilled if $A_s(s',t)$ is bounded inside the Lehmann ellipse by some arbitrary continuous function $B(s')$. So we get :

$$\frac{d}{dt} F(s_1,0) \geqslant \frac{1}{\pi}\int_{(M_A+M_B)^2}^{x} \frac{\frac{dA_s(s',0)}{dt}}{s'-s_1}\, ds' \qquad (10)$$

In Eq. (10) the right-hand integral is bounded independently of x, and $(dA_s(s',0))/dt \geqslant 0$. Therefore the integral converges for $x \to \infty$, and

$$\frac{d}{dt} F(s_1,0) \geqslant \frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \frac{\frac{dA_s(s',0)}{dt}\, ds'}{s'-s_1}$$

On the other hand we have

$$\int_{(M_A+M_B)^2+\varepsilon}^{\infty} \frac{\dfrac{A_s(s',0)-A_s(s',-\tau)}{\tau}}{s'-s_1}\,ds' = \int_{(M_A+M_B)^2+\varepsilon}^{\infty} \frac{\dfrac{d\,A_s(s',-\tau(s'))}{dt}}{s'-s_1}\,ds_1 \leqslant \int_{(M_A+M_B)^2+\varepsilon}^{\infty} \frac{\dfrac{dA_s(s',0)}{dt}}{s'-s_1}\,ds'$$

where $0 \leqslant \tau(s') \leqslant \tau$, τ small enough, so that $s', -\tau$ lies inside the physical region and where inequalities (5) have been again used. Hence

$$\frac{d}{dt} F(s_1,0) = \frac{1}{\pi} \int_{(M_A+M_B)^2}^{\infty} \frac{\dfrac{dA_s(s',0)}{dt}}{s'-s_1}\,ds' \qquad (11)$$

Now can we generalize this to any negative t, $-R \leqslant t < 0$? The answer is yes. Indeed

$$\frac{d}{dt} F(s_1,t) = \lim_{\tau\to 0} \frac{1}{\pi} \int_{(M_A+M_B)^2}^{X} \frac{\dfrac{A_s(s',t+\tau)-A_s(s',t)}{\tau}}{s'-s_1}\,ds'$$

$$+ \lim_{\tau\to 0} \frac{1}{\pi} \int_{X}^{\infty} \frac{A_s(s',t+\tau)-A_s(s',t)}{\tau(s'-s_1)}\,ds'$$

By property (3) the second integral can be bounded by

$$\frac{1}{\pi}\int_{X}^{\infty} \frac{\frac{dA_s(s',0)}{dt}}{s'-s_1}\, ds'$$

and can be made arbitrarily small by taking x big enough. The first integral is on a compact and converges to the integral of the limit. So for $-R < t \leq 0$

$$\frac{d}{dt} F(s_1,t) = \frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \frac{\frac{dA_s(s',t)}{dt}}{s'-s_1}\, ds' \qquad (12)$$

At this point it is not difficult to prove by iteration that Eqs. (11) and (12) can be generalized to the $(n+1)^{th}$ derivative. One defines :

$$\left(\frac{d}{dt}\right)^{n+1} F(s_1,0) = \lim_{\tau\to 0} \frac{1}{\pi}\int \frac{\dfrac{\left(\frac{d}{dt}\right)^n A_s(s',0) - \left(\frac{d}{dt}\right)^n A_s(s',-\tau)}{\tau}}{s'-s_1}\, ds'$$

The integral on the right is supposed to be meaningful, which is the case for $n = 0$ and $n = 1$. By using inequalities (3) applied to the n^{th} derivative, one proves in the very same way as for the first derivative that the limit can be taken under the integral. Then one extends this to negative t.

Hence we have

$$\left(\frac{d}{dt}\right)^{n} F(s_1,t) = \frac{1}{\pi}\int \frac{\left(\frac{d}{dt}\right)^n A_s(s',t)\, ds'}{s'-s_1} \qquad (13)$$

$$-R < t \leq 0$$

Now, by the Cauchy inequalities, we have :

$$\frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \frac{\left(\frac{d}{dt}\right)^n A_s(s',0)}{s'-s_1}\, ds' \leq \frac{M\, n!}{R^n}$$

We notice first of all that the function

$$\left(\frac{d}{dt}\right)^n F(s,0) = \frac{1}{\pi}\int \frac{\left(\frac{d}{dt}\right)^n A_s(s',0)}{s'-s}\, ds'$$

exists over some finite segment of the real s axis, because we can move s_1 without changing the argument. Now it can be continued to arbitrary complex s as long as the integral converges uniformly. $(d/dt)^n F(s,0)$ can also be continued to arbitrary complex s because of the existence of analyticity neighbourhoods. Both therefore coincide whenever the integral converges. Now the integral converges because

$$\left|\frac{\left(\frac{d}{dt}\right)^n A_s(s',0)}{s'-s}\right| \leq \frac{\left(\frac{d}{dt}\right)^n A_s(s',0)}{s'-s_1}\ \mu(s,s_1)$$

where

$$\mu(s,s_1) = \sup_{(M_A+M_B)^2 \leq s' < \infty} \left|\frac{s'-s_1}{s'-s}\right|$$

Notice that $\mu \to \infty$ like $1/|\mathrm{Im}\, s|$ for $\mathrm{Im}\, s \to 0$ $\mathrm{Re}\, s > (M_A+M_B)^2$. Hence we get :

$$\left|\left(\frac{d}{dt}\right)^n F(s,0)\right| < \mu(s,s_1)\ \frac{M\, n!}{R^n}$$

Therefore the series

$$\sum_0^{\infty} \frac{t^n}{n!}\left(\frac{d}{dt}\right)^n F(s,0)$$

converges for $|t| < R$. It defines an analytic function of t for fixed s. But now for any fixed t, $|t| < R$, it converges uniformly in any compact of the s plane in which $\mu(s_1, s)$ is bounded. Each term is analytic in s. It is therefore analytic in s. According to Hartog's theorem [8] it is analytic in s and t. Therefore the analyticity domain we get is the direct product of

$$|t| < R \quad \times \quad s \text{ outside the cut } (M_A+M_B)^2 + \lambda \qquad (\lambda \geq 0) \qquad (14)$$

This is the fundamental result that will be extended to the more realistic case in the next Section. It means essentially this : to each point of the s cut plane, we knew in advance that we could attach a neighbourhood in t, $|t| < \eta(s)$. But this neighbourhood could vary with s, in particular shrink to zero as $|s| \to \infty$. However, we have proved that this is not the case, that $\eta(s)$, irrespective of $|s|$ has a fixed lower limit $\eta(s) \geq R$.

Let us now deal with the problem of subtractions. Suppose we have one subtraction in dispersion relations for $-t_0 \leq t \leq 0$, then we start with two points s_1 and s_2 both real $< (M_A+M_B)^2$:

$$F(s_1,t) - F(s_2,t) = \frac{s_1 - s_2}{\pi} \int \frac{A_s(s',t)\, ds'}{(s'-s_1)(s'-s_2)} \qquad (15)$$

$F(s_1,t)$ and $F(s_2,t)$ are analytic in $|t| < R_1$ and $|t| < R_2$ respectively. Then take the smallest of the two. Along the same lines as before one can prove that

$$\left(\frac{d}{dt}\right)^n \left(F(s_1,0) - F(s_2,0)\right) = \frac{s_1 - s_2}{\pi} \int \frac{\left(\frac{d}{dt}\right)^n A_s(s',0)\, ds'}{(s'-s_1)(s'-s_2)} \leq \frac{M\, n!}{(R_{min})^n}$$

Then one proves that for any s outside the cut, the series :

$$\sum_{0}^{\infty} \frac{t^n}{n!} \left(\frac{d}{dt}\right)^n \left[F(s,0) - F(s_2,0) \right]$$

has a radius of convergence R_{min}. Therefore the same holds for the series

$$\sum \frac{t^n}{n!} \left(\frac{d}{dt}\right)^n F(s,0)$$

Any number of subtractions can be dealt with in the same way.

III. THE REALISTIC CASE

a) No subtractions

We use the standard variables s, t, and u defined in the introduction. We assume, as a consequence of field theory, that for $-t_0 < t \leq 0$

$$F(s,t) = \frac{1}{\pi} \int_{(M_A+M_B)^2}^{\infty} \frac{A_s(s',t)\,ds'}{s'-s} + \frac{1}{\pi} \int_{(M_A+M_B)^2}^{\infty} \frac{A_u(u',t)\,du'}{u' - \Sigma M^2 + t + s} \tag{16}$$

where $A_s(s',t)$ and $A_u(u',t)$ represent the absorptive parts in the s and u channel, respectively.

$F(s_1,t)$ where again s_1 is real and such that $(M_A-M_B)^2 < s_1 < (M_A+M_B)^2$ is supposed to be analytic in t in $|t| < R$, and so, as before, we get :

$$\left|\left(\frac{d}{dt}\right)^n F(s_1,0)\right| \leq \frac{M\, n!}{R^n} \tag{17}$$

The question is now to justify derivations under the integral sign to evaluate the n^{th} derivative. What causes all the trouble is that the second integral in (15) contains t in the denominator. Instead of working with $F(s_1,t)$ itself, it is convenient to work with an expression which is such that all the derivatives of the integrand at $t=0$ are positive. Such an expression is

$$\begin{aligned}\phi(s_1,t) &= \frac{F(s_1,t)}{s_1-(M_A-M_B)^2-t} \\ &= \frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \frac{A_s(s',t)\,ds'}{(s'-s_1)(s_1-(M_A-M_B)^2-t)} \\ &+ \frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \frac{A_u(u',t)\,du'}{\left[s_1-(M_A-M_B)^2\right]\left[u'-\Sigma M^2+s_1\right] -t\left[u'-(M_A+M_B)^2\right]-t^2} \\ &= \frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} A_s(s',t)\,\chi(s_1,t,s')\,ds' + \frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} A_u(u',t)\,\psi(s_1,t,u')\,du'\end{aligned} \tag{18}$$

Now we try to calculate the first derivative, which – we know – exists by taking the limit

$$\lim_{\tau\to 0} \frac{\phi(s_1,0)-\phi(s_1,-\tau)}{\tau} =$$

$$\lim \Bigg[\frac{1}{\pi}\int \frac{A_s(s',0)-A_s(s',-\tau)}{\tau}\,\chi(s_1,0,s')\,ds' + \frac{1}{\pi}\int A_s(s',-\tau)\,\frac{\chi(s_1,0,s')-\chi(s_1,-\tau,s')}{\tau}\,ds' + \frac{1}{\pi}\int \frac{A_u(u',0)-A_u(u',-\tau)}{\tau}\,\psi(s_1,0,u')\,du' + \frac{1}{\pi}\int A_u(u',-\tau)\,\frac{\psi(s_1,0,u')-\psi(s_1,-\tau,u')}{\tau}\,du' \Bigg] \tag{19}$$

The second and fourth integrals remain bounded as $\tau \to 0$. Indeed we have for $s' > (M_A+M_B)^2+\Sigma$, $|A_s(s',-\tau)| < A_s(s',0)$ and for $u' > (M_A+M_B)^2+\Sigma$ $|A_u(u',-\tau)| < A_u(u',0)$, and

$$\left.\begin{aligned} \frac{\chi(s_1,0,s')-\chi(s_1,-\tau,s')}{\tau} &< \frac{C}{s'-s_1} \\ \frac{\psi(s_1,0,u')-\psi(s_1,-\tau,u')}{\tau} &< \frac{C}{u'-u_1} \end{aligned}\right\} \tag{20}$$

for

$$|\tau| < s_1-(M_A-M_B)^2-\varepsilon$$

So the sum of these two integrals can be majorized by some constant times $\phi(s_1,0)$, plus the low energy contribution of the unphysical region which stays finite. The existence of the limit of these two integrals makes no problem because one can, using this majorization, show that one makes an arbitrarily small error uniform in τ by cutting them off at $s' = x$ and $u' = x$ and we are then back to the limit of an integral over a compact. Therefore

$$\lim \left\{ \frac{1}{\pi}\int \frac{A_s(s',0) - A_s(s',-\tau)}{\tau} \chi(s_1,0,s')\, ds' + \frac{1}{\pi}\int \frac{A_u(u',0) - A_u(u',-\tau)}{\tau} \psi(s_1,0,u')\, du' \right\}$$

exists since the derivative of $\phi(s_1,t)$ exists at $t = 0$. χ and ψ are positive, $A_s(s',0) - A_s(s',-\tau) > 0$ and $A_u(u',0) - A_u(u',-\tau) > 0$ for $s' > (M_A+M_B)^2 + \varepsilon$, $u' > (M_A+M_B)^2 + \varepsilon$. Therefore, carrying again the same argument we get first that the limit is less than

$$\frac{1}{\pi}\int_{(M_A+M_B)^2}^{x} \frac{dA_s}{dt}(s',0)\,\chi(s_1,0,s')\,ds' + \frac{1}{\pi}\int_{(M_A+M_B)^2}^{x} \frac{d}{dt}A_u(u',0)\,\psi(s_1,0,u')\,du'$$

and deduce the convergence of both integrals when $x \to \infty$. Then one proves that the limit cannot in fact exceed the integral, by using the Rolle theorem and inequalities (3). Hence we can take the derivative under the integral sign and get

$$\frac{d\phi(s_1,0)}{dt} = \frac{1}{\pi}\int \frac{dA_s(s',0)}{dt}\,\chi(s_1,0,s')\,ds' + \frac{1}{\pi}\int \frac{dA_u(u',0)}{dt}\,\psi(s_1,0,u')\,du' + \frac{1}{\pi}\int A_s(s',0)\,\frac{d}{dt}\chi(s_1,0,s')\,ds' + \frac{1}{\pi}\int A_u(u',0)\,\frac{d}{dt}\psi(s_1,0,u')\,du' \tag{21}$$

Then, following Section II, we can extend this to negative t, provided $-t_1 < t \leqslant 0$ <u>and</u> $s_1-(M_A-M_B)^2 < t \leqslant 0$. Now notice that (21) is a sum of four <u>positive</u> expressions. Since $(d\phi(s_1,0))/dt$ is bounded, each of them is separately bounded.

One can then start the iteration procedure, which we shall only outline because it is very boring : at the n^{th} step one knows that the integrals

$$\int \left(\frac{d}{dt}\right)^p A_s(s',0)\,\chi(s_1,0,s')\,ds' \qquad p \leqslant n$$

$$\int \left(\frac{d}{dt}\right)^p A_u(u',0)\,\psi(s_1,0,u')\,du'$$

converge and are bounded in terms of the bounds on

$$\phi(s_1,0),\ \frac{d}{dt}\phi(s_1,0),\ \ldots\ \left(\frac{d}{dt}\right)^n\phi(s_1,0)$$

On the other hand, one can prove that

$$\left|\left(\frac{d}{dt}\right)^p \chi(s_1, 0, s')\right| < \frac{C_p(s_1)}{s'-s_1}$$

$$\left|\left(\frac{d}{dt}\right)^p \psi(s_1, 0, u')\right| < \frac{D_p(s_1)}{u'-u_1}$$

and for $|\tau| < s_1-(M_A-M_B)^2-\varepsilon$

$$\frac{\left(\frac{d}{dt}\right)^p \chi(s_1, 0, s') - \left(\frac{d}{dt}\right)^p \chi(s_1, -\tau, s')}{\tau} < \frac{C'_p(s_1)}{s'-s_1}$$

and a similar statement for ψ .

Hence one can prove that

$$\int \left(\frac{d}{dt}\right)^p A_s(s', t) \left(\frac{d}{dt}\right)^{n-p+1} \chi(s_1, 0, s')\, ds' \qquad p \leq n$$

exists and can be considered as a certain limit. The same holds for A_u. Then from the existence of $(d/dt)^{n+1}\phi(s_1, 0)$ one shows that the limit

$$\lim_{\tau\to 0} \int \frac{\left(\frac{d}{dt}\right)^n A_s(s', 0) - \left(\frac{d}{dt}\right)^n A_s(s', -\tau)}{\tau} \chi(s_1, 0, s')\, ds'$$

can be taken under the integral (and similarly for A_u).

Finally we get

$$\left(\frac{d}{dt}\right)^n \phi(s_1,0)$$

$$= \sum_{p=0}^{n} \frac{n!}{p!(n-p)!}\,\frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \left(\frac{d}{dt}\right)^p A_s(s',0)\left(\frac{d}{dt}\right)^{n-p} \chi(s_1,0,s')\,ds'$$

$$+ \sum_{p=0}^{n} \frac{n!}{p!(n-p)!}\,\frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \left(\frac{d}{dt}\right)^p A_u(u',0)\left(\frac{d}{dt}\right)^{n-p} \psi(s_1,0,u')\,du' \qquad (22)$$

From the Cauchy inequalities we have

$$\left|\left(\frac{d}{dt}\right)^n \phi(s_1,0)\right| < \frac{M\,n!}{R^n}\;\frac{1}{s_1-(M_A-M_B)^2-R}$$

when R is taken smaller than $s_1-(M_A-M_B)^2$.

On the other hand, all terms in (22) are <u>positive</u> by the choice (18) and by the positivity properties. Hence we have :

$$\frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \left(\frac{d}{dt}\right)^p A_s(s',0)\,\frac{ds'}{s'-s_1} < \frac{M\,p!}{R^p}\;\frac{s_1-(M_A-M_B)^2}{s_1-(M_A-M_B)^2-R}$$

$$\frac{1}{\pi}\int_{(M_A+M_B)^2}^{\infty} \left(\frac{d}{dt}\right)^p A_u(u',0)\,\frac{du'}{u'-u_1} < \frac{M\,p!}{R^p}\;\frac{s_1-(M_A-M_B)^2}{s_1-(M_A-M_B)^2-R} \qquad (23)$$

with $u_1 = 2M_A^2 + 2M_B^2 - s_1$. Now consider the series :

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\int \frac{\left(\frac{d}{dt}\right)^n A_s(s',0)}{s'-s} ds' + \int \frac{\left(\frac{d}{dt}\right)^n A_u(u',0)}{u'-\Sigma M^2+s+t} du' \right] \quad (24)$$

each term of this series is an analytic function of s and t whenever

$$s \neq \lambda + (M_A+M_B)^2 \qquad \lambda \geq 0$$
$$s \neq -t + (M_A-M_B)^2 - \mu \qquad \mu \geq 0$$

The uniform convergence of the integrals is guaranteed by inequalities (23). Now each term of the series (24) is bounded according to (23) by :

$$\left[\sup_{(M_A+M_B)^2 < s' < \infty} \left| \frac{s'-s_1}{s'-s} \right| + \sup_{(M_A+M_B)^2 < u' < \infty} \left| \frac{u'-u_1}{u'-u} \right| \right] \frac{M\, n!}{R^n} \frac{s_1-(M_A-M_B)^2}{s_1-(M_A-M_B)^2-R}$$

Hence, if we take any compact region which does not contain the cuts $s > (M_A+M_B)^2$ and $u > (M_A+M_B)^2$ and lies inside $|t| < R-\varepsilon$, the series (24) converges absolutely and uniformly in all variables. It represents an analytic function of s and t which in the neighbourhood of $s = s_1$, $t = 0$ coincides with the scattering amplitude $F(s,t)$. Hence the scattering amplitude is analytic in the topological product

$$|t| < R \quad (\text{provided } R < s_1-(M_A-M_B)^2)$$

$$\times \quad s \text{ in the cut plane, with cuts } \quad s = (M_A+M_B)^2 + \lambda \quad (\lambda \geq 0) \quad (25)$$
$$s = (M_A-M_B)^2 - t - \mu \ (\mu \geq 0) \ .$$

b) Subtractions

First of all, there are at most two subtractions according to a recent paper by the author [9], then

$$F(s_1,t) = F(s_2,t) + (s_1-s_2)\frac{d}{ds}F(s_2,t)$$
$$+ (s_1-s_2)^2 \int \frac{A_s(s',t)\,ds'}{(s'-s_1)(s'-s_2)^2}$$
$$+ (s_1-s_2)^2 \int \frac{A_u(u',t)\,du'}{(u'-\Sigma M^2+s_1+t)(u'-\Sigma M^2+s_2+t)^2}$$

Then one works with the function

$$\phi(s_1,s_2,t) = \frac{F(s_1,t)-F(s_2,t)-(s_1-s_2)\frac{d}{ds}F(s_2,t)}{\left(s_1-(M_A-M_B)^2-t\right)\left(s_2-(M_A-M_B)^2-t\right)^2}$$

which is represented by an integral whose integrand has all its derivatives with respect to t positive at $t=0$.

Then one follows the same procedure as for the unsubtracted case :

1) show that one can differentiate under the integral at $t=0$;

2) find bounds on the successive derivatives;

3) sum back the expansion when s_1 is replaced by s at an arbitrary place in the cut plane.

What comes out is this : if $F(s_1,t)$, $F(s_2,t)$ and $(dF(s_2,t))/ds$ are analytic in $|t| < R$, $R < s_1-(M_A-M_B)^2$, $R < s_2-(M_A-M_B)^2$ then $F(s,t)$ is analytic in the product of $|t| < R$ and the s cut plane. One more result comes out : in the majorization of the series expansion of $F(s,t)$ one gets essentially :

$$|F(s,t)| < |F(s_2,t)| + |s-s_2| \left| \frac{d}{ds} F(s_2,t) \right|$$

$$+ |s-s_2|^2 \left[\sup \left| \frac{s_1-s'}{s-s'} \right| + \sup \left| \frac{u_1-u'}{u-u'} \right| \right] M(\varepsilon)$$

$$\text{for} \quad |t| < R-\varepsilon .$$

So for $|t| < R-\varepsilon$ and $\varepsilon < \mathrm{Arg}\, s < \pi - \varepsilon$ we see that $|F(s,t)|$ is bounded by $|s|^2$ as $s \to \infty$, which means that the <u>number of subtractions is conserved</u> when we go from $t = 0$ to any $|t| < R$. This result had already been obtained by Jin and the author [3], postulating in advance the analyticity domain.

Among the consequences which follow from this result, let us just quote the Froissart bound [3),10)] for the forward scattering amplitude : $|F(s,t=0)| < C s \log^2 s$. We shall come back to this in Section V.

IV. <u>THE PION-PION SCATTERING AMPLITUDE</u>

First of all, we shall specifically consider the $\pi^0 \pi^0 \to \pi^0 \pi^0$ scattering amplitude. Whether π^0 is a member of an isospin triplet or a pseudoscalar singlet is irrelevant for what we shall obtain in this Section. The question is : how big is R where $|t| < R$ is the analyticity domain obtained in the last Section.

We know, from axiomatic theory, that fixed t dispersion relations certainly hold for $-4\mu^2 < t \leqslant 0$ (in fact the lower limit is much further away). Therefore all the points in the triangle

$$t < 0 \qquad s < 4\mu^2 \qquad u < 4\mu^2$$

lie <u>inside</u> the domain of analyticity of the $\pi^0\pi^0$ scattering amplitude according to the results of Glaser, Epstein and Bros [6]. The $\pi^0\pi^0 \to \pi^0\pi^0$ is symmetric in s,t,u, and therefore we have similarly the domains $s<0$, $t<4\mu^2$, $u<4\mu^2$ and $u<0$, $t<4\mu^2$, $s<4\mu^2$. To each of these points is attached a neighbourhood in s and t in which $F(s,t)$ is analytic. Now, from the results of the previous Section, we know that if for $s=s_1$ we have an analyticity domain in $|t|<R$ we again have the same analyticity domain for any s, $s_1<s<4\mu^2$. Hence for fixed s, $4\mu^2-R+\mathcal{E}<s<4\mu^2$, we have for each $-s<t_0<4\mu^2$ a neighbourhood $|t-t_0|<\eta(s,t_0)$ of analyticity in t. Hence we have analyticity on the compact region $|t-t_0|<\frac{1}{2}\eta(s,t_0)$. Therefore by the Heine-Borel-Lebesgue theorem we can cover the interval $-s+\mathcal{E}\leq t\leq 4\mu^2-\mathcal{E}$ by a finite number of such intervals. Hence the fixed s amplitude is analytic in t in a region which contains the real interval $-s+\mathcal{E}\leq t\leq 4\mu^2-\mathcal{E}$.

Now we have seen in the unsubtracted case that $(F(s,t))/(s-t)$, $0<s<4\mu^2$ has all its derivatives positive at $t=0$. Therefore its power series expansion has a singularity at $t=R$. Thus, since the strip $-s+\mathcal{E}<t<s-\mathcal{E}$ is free from singularity, we deduce that $R=s$. Now s is as close as one wishes to $4\mu^2$. So the analyticity domain is in fact $|t|<4\mu^2$.

If there are subtractions (there are at most two), we can write, using crossing symmetry

$$F(s_1,t)-F(s_2,t)=\frac{(s_1-s_2)(s_1-4+s_2+t)}{\pi}\int\frac{A_s(s',t)\,[2s'-4+t]\;ds'}{(s'-s_1)(s'-s_2)(s'-4+s_1+t)(s'-4+s_2+t)}$$

then $$\frac{F(s_1,t)-F(s_2,t)}{(s_1-t)(s_2-t)}$$

has all its derivatives with respect to t positive at $t = 0$. We take

$$4\mu^2 > s_1 > s_2 > 4\mu^2 - R + \varepsilon$$

Hence

$$\frac{F(s_1, t) - F(s_2, t)}{(s_1 - t)\,(s_2 - t)}$$

is analytic in $|t| < s_2$.

Carrying through the technique developed in Section III we find that $F(s,t) - F(s_2,t)$ is analytic in $|t| < s_2$ for any s apart from the cuts $s = 4\mu^2 + \lambda$, $s = -t - \mu$.

The question is to know now whether this holds separately for $F(s,t)$ and $F(s_2,t)$. We can choose $R/2 < s < R$, in which case fixed s dispersion tells us that $F(s,t)$ is analytic in $|t| < 4\mu^2$ minus the cut $t = -R/2 - \lambda$, $\lambda > 0$. Therefore $F(s_2,t)$ is analytic in $|t| < s_2$ minus the cut $t = -R/2 - \lambda$. However, we know that for t negative $-s_2 < t < 0$ it is analytic. Therefore $F(s_2,t)$ is analytic in $|t| < s_2$ and $F(s,t)$ is analytic in $|t| < s_2$ minus the cuts. Now we can take s_2 as close as we wish to $4\mu^2$. Therefore the final domain of analyticity is

$$|t| < 4\mu^2 \quad \times \quad \text{cut plane with cuts} \quad \begin{array}{ll} s = 4\mu^2 + \lambda & (\lambda \geq 0) \\ s = -t - \mu & (\mu \geq 0) \end{array} \qquad (26)$$

plus circular permutations on s, t, u. Such a domain is not a natural domain of holomorphy. However, we shall not treat this problem in the present paper.

Among the consequences of the existence of this domain, apart from the asymptotic bounds which we already mentioned, let us indicate the existence of absolute limits on the pion-pion scattering amplitude in the triangle

$$s < 4\mu^2 \quad t < 4\mu^2 \quad u < 4\mu^2$$

and more generally at any point inside the domain (26) and hence also inside its analytic completion.

The next problem is to extend this result to scattering amplitudes involving charged pions. We notice first that the $\pi^+\pi^+ \to \pi^+\pi^+$ is connected by crossing to the $\pi^+\pi^- \to \pi^+\pi^-$ amplitude. This amplitude does not possess the invariance under permutations of s, t, u. However, all the amplitudes under consideration are elastic. By this I mean that the particles in the initial state coincide with the particles in the final state. This is enough to carry through the whole argument and hence we get analyticity in $|t| < 4\mu^2$ × cut plane and circular permutations.

A more difficult case is that of the $\pi^+\pi^0 \to \pi^+\pi^0$, $\pi^-\pi^0 \to \pi^0\pi^-$, $\pi^0\pi^0 \to \pi^+\pi^-$ amplitudes. If one accepts to use isospin invariance, each of these amplitudes can be shown to be a linear combination of the $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude, the $\pi^+\pi^- \to \pi^+\pi^-$ amplitude and the $\pi^+\pi^+ \to \pi^+\pi^+$ amplitude. In this way one finds essentially the same analyticity domain. In fact, the assumption that $\pi^+\pi^-\pi^0$ form an isospin triplet is probably not necessary. It is probably sufficient to assume that $\pi^+\pi^-\pi^0$ have equal masses, $\pi^+\pi^-$ mutual charge conjugate and π^0 self-conjugate, but we do not want to develop this point here.

Now let us proceed to enlarge the domain (26) by combining it with the original result of Lehmann on the size of the ellipse of analyticity of the absorptive part.

We know that the absorptive part is analytic in the circle $|t| < 4\mu^2$ at any energy. However, it is also analytic in the Lehmann ellipse which has foci at $t = 4-s$ and $t = 0$, and expandable in Legendre polynomials of the angle. The Legendre polynomial expansion, having positive coefficients <u>must</u> have a singularity at the extreme right of the largest ellipse of convergence. Hence $A_s(s,t)$ having no singularities for $0 \leqslant t < 4\mu^2$ must be analytic in the ellipse :

$$E_1(s): \quad \text{foci} \quad t=0 \quad t=4-s \qquad \text{extremities} \quad t=4 \quad t=-s \tag{27}$$

(from here on we take $\mu = 1$).

On the other hand Lehmann [4] tells us that the absorptive part is analytic in the ellipse :

$$E_2(s): \quad \text{foci} \quad t=0 \quad t=4-s \qquad \text{extremities} \quad t=\frac{256}{s} \quad t=4-s-\frac{256}{s} \tag{28}$$

so for $s < 64$, $E_2(s)$ contains $E_1(s)$, for $s > 64$, $E_1(s)$ contains $E_2(s)$.

Now, the right foci being held at $t = 0$, one easily sees that $E_1(s)$ for $s > 64$ contains $E_1(64)$. So by taking the intersection of all large Lehmann ellipses for $4 \leqslant s \leqslant 64$ we shall get a common domain of analyticity of the absorptive part $A_s(s,t)$ in t for all energies $4 \leqslant s < \infty$. We know already that this domain contains the real segment

$$-28 < t < +4 \quad ,$$

the right-hand inequality becoming an equality at $s = 64$ and the left-hand inequality becoming an equality at $s = 16$. Now for $4 < s < 16$ $E_2(s)$ always contains $E_2(16)$ because as s decreases :

i) the right extremity moves to the right,

ii) the left extremity moves to the left,

iii) the left focus gets closer to the fixed focus $t = 0$.

We only have to study the behaviour for $16 \leqslant s \leqslant 64$. This is a rather difficult job. One finds the following : the border of the intersection of the elliptic discs E_2 for $16 \leqslant s \leqslant 64$ is made partly of ellipse $E_2(64)$ for $\mathrm{Re}\, t > -20$, part of ellipse $E_2(16)$ and a piece of the envelope of the ellipses defined by

$$\cos\left[Ay\left(t+s-4\right)\right] = 1 - 2 \times \frac{256}{s^2} \tag{29}$$

t on the ellipse $E_2(s)$. The domain $\mathcal{D}(t)$ is represented in Fig. 1.

Now for any t inside $\mathcal{D}$ we can write a dispersion relation. We do not have to worry about the subtraction problem because $\mathcal{D}$ is contained inside $E_1(\infty)$ and since $A_s(s,t)$ is maximum for $t \to 4\mu^2$ and is then bounded <u>in the mean</u> by s^2 we have at most two subtractions. For any given s the analyticity domain will contain at least $\mathcal{D}(t)$ minus the cut $t = -s - \lambda$, $\lambda > 0$ and by crossing symmetry $\mathcal{D}(u)$ (where t and u have been interchanged) minus the cut $u = -s - \mu$, $\mu > 0$.

Let us now look at the situation for s real > 4. There are three cases :

1) $4 < s < 28$. The scattering amplitude is analytic, with respect to t in the union of the domains $\mathcal{D}(t)$ and $\mathcal{D}(u)$ minus the cuts $t \geq 4$, $u \geq 4$ which <u>penetrate</u> both into the domain;

2) $28 \leq s \leq 60$. The scattering amplitude is analytic inside $\mathcal{D}(t)$ and $\mathcal{D}(u)$, which still overlap. There are no cuts inside the domain;

3) $s > 60$. The scattering amplitude is analytic in two disconnected regions $\mathcal{D}(t)$ and $\mathcal{D}(u)$. Unitarity, however, will make it possible to connect them. Indeed the absorptive part $A_s(s,t)$ is analytic in the ellipse $E_1(s)$. So the partial waves are such that

$$\overline{\lim_{\ell=\infty}} \left(\mathrm{Im} f_\ell\right)^{1/\ell} \leq \frac{1}{X + \sqrt{X^2 - 1}}$$

where $X = 1 + 8/s-4$.

Hence, by unitarity :

$$\overline{\lim}\, |f_\ell|^{1/\ell} \leq \frac{1}{\sqrt{X+\sqrt{X^2-1}}} = \frac{1}{x+\sqrt{x^2-1}}$$

where

$$x = \sqrt{1+\frac{4\mu^2}{s-4}}$$

Thus the full amplitude is analytic in an ellipse with foci at the extremities of the physical region and a semi-major axis (measured in $\cos\theta$) which is just x. Asymptotically the extremities of this ellipse lie at $t=1$, $t=-s+4-1$. Notice that the existence of this ellipse leads to some improvement even below $s=60$.

Situations 1), 2), and 3) are represented in Fig. 2.

For complex s we have similar situations : if $|s|$ is very large, $\mathcal{D}(u)$ and $\mathcal{D}(t)$ do not overlap but now we have not anymore the unitarity argument to connect them.

If $|s|$ is not too large, the domains $\mathcal{D}(u)$ and $\mathcal{D}(t)$ overlap, and this is of special interest. Indeed, if they overlap that means that there is a complex path connecting the points $t=0$ and $t=4-s$. That means therefore that we can give an analytic meaning to the integral

$$f_\ell(s) = \frac{1}{4-s}\int_{t=0}^{t=4-s} F(s,t)\, P_\ell\left(1+\frac{2t}{s-4}\right) dt$$

So whenever $\mathcal{D}(t)$ and $\mathcal{D}(u)$ overlap, we are inside the analyticity domain of partial waves. A sufficient condition for overlap is $t=(4-s)/2$ inside $\mathcal{D}(t)$. This gives a rather large analyticity domain which extends on the real s axis up to $s=60\,\mu^2$, i.e., well above the ρ mass and the nucleon mass.

The analyticity domain of partial waves can also be extended to the left of $\mathrm{Re}\, s = 0$. Indeed, when s lies inside $\mathcal{D}(s)$ we have dispersion relations in t and except for real s the line $t = 0$, $t = 4-s$ does not intersect the cuts. Figure 3 represents the minimum analyticity domain of partial waves. It is seen that the near left-hand cut of the partial waves from $s = -28\,\mu^2$ to $s = 0$ is included in the domain.

At this point, let us say that it is completely clear that the domain we have obtained is <u>not</u> a natural domain of holomorphy. This would be the case if we did not take into account crossing symmetry, but with crossing symmetry it is not. However, the analytic completion of the domain does <u>not</u> yield Mandlestam representation. This can be shown by enlarging the domain to make the problem soluble. If there is any hope of proving the Mandelstam representation, it is by using more of unitarity than just condition (3).

V. <u>THE ROLE OF UNITARITY IN THE ELASTIC REGION</u>

Here, for simplicity, we shall restrict ourselves to neutral pseudo-scalar pions with isospin zero. However, all the results we get can be extended to real pions. For $4\mu^2 \leqslant s \leqslant 16\,\mu^2$ the full scattering amplitude is analytic in the union of $\mathcal{D}(t)$ and $\mathcal{D}(u)$ minus cuts starting at $t = 4\mu^2$ and $u = 4\mu^2$. Figure 4 illustrates the situation for $s = 8\mu^2$. From this analyticity domain we want to deduce the analyticity domain of the absorptive part. If the partial wave expansion of the amplitude reads :

$$F = \frac{\sqrt{s}}{k} \sum (2\ell+1)\, f_\ell(s)\, P_\ell(\cos\theta)$$

the partial wave expansion of the absorptive part is :

$$A = \frac{\sqrt{s}}{k} \sum (2\ell+1)\, |f_\ell(s)|^2\, P_\ell(\cos\theta)$$

Once the analyticity domain of F is known, one can compute exactly the minimum analyticity domain of A through the prescription that the possible singularities of A are given by

$$\cos\theta_i \cos\theta_j^* + \sqrt{(\cos\theta_i)^2 - 1}\ \sqrt{(\cos\theta_j^*)^2 - 1} \qquad (30)$$

where $\cos\theta_i$ and $\cos\theta_j$ are taken to be at the location of all singularities of F. However, here we shall simplify somewhat the problem, because the domain of analyticity of F is rather complicated.

We notice that for $s < 16$ a subdomain of analyticity of F is given by the union of :

1) the ellipse with foci $t = 0$, $t = -60$ and extremities $t = 4$, $t = -64$, with the restriction $\mathrm{Re}\, t > -20$;
2) the ellipse with foci $t = -s+4$, $t = 60-s+4$, extremities $t = -s$, $t = 60-s+8$, with the restriction $\mathrm{Re}\, t < 24-s$, apart from the cuts $t \geq 4$, $t \leq -s$.

This domain (<u>forgetting the cuts</u>) contains the ellipse with foci at the extremities of the physical region $t = 0$, $t = 4-s$ and which goes through the intersection of the two previously mentioned ellipses. Its right extremity lies at

$$t_m = \frac{132-s}{17} \qquad \cos\theta_m = 1 + \frac{2t_m}{s-4} \qquad (31)$$

What matters really is that for $s < 16$, t_m is larger than 4, which means that this ellipse contains parts of the cuts $t > 4$, $t < -s$. For instance, for $s = 8$ we get $t_m = 7.3$ [Fig. 4]. Then one applies prescription (30). One finds the following : if one combines a singularity on the ellipse with a singularity on the cut $\cos\theta_0 < \cos\theta < \cos\theta_m$, where $\cos\theta_0 = 1 + (8/s-4)$, one gets singularities on or outside the ellipse with semi-major axis :

$$\cos\theta_M = \cos\theta_0 \cos\theta_m + \sqrt{(\cos\theta_0)^2 - 1} \quad \sqrt{(\cos\theta_m)^2 - 1} \qquad (32)$$

If one combines singularities on the cuts between themselves, one finds cuts

$$\cos\theta > 2(\cos\theta_0)^2 - 1 \qquad \cos\theta < -\left[2\cos\theta_0^2 - 1\right] \qquad (33)$$

These cuts are just the Mandelstam cuts for the absorptive part. Equation (33) defines the border of the double spectral function in the Mandelstam representation. From Eqs. (32) and (33) we see that a large part of the Mandelstam cut, from

$$t = 16 + \frac{64}{s-4} \quad \text{to} \quad t = 2\left[\cos\theta_M - 1\right]\frac{s-4}{4}$$

is contained inside the ellipse (32). For instance, for $s = 8$ the part of the cut which lies inside the analyticity domain extends (Fig.5) from $t = 32$ to $t = 51.8$. This is already a quite large region. It could be extended further by a more careful use of the analyticity domain of F. However, what prevents us from going very far away is that the analyticity domain of F does not extend far enough in $|\mathrm{Im}\, t|$.

At this stage it is too early to interpret the discontinuity of $A_s(s,t)$ across the cuts as the double discontinuity of $F(s,t)$. To do this we should know that $F(s,t)$ is analytic for $s = s_0 \pm i\varepsilon$, $t = t_0 \pm i\varepsilon$, s_0, t_0 on the cut. It does not look impossible to establish this by replacing s and t by new variables depending non-linearly on s and t. This would allow to investigate the region of the s, t plane where one has analyticity of the absorptive part. This will be studied later. All the results obtained for the neutral case can be extended to the actual pion-pion case by treating simultaneously the various amplitudes.

VI. SOME CONSEQUENCES

We want to present a tentative list of rigorous results which follow from the existence of the analyticity domain and of the limitations which follow on the polynomial growth at infinity :

a) Asymptotic bounds

The Froissart bound [3),10)] is of course valid

$$|F(s, \cos\theta = 1)| < C s (\log s)^2$$

not only for elastic amplitudes, but also for any two-body → two-body amplitude, via unitarity. Similarly

$$|F(s, \cos\theta)| < \frac{s^{3/4} (\log s)^{3/2}}{(\sin\theta)^{1/2}}$$

A correlated result [11)] is

$$\sigma_{elastic} > \frac{(\sigma_{total})^2}{(\log s)^2}$$

For t inside $|t| < R$ or $\mathcal{D}(t)$ for the pion-pion case, we have

$$\lim_{s \to \infty} \frac{|F(s,t)|}{|s|^2} = 0$$

for $\varepsilon < \mathrm{Arg}\, s < \pi - \varepsilon$.

b) Partial waves at threshold

The D waves and higher waves [3),12)] of the $\pi^0 \pi^0 \to \pi^0 \pi^0$ amplitude have positive scattering lengths, i.e.,

$$\lim_{s \to 4} \frac{f_\ell(s)}{(s-4)^\ell} > 0$$

The same result holds for the $T = 0$ partial waves.

c) Absolute limits on the pion-pion amplitude [13)]

The $\pi^0\pi^0$ amplitude $F(s,t,u)$ for $s < 4\mu^2$ $t < 4\mu^2$ $u < 4\mu^2$ can be shown to be bounded numerically. In particular, the value $F(4/3,4/3,4/3)$ is bounded. At present one already knows that λ, as defined by Chew and Mandelstam, is limited by $-4 < \lambda < +25$. In the physical region one has also absolute limits on averages of the amplitude or of the cross-section.

This shows that the basic principles of field theory, once the information that the $\pi\pi$ system has no bound state is fed in, induce a limitation on the strength of the $\pi\pi$ interaction and, we hope, of any interaction.

d) Determination of coupling constants by extrapolation

The results we have obtained for pseudoscalar pions are also valid for a scalar theory. Clearly, the pole term $t = \mu^2$ will be inside the analyticity domain, irrespective of the energy. We expect that the same situation will occur for more realistic cases.

Now, concerning approximate calculations, the situation is rather good. Notice that the ρ mass corresponds to $s \simeq 28$, and that the beginning of the left-hand cut due to the exchange of the ρ corresponds to something like $s = -24$. These two points lie well inside the analyticity domain of the partial waves. Also, let us mention that the part of the cuts which lies inside the analyticity domain of $F(s,t)$ in t for $4 < s < 28$ can be calculated from the partial wave expansion of the absorptive part in the crossed channels.

VII. CONCLUDING REMARKS

The results we have obtained so far seem to indicate that the Mandelstam representation is a good approximation for the description of low energy pion-pion scattering. Undoubtedly, these results are not the optimum results. Analytic completion will certainly lead to a rather big enlargement of the domain [14]. In particular, the analyticity domain for partial waves is certainly bigger than what we have found. However, analytic completion alone will not enable us to get new points on the distinguished boundary of Mandelstam representation. The only hope to do this is to use again unitarity, especially in the elastic region. It is very unlikely, however, due to our ignorance of the many-particle structure of the unitarity condition, that Mandelstam representation could be proved exactly by this method. However, if the positivity properties of the absorptive part are used <u>off</u> the mass shell, as has been pointed out to me by Glaser, there is a non-zero hope to prove the Mandelstam representation.

Concerning the starting point, let me say this. We start from results which follow from axiomatic field theory. However, it seems that these results can be obtained from a more general basis which does not assume the existence of local fields [15]. So it seems that if what has been obtained here turned out to be wrong, it would seriously endanger the principle of causality.

Finally, let me say that we have not used yet all the existing results of field theory, in particular the recent results of Bros, Epstein and Glaser [6] on the analytic structure of the scattering amplitude for fixed, arbitrarily large, negative transfers. This also should be included in the programme.

ACKNOWLEDGEMENTS

I am extremely grateful to Professor V. Glaser who kept me informed of the latest developments of field theory and for many illuminating discussions. I thank Dr. J.D. Bessis for useful information and discussions.

R E F E R E N C E S

1) M.L. Goldberger, Phys.Rev. 99, 979 (1955);

K. Symanzik, Phys.Rev. 100, 743 (1957);

N.N. Bogoliubov, B. Medvedev and M. Polivanov, "Voprossy teorii dispersionnykh sootnosheni", Moscow (1958);

H.J. Bremermann, R. Oehme and J.G. Taylor, Phys.Rev. 109, 2178 (1958).

2) S. Mandelstam, Phys.Rev. 112, 1344 (1958); Phys.Rev. 115, 1741 (1959); Phys.Rev. 115, 1752 (1959).

3) A. Martin, Phys.Rev. 129, 1432 (1963);

Y.S. Jin and A. Martin, Phys.Rev. 135, B1375 (1964).

4) H. Lehmann, Nuovo Cimento 10, 579 (1958).

5) J. Bros, H. Epstein and V. Glaser, Nuovo Cimento 31, 1265 (1964).

6) J. Bros, H. Epstein and V. Glaser, to appear in Communications in Mathematical Physics.

7) G. Valiron, "Théorie des functions", p.137 (2nd Ed., Masson, Paris 1942).

8) S. Bochner and W.T. Martin, "Several complex variables", pp.137-139, Princeton University Press (1948).

9) A. Martin, to be published in Nuovo Cimento, CERN preprint 65/1699/5-TH.630.

10) M. Froissart, Phys.Rev. 123, 1053 (1961).

11) A. Martin, Nuovo Cimento 29, 993 (1963).

12) A. Martin, CERN preprint 65/1363/5-TH.606.

13) A. Martin, Preprint Stanford University (1964) and Proceedings of the 1965 Trieste International Seminar on High Energies and Elementary Particles, to be published.

14) For instance, S. Mandelstam, in Nuovo Cimento 15, 658 (1960), gets a domain of analyticity partly outside and partly inside our domain.

15) V. Glaser, Private communication.

FIGURE CAPTIONS

Figure 1 : The domain $\mathcal{D}(t)$ of validity of dispersion relations.

Figure 2 : Sections for s real of the analyticity domain.

Figure 3 : The minimum analyticity domain for partial waves.

Figure 4 : The analyticity domain of the amplitude for s below 16.

Figure 5 : The analyticity domain of the absorptive part for s below 16.

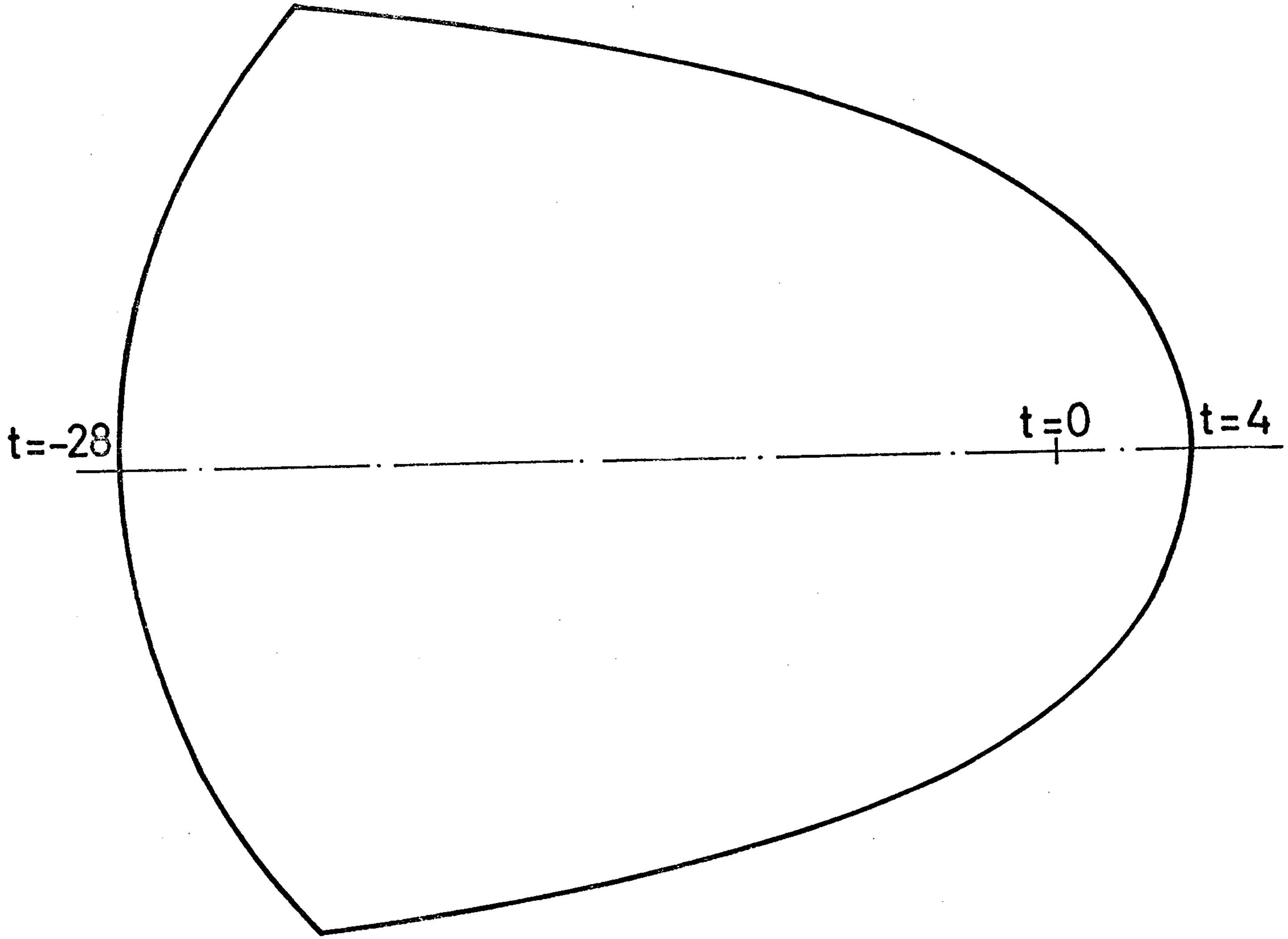
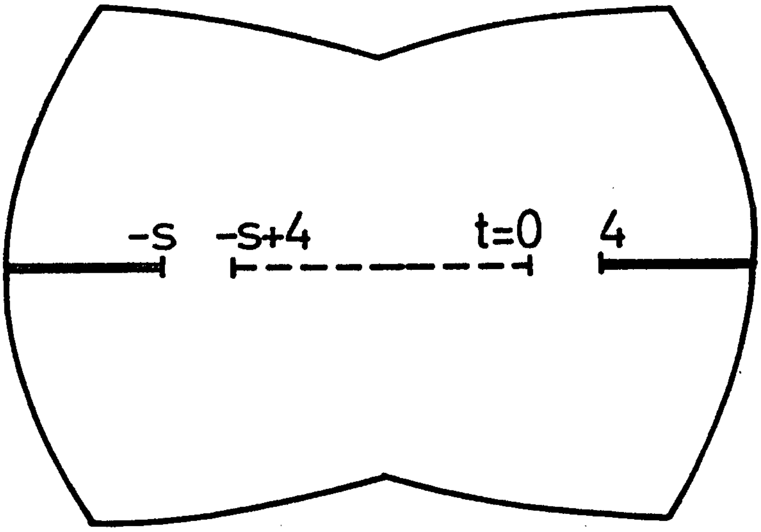
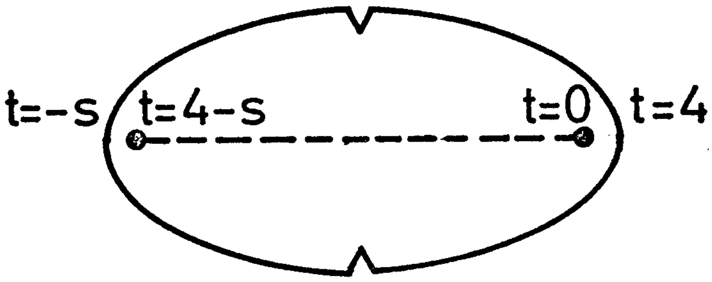


FIG. 1

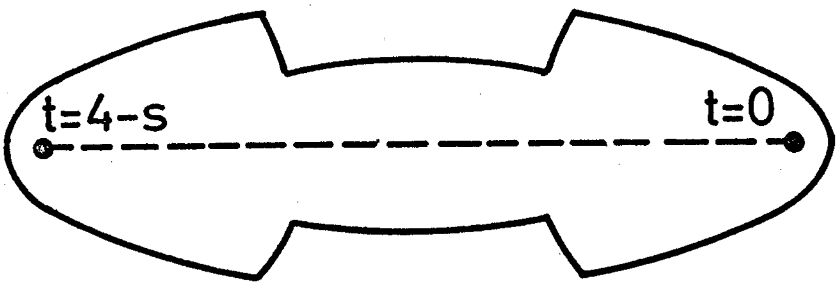
P/66/4



a) $4 < s < 28$ $(s=20)$



b) $28 < s < 60$ $(s = 52)$



c) $60 < s$ $(s=84)$

FIG.2

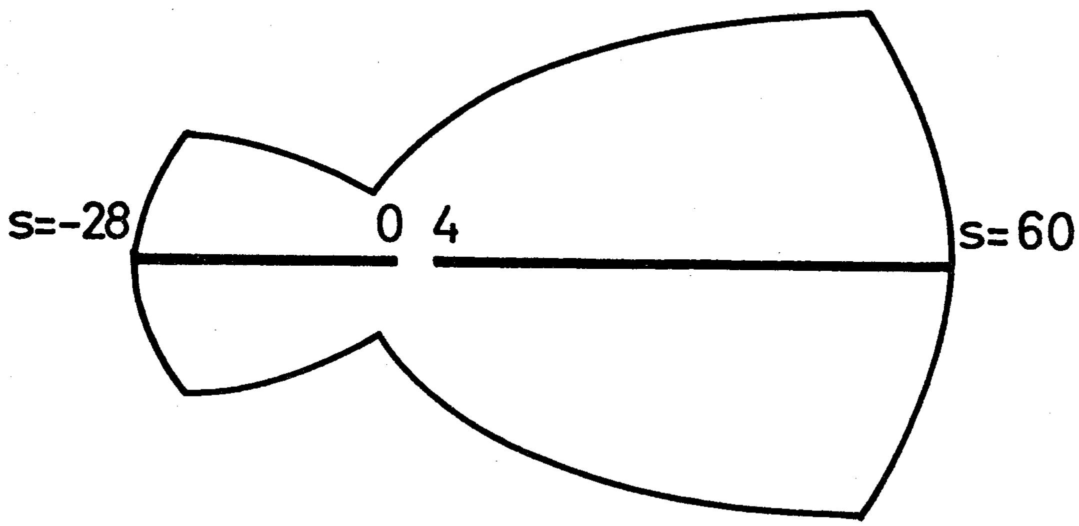


FIG. 3

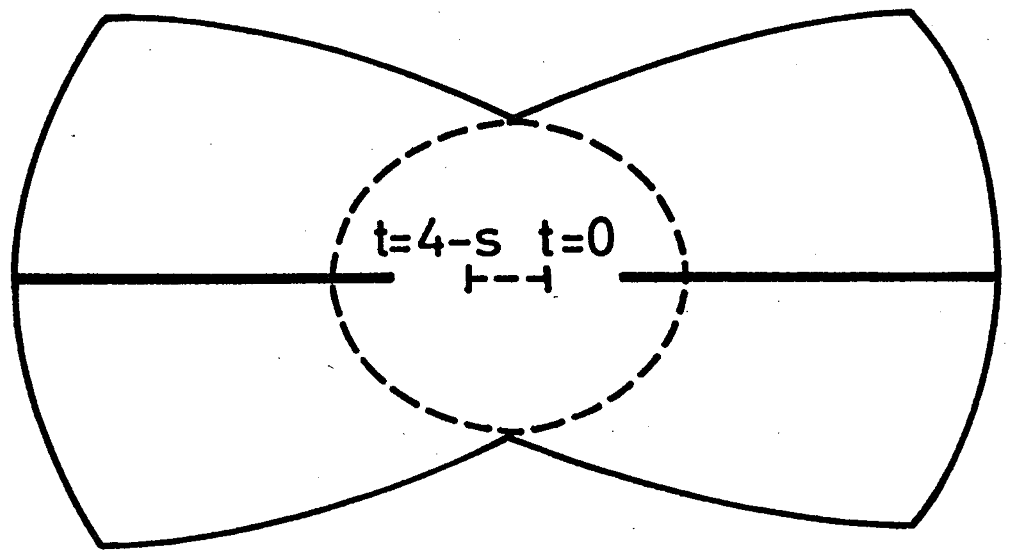


FIG. 4

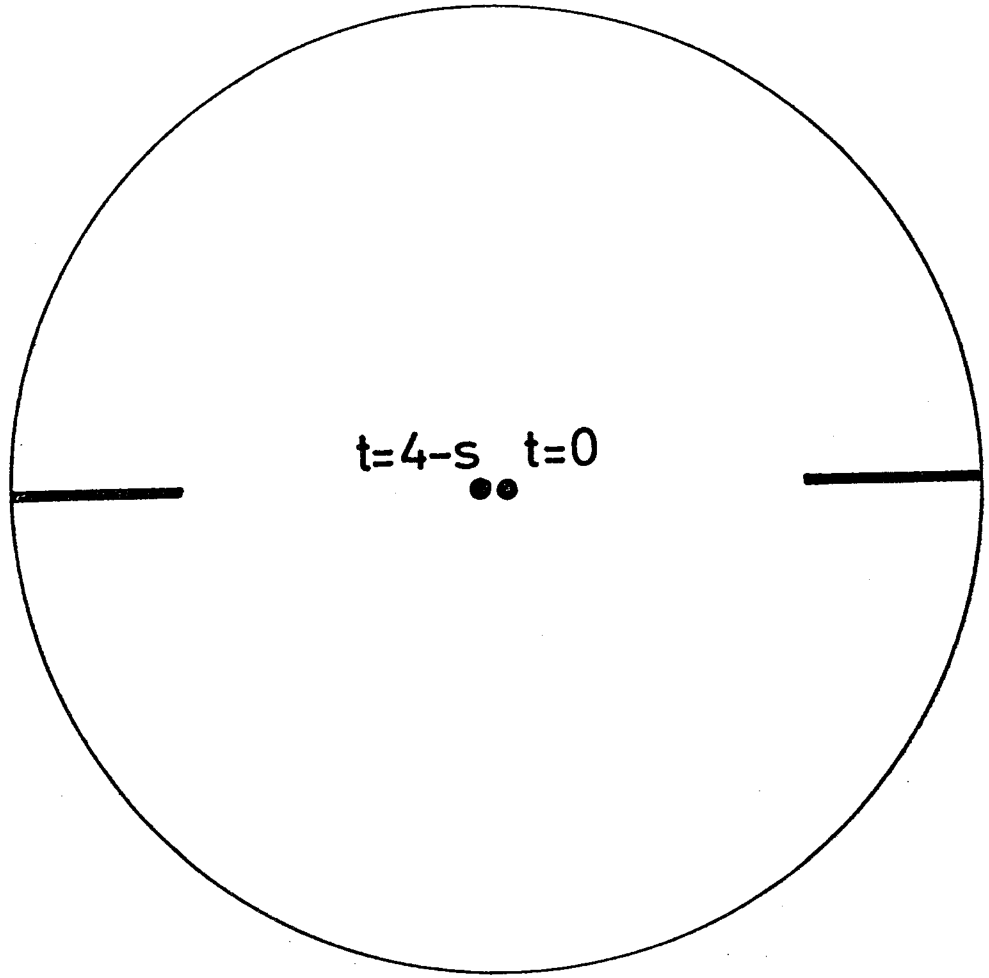


FIG. 5