

EXTENSION OF THE CAYLEY-HAMILTON THEOREM TO CONTINUOUS-TIME LINEAR SYSTEMS WITH DELAYS

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The classical Cayley-Hamilton theorem is extended to continuous-time linear systems with delays. The matrices $A_0, A_1, \dots, A_h \in \mathbb{R}^{n \times n}$ of the system with h delays $\dot{x}(t) = A_0x(t) + \sum_{i=1}^h A_i x(t - hi) + Bu(t)$ satisfy $nh + 1$ algebraic matrix equations with coefficients of the characteristic polynomial $p(s, w) = \det [I_n s - A_0 - A_1 w - \dots - A_h w^h]$, $w = e^{-hs}$.

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1. Introduction

The classical Cayley-Hamilton theorem (Gantmacher, 1974; Lancaster, 1969) says that every square matrix satisfies its own characteristic equation. The Cayley-Hamilton theorem was extended to rectangular matrices (Kaczorek, 1995c), block matrices (Kaczorek, 1995b; Victoria, 1982), pairs of commuting matrices (Chang and Chan, 1992; Lewis, 1982; 1986; Mertizios and Christodoulous, 1986), pairs of block matrices (Kaczorek, 1998), and standard and singular two-dimensional linear (2-D) systems (Kaczorek, 1992/93, 1994; 1995a; Smart and Barnett, 1989; Theodoru, 1989).

The Cayley-Hamilton theorem and its generalizations have been used in control systems, electrical circuits, systems with delays, singular systems, 2-D linear systems, etc., (Gałkowski, 1996; Kaczorek, 1992/93; 1995c; Lancaster, 1969).

In (Kaczorek, 2005), the Cayley-Hamilton theorem was extended to n -dimensional (n -D) real polynomial matrices. An extension of the Cayley-Hamilton theorem to discrete-time linear systems with delay was given in (Busłowicz and Kaczorek, 2004).

In this note the classical Cayley-Hamilton theorem is extended to continuous-time linear systems with delays. It will be shown that matrices of the n -th order system with h delays satisfy $(nh + 1)$ algebraic equations.

2. Main Result

Let $\mathbb{R}^{n \times m}$ be the set of $n \times m$ real matrices and $\mathbb{R}^n := \mathbb{R}^{n \times 1}$. Consider the continuous-time linear system with h delays described by the equation

$$\begin{aligned} \dot{x}(t) = & A_0x(t) + A_1x(t-d) + \dots \\ & + A_hx(t-hd) + Bu(t), \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are respectively the state and input vectors, $A_k \in \mathbb{R}^{n \times n}$, $k = 0, 1, \dots, h$, $B \in \mathbb{R}^{n \times m}$, and d is the delay.

The characteristic polynomial of (1) has the form

$$\begin{aligned} p(s, w) = & \det [I_n s - A_0 - A_1 w - \dots - A_h w^h] \\ = & s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0, \end{aligned} \quad (2)$$

where $w = e^{-ds}$ and

$$\begin{aligned} a_{n-1} = a_{n-1}(w) = & a_{n-1,h} w^h + \dots + a_{n-1,1} w \\ & + a_{n-1,0} \\ a_{n-2} = a_{n-2}(w) = & a_{n-2,2h} w^{2h} + \dots + a_{n-2,1} w \\ & + a_{n-2,0} \\ & \vdots \\ a_0 = a_0(w) = & a_{0,nh} w^{nh} + \dots + a_{0,1} w + a_{0,0}. \end{aligned} \quad (3)$$

The coefficients a_{kj} , $k = 0, 1, \dots, n - 1$ and $j = 0, 1, \dots, nh$, depend on the entries of matrices A_0, A_1, \dots, A_h .

Let

$$\begin{aligned} & [I_n s - (A_0 + A_1 w + \dots + A_h w^h)]^{-1} \\ &= I_n s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-3} + \dots, \end{aligned} \quad (4)$$

where

$$\Phi_i = \Phi_i(w) = (A(w))^i \quad \text{for } i = 1, 2, \dots$$

and

$$A(w) = A_0 + A_1 w + \dots + A_h w^h. \quad (5)$$

Using the well-known relation $\text{Adj } M = M^{-1} \det M$ between the adjoint matrix $\text{Adj } M$, the inverse matrix M^{-1} and its determinant $\det M$, taken in conjunction with (2) and (4), we can write

$$\begin{aligned} \text{Adj } A(w) &= [I_n s^{-1} + \Phi_1 s^{-2} + \Phi_2 s^{-3} + \dots] \\ &\times (s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0). \end{aligned} \quad (6)$$

Note the adjoint matrix $\text{Adj } A(w)$ is a polynomial matrix in non-negative powers of s . Thus equating the coefficients at the same powers of s^{-1} of (6) yields

$$\begin{aligned} & \Phi_{n+k-1} + a_{n-1} \Phi_{n+k-2} + \dots + a_1 \Phi_k + a_0 \Phi_{k-1} = 0 \\ & \text{for } k = 1, 2, \dots \quad (\Phi_0 = I_n). \end{aligned} \quad (7)$$

From (7) for $k = 1$ we have (cf. the Cayley-Hamilton theorem):

$$\Phi_n + a_{n-1} \Phi_{n-1} + \dots + a_1 \Phi_1 + a_0 I_n = 0 \quad (8)$$

with coefficients a_k depending on w .

From (5) we have

$$\begin{aligned} \Phi_i &= (A_0 + A_1 w + \dots + A_h w^h)^i \\ &= A_0^i + (A_0 A_1 A_0^{i-2} + A_1 A_0^{i-1} \\ &+ \dots + A_0^{i-2} A_1 A_0) w \\ &+ (A_0 A_2 A_0^{i-2} + A_1^2 A_0^{i-2} + A_2 A_0^{i-1} \\ &+ A_1 A_0 A_1 A_0^{i-3} + A_0 A_1 A_0 A_1^{i-3} \\ &+ A_1 A_0^2 A_1^{i-3} + \dots + A_0^2 A_1^{i-2} A_0^2) w^2 \\ &+ \dots + A_h^i w^{hi} \quad \text{for } i = 1, 2, \dots \end{aligned} \quad (9)$$

The substitution of (9) and (3) into (8) yields

$$\begin{aligned} & A_0^n + (A_0 A_1 A_0^{n-2} + A_1 A_0^{n-1} + \dots + A_0^{n-2} A_1 A_0) w \\ &+ (A_0 A_2 A_0^{n-2} + A_1^2 A_0^{n-2} + \dots + A_0^2 A_1^{n-2} A_0^2) w^2 \\ &+ \dots + A_h^n w^{nh} \\ &+ (a_{n-1, h} w^h + \dots + a_{n-1, 1} w + a_{n-1, 0}) \\ &\times [A_0^{n-1} + (A_0 A_1 A_0^{n-3} + A_1 A_0^{n-2} \\ &+ \dots + A_0^{n-3} A_1 A_0) w \\ &+ (A_0 A_2 A_0^{n-3} + A_1^2 A_0^{n-3} + \dots + A_0^2 A_1^{n-3} A_0^2) w^2 \\ &+ \dots + A_h^{n-1} w^{(n-1)h} + \dots + \\ &+ (a_{1, n} w^{(h-1)} + \dots + a_{1, 1} w + a_{1, 0})] \\ &\times [A_0 + A_1 w + \dots + A_h w^h] \\ &+ (a_{0, nh} w^{nh} + \dots + a_{0, 1} w + a_{0, 0}) I_n = 0. \end{aligned} \quad (10)$$

From (10) we have the following $nh + 1$ equations:

$$\begin{aligned} & A_0^n + a_{n-1, 0} A_0^{n-1} + \dots + a_{1, 0} A_0 + a_{0, 0} I_n = 0, \\ & A_0 A_1 A_0^{n-2} + A_1 A_0^{n-1} + \dots + A_0^{n-2} A_1 A_0 \\ &+ a_{n-1, 0} (A_0 A_1 A_0^{n-3} + A_1 A_0^{n-2} \\ &+ \dots + A_0^{n-3} A_1 A_0) + a_{n-1, 1} A_0^{n-1} \\ &+ \dots + a_{1, 0} A_1 + a_{1, 1} A_0 + a_{0, 1} I_n = 0, \\ & \vdots \\ & A_h^n + a_{0, nh} I_n = 0. \end{aligned} \quad (11)$$

Therefore, the following theorem has been proved:

Theorem 1. Matrices $A_k \in \mathbb{R}^{n \times n}$, $k = 0, 1, \dots, h$ of the continuous-time linear system with h delays (I) satisfy the $nh + 1$ algebraic matrix equations (II).

Note that the first equation of (11) expresses the Cayley-Hamilton theorem for the system (1) without delay ($h = 0$).

Example 1. Consider the system with

$$\begin{aligned} A_0 &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \quad (12)$$

In this case the characteristic polynomial (2) has the form

$$\begin{aligned}
 p(s, w) &= \det [I_2 s - A_0 - A_1 w - A_2 w^2] \\
 &= \begin{vmatrix} s - 1 - w & -2 - w^2 \\ -1 - w^2 & s - w \end{vmatrix} \\
 &= s^2 - (2w + 1)s - (w^4 + 2w^2 - w + 2)
 \end{aligned}$$

$$\begin{aligned}
 a_1 &= a_1(w) = a_{11}w + a_{10} = -2w - 1, \\
 a_0 &= a_0(w) = a_{04}w^4 + a_{03}w^3 + a_{02}w^2 + a_{01}w \\
 &\quad + a_{00} = -w^4 - 2w^2 + w - 2.
 \end{aligned}$$

Taking into account the fact that $n = h = 2$, from (11) we obtain the following equations:

$$\begin{aligned}
 &A_0^2 + a_{10}A_0 + a_{00}I_2 \\
 &= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^2 - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 &A_0A_1 + A_1A_0 + a_{11}A_0 + a_{10}A_1 + a_{01}I_2 \\
 &= 2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &\quad + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 &A_1^2 + A_0A_2 + A_2A_0 + a_{10}A_2 + a_{11}A_1 + a_{02}I_2 \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \\
 &\quad - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
 \end{aligned}$$

$$\begin{aligned}
 &A_1A_2 + A_2A_1 + a_{11}A_2 + a_{03}I_2 \\
 &= 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$A_2^2 + a_{04}I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. Concluding Remarks

The classical Cayley-Hamilton theorem was extended to continuous-time linear systems with delays. It was shown that the matrices $A_k \in \mathbb{R}^{n \times n}$, $k = 0, 1, \dots, h$ of the system (1) satisfy the $nh + 1$ algebraic equations (11) with coefficients a_{kj} , $k = 0, 1, \dots, n - 1$ and $j = 0, 1, \dots, nh$, of the characteristic polynomial (2). The proposed extension can be generalized to rectangular matrices and block matrices (Kaczorek, 1995b; Kaczorek, 1995c; Victoria, 1982). An open problem is the extension of the theorem to singular continuous-time linear systems with delays.

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