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Abstract

For mass transfer to the wall of a tube with fully developed, laminar flow and a constant wall concentration, Lévêque's solution is used as a basis for an expansion valid near the entrance to the mass-transfer section.

For mass transfer to the wall of a tube with fully developed, laminar flow, the concentration satisfies the equation

$$v_z \frac{\partial c_i}{\partial z} = 2\langle v_z \rangle \left(1 - \frac{r^2}{R^2}\right) \frac{\partial c_i}{\partial z} = \frac{D}{r} \frac{\partial}{\partial r} \left(r \frac{\partial c_i}{\partial r}\right) + D \frac{\partial^2 c_i}{\partial z^2} . \tag{1}$$

For any appreciable flow rates, the axial diffusion term $D\partial^2 c_i/\partial z^2$ can be neglected. With this approximation and a constant concentration at the wall, one has the classical Graetz problem.[1] The boundary conditions thus become

$$\left.\begin{aligned} c_i &= c_b \text{ at } z = 0. \\ c_i &= c_o \text{ at } r = R. \\ \partial c_i/\partial r &= 0 \text{ at } r = 0. \end{aligned}\right\} \tag{2}$$

Lévêque[2] used a similarity transformation to obtain the solution near the entrance to the mass-transfer section. In order to use Lévêque's solution as a basis for obtaining higher order terms in an expansion valid for small values of z, it is appropriate to use Lévêque's similarity variable

$$\xi = (R-r)(4\langle v_z \rangle/9DRz)^{1/3} \tag{3}$$

as one variable and a dimensionless axial distance

$$\zeta = 9Dz/4\langle v_z \rangle R^2 \tag{4}$$

as the second variable. In terms of the dimensionless concentration

$$\Theta = (c_i - c_o)/(c_b - c_o) , \tag{5}$$

the problem then becomes

$$\frac{\partial^2 \Theta}{\partial \xi^2} + 3\xi^2 \frac{\partial \Theta}{\partial \xi} - 3\xi\zeta^{1/3} \frac{\partial \Theta}{\partial \zeta^{1/3}} = \left(\frac{3}{2} \xi^3 \zeta^{1/3} + \frac{\zeta^{1/3}}{1-\xi\zeta^{1/3}}\right) \frac{\partial \Theta}{\partial \xi} - \frac{3}{2} \xi^2 \zeta^{2/3} \frac{\partial \Theta}{\partial \zeta^{1/3}} , \tag{6}$$

with the boundary conditions

$$\Theta = 0 \text{ at } \xi = 0, \quad \Theta = 1 \text{ at } \xi = \infty, \quad \Theta = 1 \text{ at } \zeta = 0. \tag{7}$$

The solution can be expressed as a power series in $\zeta^{1/3}$:

$$\Theta = \Theta_o(\xi) + \zeta^{1/3}\Theta_1(\xi) + \zeta^{2/3}\Theta_2(\xi) + O(\zeta) , \tag{8}$$

where the leading term $\Theta_o(\xi)$ is Lévêque's solution. The ordinary differential equations for Θ_o, Θ_1, and Θ_2 are

$$\Theta_o'' + 3\xi^2\Theta_o' = 0 , \tag{9}$$

$$\Theta_1'' + 3\xi^2\Theta_1' - 3\xi\Theta_1 = (1 + \frac{3}{2}\xi^3)\,\Theta_o' , \tag{10}$$

$$\Theta_2'' + 3\xi^2\Theta_2' - 6\xi\Theta_2 = (1 + \frac{3}{2}\xi^3)\,\Theta_1' + \xi\Theta_o' - \frac{3}{2}\xi^2\,\Theta_1 , \tag{11}$$

with boundary conditions

$$\Theta_o = \Theta_1 = \Theta_2 = 0 \text{ at } \xi = 0 . \tag{12}$$

$$\Theta_o = 1, \quad \Theta_1 = \Theta_2 = 0 \text{ at } \xi = \infty . \tag{13}$$

The solutions for Θ_o and Θ_1 are

$$\Theta_o = \frac{1}{\Gamma(4/3)} \int_o^{\xi} e^{-x^3} dx . \tag{14}$$

$$\Theta_1 = -\frac{\xi^2 e^{-\xi^3}}{10\Gamma(4/3)} - \frac{3/5}{\Gamma(4/3)}\,\xi \int_{\xi}^{\infty} e^{-x^3} dx . \tag{15}$$

Hence the differential equation (11) for Θ_2 becomes

$$\Theta_2'' + 3\xi^2\Theta_2' - 6\xi\Theta_2 = \left(\frac{7}{5}\xi + \frac{21}{20}\xi^4 + \frac{9}{20}\xi^7\right)\frac{e^{-\xi^3}}{\Gamma(4/3)} - \frac{3/5}{\Gamma(4/3)}\int_{\xi}^{\infty} e^{-x^3} dx , \tag{16}$$

with the solution

$$\Theta_2 = -\frac{3/10}{\Gamma(4/3)}\,\xi^2 \int_{\xi}^{\infty} e^{-x^3} dx - \left(\frac{11}{210} + \frac{1}{14}\xi^3 + \frac{3}{200}\xi^6\right)\frac{e^{-\xi^3}}{\Gamma(4/3)}$$

$$+ \frac{11}{630}\,\frac{\Gamma(5/3)}{[\Gamma(4/3)]^3} \int_o^1 \frac{x^{1/3}}{(1-x)^{2/3}}\, e^{-\xi^3/(1-x)}\, dx . \tag{17}$$

For the derivatives at the surface we then obtain

$$\begin{aligned} \Theta_0'(0) &= 1/\Gamma(4/3) = 1.11984652 \; . \\ \Theta_1'(0) &= -0.6 \; . \\ \Theta_2'(0) &= -\frac{11/140}{\Gamma(4/3)}\left[\frac{\Gamma(5/3)}{\Gamma(4/3)}\right]^2 = -0.0899230 \; . \end{aligned} \tag{18}$$

Hence, the average Nusselt number referred to the concentration difference at the inlet is

$$\mathrm{Nu} = \frac{J}{\pi z D(c_b - c_o)} = 1.6151\left(\frac{\mathrm{Pe}}{z/2R}\right)^{1/3} - 1.2 - 0.28057\left(\frac{z/2R}{\mathrm{Pe}}\right)^{1/3} + \ldots, \tag{19}$$

where

J is the amount transferred to the wall in a length z.

$\mathrm{Pe} = 2R\langle v_z\rangle/D$ is the Péclet number.

Discussion

In order to determine the applicable range of the Lévêque series, and also to check for algebraic errors, it is desirable to compare this series with the Graetz series. For this purpose we use the mass transfer J in a length z divided by the total mass transfer for an infinite length. The Lévêque series for this quantity is

$$\frac{J}{\pi R^2(c_b - c_o)\langle v_z\rangle} = 2\mathrm{Nu}Z = 4.069792\,Z^{2/3} - 2.4\,Z - 0.4453787\,Z^{4/3} + O(Z^{5/3}), \tag{20}$$

where

$$Z = zD/2\langle v_z\rangle R^2 \; , \tag{21}$$

and the Graetz series is

$$\frac{J}{\pi R^2(c_b - c_o)\langle v_z\rangle} = 1 - \sum M_k e^{-\lambda_k^2 Z} \; . \tag{22}$$

This comparison is made in table 1. Values of the first five eigenvalues λ_k were taken from Abramowitz[3], and the first five values of M_k were

Table 1. Comparison of Lévêque series and Graetz series

$J/\pi R^2(c_b - c_o)\langle v_z \rangle$

Z	Graetz series	Lévêque series 1 term	Lévêque series 2 terms	Lévêque series 3 terms	ϵ
0	0.002508	0	0	0	-----
0.001	0.038190	0.040698	0.038298	0.038253	0.165
0.002	0.059614	0.064604	0.059804	0.059692	0.131
0.003	0.077172	0.084655	0.077455	0.077262	0.117
0.004	0.092566	0.102552	0.092952	0.092669	0.111
0.005	0.106502	0.119001	0.107001	0.106621	0.112
0.006	0.119360	0.134382	0.119982	0.119496	0.114
0.007	0.131375	0.148926	0.132126	0.131530	0.118
0.008	0.142703	0.162792	0.143592	0.142879	0.123
0.009	0.153457	0.176090	0.154490	0.153656	0.130
0.01	0.163719	0.188903	0.164903	0.163943	0.137
0.015	0.209501	0.247533	0.211533	0.209885	0.183
0.02	0.248849	0.299865	0.251865	0.249447	0.240
0.025	0.283844	0.347962	0.287962	0.284706	0.304
0.03	0.315603	0.392934	0.320934	0.316782	0.374

ϵ is the per cent deviation of the three-term Lévêque series from the Graetz series.

calculated from the results of Lipkis.[4] Values of λ_k and M_k for the sixth through thirtieth terms of the Graetz series were calculated from the formulas of Sellars, Tribus, and Klein,[5] which formulas are valid in the asymptotic limit of large λ_k.

The per cent deviation ϵ of the three-term Lévêque series from the Graetz series increases as Z approaches zero. This can be attributed to errors in the Graetz series since the results of Lipkis are given to only five significant figures. Truncation errors due to carrying only thirty terms in the Graetz series show up only in the value for Z=0, not in the value for Z=0.001. The Graetz series, as calculated, is appreciably in error up to Z=0.01.

With due allowance for the errors in the Graetz series, the remaining deviations in table 1 can be attributed to terms neglected in the Lévêque series. From the behavior of these deviations one can estimate that the first neglected term in equation (20) is $-0.3\ Z^{5/3}$, corresponding to the next term in equation (19) of $-0.25\ (z/2R\ Pe)^{2/3}$.

It is occasionally stated that the Lévêque solution should be good for $Z < 0.02$. We see from table 1 that for this value of Z the Lévêque solution predicts an average rate of mass transfer that is too high by 20.5%, while the three-term Lévêque series is accurate to 0.24%. At this point 25% of the possible mass transfer has already occurred.

If one uses the values of λ_k and M_k obtained from the formulas of Sellars, Tribus, and Klein for all terms of the Graetz series instead of just the sixth and higher terms, the error is considerably greater than that shown in table 1. In fact, replacing the values for the first term by those of Jakob[6] leads to a considerable improvement. It is worthwhile to point this out since at least one text, that of Knudsen and Katz[7], quotes values from Sellars, Tribus, and Klein for ten terms without mentioning that the formulas on which they are based are valid only for large values of λ_k.

We might note that Lipkis suggests the following formula for the local Nusselt number for small values of Z:

$$Nu_{loc}(Z) = 1.357\, Z^{-1/3} - 1.7. \tag{23}$$

The term -1.7 can be compared directly with the term -1.2 of equation (19).

Finally, the Lévêque series might be expected to break down when the diffusion layer reaches the middle of the pipe, since there seems to be no way to take this into account in the analysis. This problem would arise beginning with a value of Z in the range from 0.05 to 0.08. On the other hand, the Graetz series is uniformly convergent for any non-zero value of Z (although more accurate values of λ_k and M_k are required for small Z).

Conclusions

The Lévêque solution for mass transfer in fully developed, laminar pipe flow can be extended to higher terms, with a considerable improvement in the accuracy. Near the beginning of the mass-transfer section this series works better than the Graetz series.

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Nomenclature

c_i - concentration of reactant.

c_b - inlet concentration.

c_o - wall concentration.

D - diffusion coefficient.

J - amount of material transferred to the wall.

Nu - Nusselt number.

Pe - Péclet number.

r - radial position.

R - radius of pipe.

v_z - axial velocity.

$\langle v_z \rangle$ - average axial velocity.

z - axial distance.

Γ - gamma function.

ζ - dimensionless axial distance.

Θ - dimensionless concentration.

ξ - Lévêque similarity variable.

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