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Extension Spaces of Oriented Matriods

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Abstract. We study the space of all extensions of a real hyperplane arrangement by a new pseudo-hyperplane, and, more generally, of an oriented matroid by a new element. The question whether this space has the homotopy type of a sphere is a special case of the "Generalized Baues Problem" of Billera, Kapranov & Sturmfels, via the Bohne-Dress Theorem on zonotopal tilings.

We prove that the extension space is spherical for the class of strongly euclidean oriented matroids. This class includes the alternating matroids and all oriented matroids of rank at most 3 or of corank at most 2. In general it is not even known whether the extension space is connected. We show that the subspace of realizable extensions is always connected but not necessarily spherical.

1. Introduction

Let $\mathbf{X}$ be a central arrangement of oriented hyperplanes in $\mathbb{R}^r$. Consider the set $\mathcal{E}(\mathbf{X})$ of all (equivalence classes of) extensions of $\mathbf{X}$ by a new oriented hyperplane H. This set is partially ordered, with $H \leq H'$ whenever H is obtained by moving H' into more special position. The poset $\mathcal{E}(\mathbf{X})$ is isomorphic to the face poset of a convex r-polytope and hence it is homeomorphic to the $(r-1)$-sphere.

This homeomorphism is understood in the following sense. With any poset P we associate the geometric realization $|\Delta(P)|$ of the simplicial complex $\Delta(P)$ of chains in P. By the *homotopy type* of P we mean the homotopy type of the topological space $|\Delta(P)|$. Basic tools for determining homotopy types of posets are Quillen's fiber theorem and its relatives. These will be reviewed in Lemmas 3.1 to 3.3.

From a combinatorial point of view, it is natural to consider the larger poset of all extensions of $\mathbf{X}$ by a new *pseudo-hyperplane*. This poset is an invariant of the oriented matroid $\mathcal{M}$ of $\mathbf{X}$. We call this poset the *extension poset* $\mathcal{E}(\mathcal{M})$ of $\mathcal{M}$.

More generally, for any (not necessarily realizable) oriented matroid $\mathcal{M}$ of rank r, the *extension poset* $\mathcal{E}(\mathcal{M})$ is the set of all one-element extensions of $\mathcal{M}$, excluding loops and coloops, with their natural partial order [2]. The *extension space* of $\mathcal{M}$ is the space $|\Delta(\mathcal{E}(\mathcal{M}))|$, which we usually identify with $\mathcal{E}(\mathcal{M})$. For an introduction to the theory of oriented matroids and complete references to the original literature we refer to [6].

The theme of the present article is the following question: *Does the extension space $\mathcal{E}(\mathcal{M})$ have the homotopy type of the $(r-1)$-dimensional sphere; in symbols $\mathcal{E}(\mathcal{M}) \simeq S^{r-1}$?* An affirmative answer will be given for certain special cases; however, in general it remains unknown whether $\mathcal{E}(\mathcal{M})$ is even connected.

We start out in Section 2 with a discussion of the case where $\mathcal{M}$ is realizable. Here the question "$\mathcal{E}(\mathcal{M}) \simeq S^{r-1}$?" is seen to be an instance of the *Generalized Baues Problem* [3], which asks whether the poset $\omega(P,Q)$ of cellular strings of a projection of polytopes $P \to Q$ is homotopy equivalent to a sphere of dimension $\dim(P) - \dim(Q) - 1$. Again, it is generally unknown whether $\omega(P,Q)$ is a connected space. Using the Bohne-Dress theorem on zonotopal tilings [7], we will see that $\mathcal{E}(\mathcal{M})$ is isomorphic to $\omega(P,Q)$ when P is a regular n-cube projecting onto an $(n-r)$-zonotope Q, and $\mathcal{M}$ is the rank r oriented matroid determined by the kernel of this projection.

For realizable $\mathcal{M}$ we may also consider the subposet $\mathcal{E}(\mathcal{M})_{real}$ of realizable extensions, which is proved to be connected. Using an non-isotopic oriented matroid due to Suvorov [22], we will show that $\mathcal{E}(\mathcal{M})_{real}$ is in general not a nice space.

Theorem 1.1. *There exists a realizable rank 3 oriented matroid $\mathcal{M}$ such that $\mathcal{E}(\mathcal{M})_{real}$ does not have the homotopy type of a sphere.*

In Section 3 we relate the topology of extension spaces to linear programming in oriented matroids. A special role will be played by oriented matroid programs which are *euclidean* in the sense of Edmonds & Mandel [10]. In fact, euclidean programs are characterized by the contractibility of a suitable extension poset (Corollary 3.12).

In Section 4 we introduce the class of *strongly euclidean* oriented matroids. The following is the main result of this paper.

Theorem 1.2. *Let $\mathcal{M}$ be a strongly euclidean rank r oriented matroid. Then the extension poset $\mathcal{E}(\mathcal{M})$ is homotopy equivalent to the $(r-1)$-sphere S^{r-1}.*

Theorem 1.2 applies to all oriented matroids of rank at most 3 or corank at most 2. We conjecture that there exist oriented matroids of rank 4 which are not strongly euclidean. In order to establish strong euclideanness results, we analyze the structure of minimally non-euclidean programs in terms of their *inseparability graphs*. As an application we prove that the alternating matroid $C^{n,r}$ is strongly euclidean, and hence its extension space $\mathcal{E}(C^{n,r})$ is spherical. By definition, $C^{n,r}$ is the oriented matroid of the cyclic $(r-1)$-polytope with n vertices.

The strong euclideanness of $C^{n,r}$ will be used in a subsequent paper to study the *Higher Bruhat Orders* $B(n, n-r)$ of Manin & Schechtman [18]. These can be viewed as posets of uniform extensions of $C^{n,r}$.

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2. Realizable extensions and the Generalized Baues Problem

Throughout this section $\mathcal{M}$ denotes a realizable oriented matroid of rank $r \geq 2$ on $[n] = \{1, 2, \ldots, n\}$. Let $\mathcal{E}(\mathcal{M})_{real}$ be the subposet of $\mathcal{E}(\mathcal{M})$ consisting of all realizable rank r oriented matroids $\mathcal{M}'$ on $[n+1]$ such that $\mathcal{M}' \setminus \{n+1\} = \mathcal{M}$ and $n+1$ is not a loop of $\mathcal{M}'$. For two elements $\mathcal{M}'$ and $\mathcal{M}''$ of $\mathcal{E}(\mathcal{M})_{real}$ we have $\mathcal{M}' \leq \mathcal{M}''$ if and only if $\mathcal{M}'$ is a weak image of $\mathcal{M}''$, i.e., every signed basis of $\mathcal{M}'$ is a basis with the same sign in $\mathcal{M}''$ (see [6, Sect. 7.7]).

Proposition 2.1. *The space $\mathcal{E}(\mathcal{M})_{real}$ of realizable extensions of $\mathcal{M}$ is connected.*

Proof. Let $\mathbf{X}$ be any realization of $\mathcal{M}$, considered as a configuration of n vectors in $\mathbb{R}^r$. Let $\mathbf{X}^{ad}$ be the configuration of linear hyperplanes spanned by vectors in $\mathbf{X}$, and let $\mathcal{M}^{ad}(\mathbf{X})$ denote the rank r oriented matroid of this hyperplane arrangement. Thus $\mathcal{M}^{ad}(\mathbf{X})$ is the *adjoint* of $\mathcal{M}$ determined by the specific realization $\mathbf{X}$ (see [2], [6, Sect. 5.3]). The regions of the arrangement $\mathbf{X}^{ad}$ are in order preserving bijection with the one-element extension of $\mathcal{M}$ which can be realized in $\mathbf{X}$. In other words, the poset $\mathcal{E}(\mathbf{X})$ is isomorphic to the poset of nonzero covectors $\mathcal{L}(\mathcal{M}^{ad}(\mathbf{X}))$.

We call two realizations $\mathbf{X}$ and $\mathbf{Y}$ of $\mathcal{M}$ *adjoint equivalent* if $\mathcal{M}^{ad}(\mathbf{X}) = \mathcal{M}^{ad}(\mathbf{Y})$. There are only finitely many adjoint equivalence classes. We choose a system of representatives $\mathbf{X}_1, \ldots, \mathbf{X}_m$, and we abbreviate $\mathcal{L}_i := \mathcal{E}(\mathbf{X}_i) = \mathcal{L}(\mathcal{M}^{ad}(\mathbf{X}_i))$. Clearly, each $\mathcal{L}_i$ is homeomorphic to the $(r-1)$-sphere S^{r-1}.

Let $\phi_i : \mathcal{L}_i \hookrightarrow \mathcal{E}(\mathcal{M})_{real}$ denote the inclusion of the spherical poset $\mathcal{L}_i$. By construction, each element of $\mathcal{E}(\mathcal{M})_{real}$ lies in $\phi_i(\mathcal{L}_i)$ for some $i \in [m]$. Now consider the subposet $\cap_{i=1}^m \phi_i(\mathcal{L}_i)$ of those extensions which can be realized in every realization of $\mathcal{M}$. This poset is non-empty since it contains all parallel extensions of $\mathcal{M}$. More generally, all *lexicographic extensions* of $\mathcal{M}$ lie in $\cap_{i=1}^m \phi_i(\mathcal{L}_i)$ by [6, Prop. 8.2.2].

We have shown that the poset $\mathcal{E}(\mathcal{M})_{real}$ is the union of the connected posets $\phi_i(\mathcal{L}_i)$ whose intersection is non-empty. This implies that the space $\mathcal{E}(\mathcal{M})_{real}$ is connected. □

We will now show that $\mathcal{E}(\mathcal{M})_{real}$ is in general not homotopy equivalent to a sphere. Theorem 1.1 is implied by the following more specific result.

Proposition 2.2. *There exists a rank 3 oriented matroid $\mathcal{M}$ on 14 elements such that $\mathcal{E}(\mathcal{M})_{real}$ is not homotopy equivalent to a sphere.*

Proof. We choose $\mathcal{M}$ to be Suvorov's rank 3 oriented matroid Suv(14) on 14 elements [22], in the version presented in [6, Sect. 8.6]. This oriented matroid has the property that its projective realization space consists of precisely two points, $\mathbf{X}_1$ and $\mathbf{X}_2$.

In the above notation we have $m = 2$, that is, $\mathcal{M}$ has precisely two realizable adjoints $\mathcal{L}_1 = \mathcal{M}^{ad}(\mathbf{X}_1)$ and $\mathcal{L}_2 = \mathcal{M}^{ad}(\mathbf{X}_2)$. Both $\mathbf{X}_1^{ad}$ and $\mathbf{X}_2^{ad}$ are central arrangements of 45 planes in $\mathbb{R}^3$. The corresponding 45 lines in the projective plane are constructed as the connecting lines of the 14 points in [6, Fig. 8.6.1]. By identifying corresponding regions, we can determine how the space $\mathcal{E}(\mathcal{M})_{real}$ is glued from two 2-spheres $\mathcal{L}_1$ and $\mathcal{L}_2$.

We now zoom in on a particular local situation in the two projective line arrangements $\mathcal{L}_1$ and $\mathcal{L}_2$. In both adjoints we consider the triangle spanned by the points 9, 10 and 11. In Figure 1 we depict all the lines of $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) which pass through this triangle. This diagram has been derived from the two "big pictures" in [6, Fig. 8.6.1].

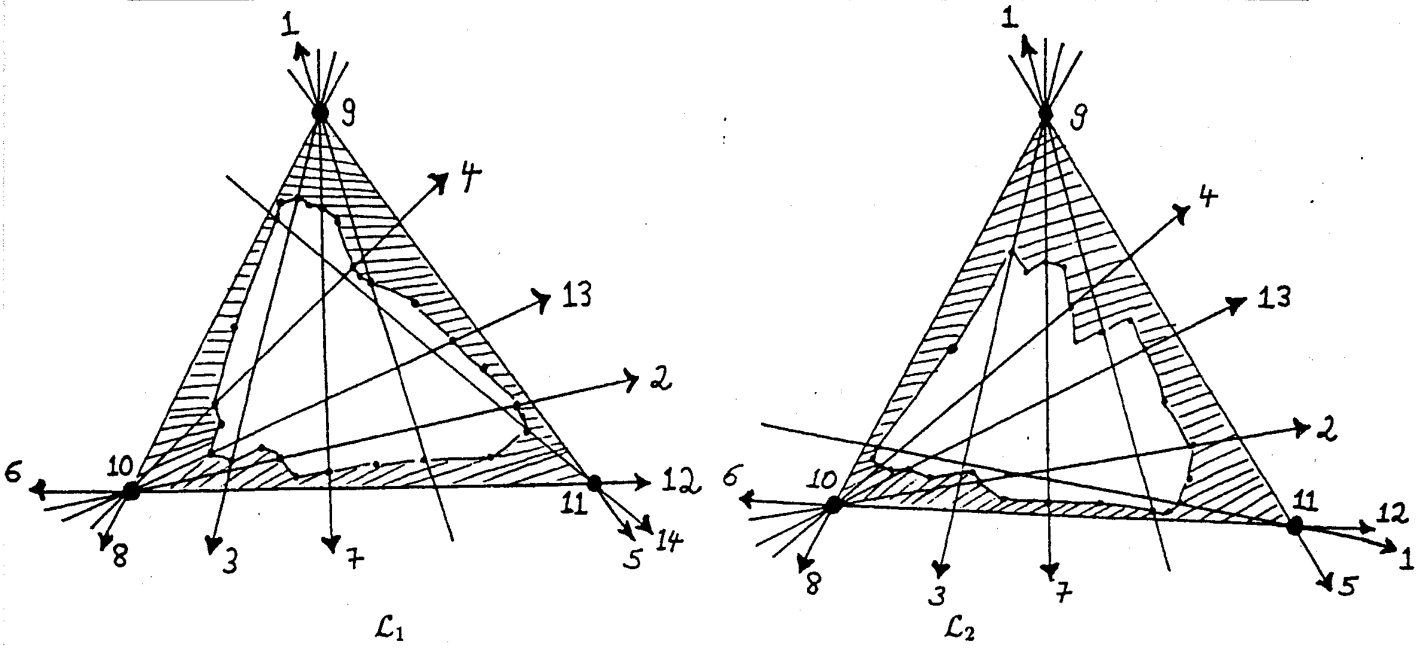


Figure 1. A triangle in the two realizable adjoints of the Suvorov oriented matroid.

The lines which cross the boundary of the triangle $\{9, 10, 11\}$ are the same in both $\mathcal{L}_1$ and $\mathcal{L}_2$, and each triangle edge is crossed in the same order. This shows that the cells on the boundaries of the two triangles are labeled identically in both adjoints. Hence the boundaries of the two triangles are identified in the space $\mathcal{E}(\mathcal{M})_{real}$.

This identification of the boundaries extends throughout the shaded part of the interiors in Figure 1. On the other hand, the unshaded interior regions in the two triangles are disjoint in $\mathcal{E}(\mathcal{M})_{real}$. Note that $\Delta(\mathcal{L}_i)$ is the barycentric subdivision of the cell complex of $\mathbf{X}_i^{ad}$, and the shaded parts are subcomplexes of these barycentric subdivisions.

Our discussion shows that the two partly-glued triangles in Figure 1 constitute a non-trivial 2-cycle in the space $\mathcal{E}(\mathcal{M})_{real}$. By considering the negatives of all involved covectors of $\mathcal{L}_1$ and $\mathcal{L}_2$, we find yet another non-trivial 2-cycle. This proves that the rank of the second homology group of $\mathcal{E}(\mathcal{M})_{real}$ is at least 3, which completes the proof of Proposition 2.2 and of Theorem 1.1. □

Our results in Section 4 imply that the space $\mathcal{E}(\mathcal{M})$ of all (not necessarily realizable) extensions of the Suvorov oriented matroid $\mathcal{M}$ is homotopy equivalent to S^2. In particular, the 2-cycle coming from the triangle $\{9, 10, 11\}$ is a 2-boundary in the extension space $\mathcal{E}(\mathcal{M})$.

An important motivation for studying the extension space of a realizable oriented matroid $\mathcal{M}$ comes from recent developments in the theory of convex polytopes. The remainder of Section 2 is devoted to this connection. It is independent of the rest of this paper, and it assumes that the reader is familiar with the theory of *fiber polytopes* [4] and the "Generalized Baues Problem" which was posed in [3, Sect. 3].

There is an isomorphism from the extension poset $\mathcal{E}(\mathcal{M})$ to a poset of *cellular strings* on the n-cube. Let $C^n = \text{conv}\,\{-1,+1\}^n$ denote the standard n-cube. The face poset of C^n will be identified with the poset of sign vectors $\{-,0,+\}^n$, which is partially ordered componentwise via "$- < 0$" and "$+ < 0$". Fix an r-dimensional linear subspace ξ in $\mathbb{R}^n$. Its oriented matroid $\mathcal{M} = ([n], \mathcal{V})$ is given by the set of *vectors* $\mathcal{V} = \{\text{sign}(v) : v \in \xi\}$ (see [6, Sections 1.2, 3.7]).

For $a \in \mathbb{R}^n$ we consider the polytope $P_a = (a + \xi) \cap C^n$. Let $\mathcal{F}_a$ denote the poset of proper faces of P_a. We get a poset embedding of $\mathcal{F}_a$ into $\{-,0,+\}^n$ by mapping each face of P_a to the smallest face of C^n containing it. We say that $a \in \mathbb{R}^n$ is *admissible* if $P_a \neq \emptyset$, and we call two admissible vectors $a, b \in \mathbb{R}^n$ *equivalent* if $\mathcal{F}_a = \mathcal{F}_b$. Thus for each equivalence class σ we get a subposet $\mathcal{F}_\sigma$ of $\{-,0,+\}^n$. Note that P_a and $\mathcal{F}_a$ depend only on the image of a in $\mathbb{R}^n/\xi$. The set of admissible vectors in $\mathbb{R}^n/\xi$ is a zonotope $\mathcal{Z}$. The oriented matroid of $\mathcal{Z}$ [6, Sect. 2.2] is the dual $\mathcal{M}^*$. The equivalence classes form a polyhedral cell decomposition Γ of $\mathcal{Z}$, called the *chamber complex*.

Let $\sigma, \tau \in \Gamma$ be chambers with $\sigma \subseteq \overline{\tau}$. Then there is a surjective order-preserving map $\rho_{\sigma\tau} : \mathcal{F}_\tau \to \mathcal{F}_\sigma$, which can be described as follows. Given $F_\tau \in \mathcal{F}_\tau$, then $\rho_{\sigma\tau}(F_\tau)$ is the join in $\{-,0,+\}^n$ of all sign vectors $G \in \mathcal{F}_\sigma$ such that $G \leq F_\tau$.

Let $\omega = \omega(C^n, \mathcal{Z})$ denote the inverse limit of the inverse system $\{\mathcal{F}_\sigma\}_{\sigma\in\Gamma}$ in the category of posets. By definition, the elements of ω are the tuples $(F_\sigma)_{\sigma\in\Gamma}$ such that $\rho_{\sigma\tau}(F_\tau) = F_\sigma$ whenever $\sigma \subseteq \overline{\tau}$. These tuples are called *cellular strings* in [3]. The following theorem is a reformulation of a result of Bohne & Dress on zonotopal tilings [7]; a uniform version has also been given by Kapranov & Voevodsky [15, Sect. 4].

Theorem 2.3. *The poset $\omega(C^n, \mathcal{Z})$ is isomorphic to the extension poset $\mathcal{E}(\mathcal{M})$.*

The theorem of Bohne & Dress [7] states that the zonotopal tilings of $\mathcal{Z}$ are in order-preserving bijection with the one-element extensions of $\mathcal{M}$ (see also [6, Thm. 2.2.13]). It can be seen that this statement is equivalent to Theorem 2.3 using the constructions in [4, Sect. 5] and [3, Sect. 3]. The proof of the Bohne-Dress theorem is difficult. Here we only describe the maps for the promised isomorphism.

We first construct a map from ω to $\mathcal{E}(\mathcal{M})$. For any cellular string $\mathbf{F} = (F_\sigma)_{\sigma\in\Gamma}$ in ω we define the following subset of $\{-,0,+\}^{n+1}$:

$$\mathcal{V}_\mathbf{F} \quad := \quad \{(X,0) : X \in \mathcal{V}\} \;\cup\; \{\pm(F_\sigma,+) : \sigma \in \Gamma\}.$$

Lemma 2.4. *The set $\mathcal{V}_\mathbf{F} \subset \{-,0,+\}^{n+1}$ is the set of vectors of a rank r oriented matroid $\mathcal{M}_\mathbf{F}$ with $\mathcal{M}_\mathbf{F} \setminus \{n+1\} = \mathcal{M}$. The map $\omega \to \mathcal{E}(\mathcal{M})$, $\mathbf{F} \mapsto \mathcal{M}_\mathbf{F}$, is order-preserving and injective.*

For the inverse map one has the following lemma.

Lemma 2.5. *Let $\mathcal{M}' \in \mathcal{E}(\mathcal{M})$. For each equivalence class σ in $\mathbb{R}^n$, there exists a unique vector X of $\mathcal{M}'$ such that σ intersects the face of the cube C^n defined by X.*

Given σ and X as above, then X defines a unique face F_σ of the polytope $\mathcal{F}_\sigma = C^n \cap \sigma$. It now remains to be verified that the tuple of faces $\mathbf{F} = (F_\sigma)_{\sigma \in \Gamma}$ satisfies the gluing relation $\rho_{\sigma\tau}(F_\tau) = F_\sigma$ whenever $\sigma \subseteq \overline{\tau}$. Thus the assignment $\mathcal{M}' \mapsto \mathbf{F}$ is a well-defined map $\mathcal{E}(\mathcal{M}) \to \omega$. It is the inverse to the map in Lemma 2.4.

3. Extension spaces of oriented matroid programs

In this section we introduce extension spaces of oriented matroid programs and we discuss their homotopy types. An oriented matroid program is euclidean if and only if its extension space is contractible (Corollary 3.12). For an introduction to linear programming in oriented matroids we refer to [6, Ch. 10].

We begin by collecting criteria for establishing homotopy equivalence. For a survey and general discussion of such criteria see [5]. Every poset P is trivially homeomorphic to its order dual P^{op}. For every order preserving map $f : P \to Q$, there is an induced order preserving map $f^{op} : P^{op} \to Q^{op}$. This leads to dual versions of the results below which will not be stated separately. For the set of all elements above x we write $P_{\geq x} := \{z \in P : z \geq x\}$, and similarly we use $P_{\leq x}$, $P_{>x}$, etc.

Lemma 3.1. *Let $f : P \to Q$ be an order preserving map of posets. If $f^{-1}(Q_{\geq x})$ is contractible for all $x \in Q$, then f induces a homotopy equivalence between P and Q.*

Lemma 3.1 is the "Quillen Fiber Theorem" [19, Thm. A], [5, (10.5)]. The following variant of Lemma 3.1 is due to E.K. Babson [1]. The proof below is due to A. Björner. We thank both of them for the possibility to include this material here.

Lemma 3.2. [1] *Let $f : P \to Q$ be an order preserving map of posets. If*
(i) *$f^{-1}(x)$ is contractible for all $x \in Q$, and*
(ii) *$P_{\geq y} \cap f^{-1}(x)$ is contractible for all $x \in Q$ and $y \in P$ with $f(y) < x$,*
then f induces a homotopy equivalence between P and Q.

Proof. By Lemma 3.1 dualized, it suffices to show that $F(x) := f^{-1}(Q_{\leq x})$ is contractible for all $x \in Q$. We now apply Lemma 3.1 to the inclusion $g : f^{-1}(x) \hookrightarrow F(x)$. By (ii), we know that $g^{-1}(F(x)_{\geq y}) = f^{-1}(x) \cap P_{\geq y}$ is contractible for all $y \in F(x)$. Hence g is a homotopy equivalence. Since $f^{-1}(x)$ is contractible by (i), also $F(x)$ is contractible. □

Babson's Lemma 3.2 will be our principal tool for relating the extension spaces $\mathcal{E}(\mathcal{M})$ and $\mathcal{E}(\mathcal{M}/g)$ in Section 4. In this section we need the following version of the Quillen Fiber Theorem.

Lemma 3.3. *Let $f : P \to P$ be an order preserving map which satisfies $f(f(x)) = f(x) \leq x$ for all $x \in P$. Then the surjection $f : P \to f(P)$ is a homotopy equivalence.*

Proof. We derive this from Lemma 3.1. Let $x \in f(P)$. Then the fiber $f^{-1}(f(P)_{\geq x})$ has a unique minimal element (namely, x). Thus it is a cone and hence contractible. $\square$

We next define the posets to be investigated in this section. Let $\mathcal{M}$ denote a loopless oriented matroid of rank r on a ground set E. By the Topological Representation Theorem of Folkman & Lawrence, we can represent $\mathcal{M}$ by an arrangement $\mathcal{A}_{\mathcal{M}}$ of signed pseudospheres $\{S_e\}_{e\in E}$ [11], [6, Ch. 4].

We distinguish a special element $g \in E$. The pair $(\mathcal{M}, g)$ is an *affine oriented matroid.* In the pseudosphere arrangement we interpret $(\mathcal{M}, g)$ as the "affine space" S_g^+, which consists of all covectors Y with $Y_g = +$. Covectors Y with $Y_g = 0$ are said to be *at infinity.* Each element $e \in E$ which is not parallel to g is represented by a pseudohyperplane $S_e \cap S_g^+$ in the affine space $(\mathcal{M}, g)$. An *oriented matroid program* is a triple $(\widetilde{\mathcal{M}}, g, f)$, where $\widetilde{\mathcal{M}}$ is an oriented matroid and f, g are distinct elements of $\widetilde{\mathcal{M}}$ such that f is not a coloop and g is not a loop [9], [6, Sect. 10.1].

Given $(\mathcal{M}, g)$ as above, then every extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ yields an oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ on $E \cup \{f\}$. This program is *non-trivial* if f is neither parallel nor antiparallel to g. By a result of Las Vergnas each extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ can be identified with its *localization* [16], [6, Prop. 7.1.4]. This is the unique function σ which assigns to each cocircuit Y of $\mathcal{M}$ a sign $\sigma(Y) \in \{-, 0, +\}$ such that $(Y, \sigma(Y))$ is a cocircuit of $\widetilde{\mathcal{M}}$. There is a natural partial order on localizations and hence on extensions which is induced by "$0 < -$" and "$0 < +$".

Definition 3.4.

(i) The *extension poset* $\mathcal{E}(\mathcal{M})$ consists of all extensions of $\mathcal{M}$ by a single element which is neither a loop nor a coloop.

(ii) The *extension poset* $\mathcal{E}(\mathcal{M}, g)$ *of an affine oriented matroid* $(\mathcal{M}, g)$ consists of all extensions in $\mathcal{E}(\mathcal{M})$ that are not parallel or antiparallel to g.

(iii) The *extension poset* $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ *of an oriented matroid program* $(\widetilde{\mathcal{M}}, g, f)$ consists of all extensions of $\mathcal{M} := \widetilde{\mathcal{M}}\backslash f$ by an element f' such that $\widetilde{\mathcal{M}}/g = (\mathcal{M} \cup f')/g$.

In the topological representation $\mathcal{A}_{\mathcal{M}}$, the extensions in $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ correspond to those affine pseudo-hyperplanes $S_{f'}$ which agree "at infinity" with S_f, that is, $S_g \cap S_f = S_g \cap S_{f'}$.

We define the graph G_f of an oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ as follows, slightly modifying [6, Def. 10.1.16]. The vertices of G_f are the cocircuits Y of $\mathcal{M}$ with $Y_g = +$. These correspond to 0-cells of the arrangement $\mathcal{A}_{\mathcal{M}} = (S_e)_{e\in E}$ which lie in the affine space S_g^+. Two such vertices are connected by an edge in G_f if and only if they are connected by a 1-cell in $\mathcal{A}_{\mathcal{M}}$. Some of the edges (Y^0, Y^1) of G_f are directed, as follows. Let Z be the unique cocircuit obtained by elimination of g from Y^1 and $-Y^0$. The edge (Y^0, Y^1) is directed from Y^0 to Y^1 if $\sigma(Z) = +$, it is directed from Y^1 to Y^0 if $\sigma(Z) = -$, and it is undirected if $\sigma(Z) = 0$, where σ denotes the localization of f.

A *path* in G_f is a sequence of vertices $P = (Y^0, Y^1, \ldots, Y^k)$ such that (Y^{i-1}, Y^i) is an edge which is either undirected or directed from Y^{i-1} to Y^i. A path P is *directed* if at least one edge in P is directed, and *undirected* otherwise [6, Def. 10.5.4].

Two vertices Y and Y' of G_f are said to be *equivalent* if there is a path from Y to Y' and a path from Y' to Y. A *strong component* of G_f is the induced subgraph of an equivalence class of vertices. A strong component is said to be *very strong* if it contains at least one directed edge. Let $\mathtt{SC} = \mathtt{SC}(\widetilde{\mathcal{M}}, g, f)$ denote the set of strong components, and let $\mathtt{VSC} = \mathtt{VSC}(\widetilde{\mathcal{M}}, g, f)$ be the subset of very strong components. We define the following natural partial order on SC. For two strong components $\mathbf{c}$ and $\mathbf{c}'$ we set $\mathbf{c} < \mathbf{c}'$ whenever there exists a directed path from a vertex Y in $\mathbf{c}$ to a vertex Y' in $\mathbf{c}'$. The set VSC is a subposet of SC, with the induced partial order.

Example 3.5. Let $(\widetilde{\mathcal{M}}, g, f)$ be the rank 4 oriented matroid program on $\{1, 2, \ldots, 6, g, f\}$ which is defined by the coordinate matrix

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & f & g \end{array}$$
$$\begin{pmatrix} 0 & -1 & 0 & 1 & -1 & -1-\epsilon & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 1+2\epsilon^2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -\epsilon^2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1+\epsilon & 1+2\epsilon+\epsilon^2+\epsilon^3 & 1000 & 1 \end{pmatrix} \tag{3.1}$$

where $0 < \epsilon < \frac{1}{4}$. Geometrically, this is an arrangement of six planes in affine 3-space (with g as the plane at infinity) together with a linear objective function f. The intersection point of the planes i, j and k defines a cocircuit Y_{ijk}, for $1 \le i < j < k \le 6$. Two of the cocircuits, namely, $Y_{125} = (00{-}{+}0{+}{-}0)$ and $Y_{345} = ({-}{+}000{+}{-}0)$, lie at infinity. The remaining 18 cocircuits $Y_{134} = (0{+}00{+}{+}{+}{+})$, $Y_{236} = ({-}00{+}{-}0{+}{+})$, $Y_{235} = ({-}00{+}0{+}{+}{+})$, $Y_{123} = (000{+}{+}{+}{+}{+}), \ldots$ lie in affine space. They are the vertices of the graph G_f of the program $(\widetilde{\mathcal{M}}, g, f)$. The graph G_f has 39 edges, all of which are directed:

$$\begin{array}{l} \underline{(Y_{123}Y_{124})}(Y_{124}Y_{126})\underline{(Y_{134}Y_{123})}(Y_{123}Y_{135})(Y_{135}Y_{136})\underline{(Y_{134}Y_{124})}(Y_{124}Y_{146})(Y_{146}Y_{145}) \\ (Y_{135}Y_{156})(Y_{156}Y_{145})(Y_{136}Y_{156})(Y_{156}Y_{146})(Y_{146}Y_{126})(Y_{236}Y_{235})(Y_{235}Y_{123})\underline{(Y_{123}Y_{234})} \\ (Y_{245}Y_{246})(Y_{246}Y_{124})\underline{(Y_{124}Y_{234})}(Y_{235}Y_{256})(Y_{256}Y_{245})(Y_{236}Y_{256})(Y_{256}Y_{246}) \\ (Y_{246}Y_{126})\underline{(Y_{134}Y_{234})}(Y_{234}Y_{346})(Y_{235}Y_{135})(Y_{135}Y_{356})(Y_{236}Y_{136})(Y_{136}Y_{356})(Y_{356}Y_{346}) \\ (Y_{456}Y_{245})(Y_{245}Y_{145})(Y_{456}Y_{246})(Y_{246}Y_{146})(Y_{146}Y_{346})(Y_{356}Y_{156})(Y_{156}Y_{256})(Y_{256}Y_{456}) \end{array} \tag{3.2}$$

The following method [20, Lemma 4.1] was used for computing the list (3.2). An edge $\{Y_{ijk}, Y_{ijl}\}$ of the arrangement is directed from Y_{ijk} to Y_{ijl} if and only if

$$\chi(i,j,k,l) \cdot \chi(i,j,g,k) \cdot \chi(i,j,g,l) \cdot \chi(i,j,g,f) \quad = \quad +1 \tag{3.3}$$

Here χ denotes the *chirotope* of $\widetilde{\mathcal{M}}$, that is, $\chi(i,j,k,l)$ equals the sign of the 4×4-minor with column indices i, j, k, l in (3.1).

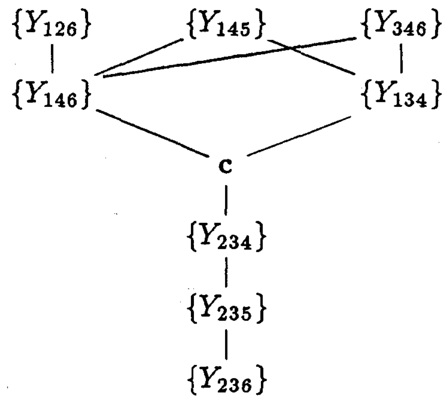
The oriented matroid $\widetilde{\mathcal{M}}$ is realizable and all edges of the graph G_f are directed. Thus G_f is acyclic, VSC is empty, each vertex itself is a strong component, and the poset SC is the transitive closure of G_f. ¿From the underlined edges in (3.2) we see that the basis 1234 is a *mutation* of $\widetilde{\mathcal{M}}$. This means geometrically that the planes $1, 2, 3$ and 4 bound a simplicial region (tetrahedron) in the arrangement.

Let $\widetilde{\mathcal{M}}'$ be the oriented matroid obtained from $\widetilde{\mathcal{M}}$ by reversing the orientation of the basis 1234. The graph G'_f of the corresponding oriented matroid program $(\widetilde{\mathcal{M}}', g, f)$ is obtained by reversing all six underlined edges in (3.2). We find the following cycle in G'_f:

$$Y_{124} \to Y_{123} \to Y_{135} \to Y_{136} \to Y_{356} \to Y_{156} \to Y_{256} \to Y_{456} \to Y_{245} \to Y_{246} \to Y_{124} \quad (3.4)$$

This shows in particular that $\widetilde{\mathcal{M}}'$ is not realizable [6, Cor. 10.1.17].

The ten cocircuits in (3.4) form one very strong component $\mathbf{c}$ of G'_f, while each of the other eight cocircuits is a strong component all by itself. Hence $\mathtt{VSC}' = \{\mathbf{c}\}$, and the poset $\mathtt{SC}'$ is given by the following Hasse diagram:



(3.5)

Our construction of $\widetilde{\mathcal{M}}'$ from $\widetilde{\mathcal{M}}$ is based on a general technique due to N. Mnëv (personal communication) for producing cycles in graphs of oriented matroid programs. The crucial property is that 1234 is a simplicial region of $\widetilde{\mathcal{M}}$ whose vertices $\{Y_{123}, Y_{124}, Y_{134}, Y_{234}\}$ do <u>not</u> form an interval in $\mathtt{SC}$. In Theorem 3.13 this technique will be applied to construct oriented matroid programs with many very strong components. □

The following sequence of lemmas will relate the extension space of an oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ to the order ideals in its poset $\mathtt{VSC}$ of very strong components.

Lemma 3.6. *Let $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ and $\mathbf{c} \in \mathtt{SC}$.*

(i) The localization σ has the same value on each cocircuit in $\mathbf{c}$, so that $\sigma(\mathbf{c})$ is well-defined.

(ii) The sets $I' := \{\mathbf{c} \in \mathtt{SC} : \sigma(\mathbf{c}) = -\}$ and $I := \{\mathbf{c} \in \mathtt{SC} : \sigma(\mathbf{c}) \neq +\}$ are order ideals in the poset $\mathtt{SC}$.

(iii) For each very strong component $\mathbf{c} \in \mathtt{VSC}$, we have $\sigma(\mathbf{c}) \neq 0$.

Proof. This follows from [6, Lemma 10.5.3]. Parts (i) and (ii) hold because σ is constant on undirected edges and weakly increasing (in the order $- \prec 0 \prec +$) along directed edges of G_f. Part (iii) holds because no localization in $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ assigns a zero to two cocircuits which are connected by a directed edge in G_f. □

Lemma 3.7. *Let $I' \subseteq I$ be order ideals of* SC *such that $I \backslash I'$ is an antichain in* SC *which does not intersect* VSC. *Then there is a unique localization $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ such that for all $\mathbf{c} \in$* SC,

$$\sigma(\mathbf{c}) = \begin{cases} - & \text{if } \mathbf{c} \in I', \\ 0 & \text{if } \mathbf{c} \in I \backslash I', \\ + & \text{otherwise.} \end{cases}$$

Proof. The localization σ is determined by the requirements that

$$\sigma(Y) = \begin{cases} Y_f & \text{if } Y_g = 0, \\ - & \text{if } Y_g = + \text{ and } [Y] \in I', \\ 0 & \text{if } Y_g = + \text{ and } [Y] \in I \backslash I', \\ + & \text{if } Y_g = + \text{ and } [Y] \notin I, \end{cases}$$

where $[Y] \in$ SC denotes the equivalence class of the cocircuit Y. The proof that every such σ is a localization is by reduction to corank 2 as in the proof of [6, Thm. 10.5.5]. Our assumptions on $I \backslash I'$ are equivalent to the fact that there is no directed edge both of whose vertices are in $I \backslash I'$. □

Definition 3.8.

(i) An oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ is *euclidean* if for every cocircuit Y of $\mathcal{M} = \widetilde{\mathcal{M}} \backslash f$ with $Y_g \neq 0$ there is an extension $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ with $\sigma(Y) = 0$.

(ii) An oriented matroid $\mathcal{M}$ is *strongly euclidean* if it has rank 1, or if it possesses an element g such that $\mathcal{M}/g$ is strongly euclidean and the program $(\widetilde{\mathcal{M}}, g, f)$ is euclidean for every extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$.

Part (i) of Definition 3.8 is due to Edmonds & Mandel [10, Sect. 9.III]. The concept in part (ii) is new. Note that realizable oriented matroid programs are always euclidean. The following result is due to Edmonds & Mandel [10].

Proposition 3.9. *An oriented matroid program $(\widetilde{\mathcal{M}}, g, f)$ is euclidean if and only if its poset of very strong components* VSC *is empty.*

Proof. The "only if" part follows from Lemma 3.6(iii). For the "if" part let $I = \mathrm{SC}_{\leq [Y]}$ be the order ideal generated by $[Y]$ and $I' := I \backslash \{[Y]\}$. Now apply Lemma 3.7. □

In order to study the topology of the space $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$, we introduce the following filtration by subspaces. For any two order ideals $I' \subseteq I$ in SC, we define

$$\mathcal{E}(I', I) \quad := \quad \{\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f) \ : \ \sigma(\mathbf{c}) = + \text{ for } \mathbf{c} \notin I, \ \sigma(\mathbf{c}) = - \text{ for } \mathbf{c} \in I'\}. \qquad (3.6)$$

This contains as special cases $\mathcal{E}(\emptyset, \mathrm{SC}) = \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ and $\mathcal{E}(I, I)$, which is a single point by Lemma 3.7.

Lemma 3.10. *Let $I' \subset I$ be order ideals in* SC *and let* $\mathbf{c}_0$ *be a maximal element in* $I \setminus I'$. *If* $\mathbf{c}_0$ *is not in* VSC, *then the inclusion* $\mathcal{E}(I', I \setminus \{\mathbf{c}_0\}) \subseteq \mathcal{E}(I', I)$ *is a homotopy equivalence.*

Proof. We define a surjective map $\pi : \mathcal{E}(I', I) \to \mathcal{E}(I', I \setminus \{\mathbf{c}_0\})$, $\tau \mapsto \pi(\tau)$ by the rule

$$\pi(\tau)(Y) \quad := \quad \begin{cases} \tau(Y) & \text{for } [Y] \in I \setminus \{\mathbf{c}_0\}, \\ + & \text{otherwise,} \end{cases}$$

for all cocircuits Y with $Y_g = +$. The fact that $\pi(\tau)$ is a localization of $\mathcal{M}$ is verified by reduction to corank 2 [16], [6, Thm. 7.1.8]. Indeed, $\pi(\tau)$ is either identical to τ or a *perturbation* of τ or a *flipping* of τ, in a more general sense than those considered by Fukuda & Tamura [12].

The map π is onto because $\pi(\tau) = \tau$ for $\tau \in \mathcal{E}(I', I \setminus \{\mathbf{c}_0\})$. To conclude that π is a homotopy equivalence, we factor $\pi = \hat{\pi} \circ \check{\pi}$ into a *reverse perturbation* $\check{\pi}$ followed by a *perturbation* $\hat{\pi}$. These two maps are defined by

$$\check{\pi}(\tau)(Y) \quad := \quad \begin{cases} \tau(Y) & \text{for } [Y] \in I \setminus \{\mathbf{c}_0\}, \\ 0 & \text{for } [Y] = \mathbf{c}_0 \text{ and } \tau(Y) \in \{-, 0\}, \\ + & \text{otherwise,} \end{cases}$$

$$\hat{\pi}(\tau)(Y) := \begin{cases} \tau(Y) & \text{for } [Y] \in I \setminus \{\mathbf{c}_0\}, \\ + & \text{otherwise.} \end{cases}$$

for all cocircuits Y with $Y_g = +$. Note that $\hat{\pi}$ is the restriction of π to the image of $\check{\pi}$.

These maps are order preserving, and they satisfy $\check{\pi}\check{\pi}(\tau) = \check{\pi}(\tau) \leq \tau$, and $\hat{\pi}\hat{\pi}(\tau') = \hat{\pi}(\tau') \geq \tau'$ for all $\tau \in \mathcal{E}(I', I)$ and $\tau' = \check{\pi}(\tau)$. In this situation we can apply Lemma 3.3. This shows that both $\check{\pi}$ and $\hat{\pi}$ are homotopy equivalences, and hence so is π. □

Theorem 3.11. *Let $(\widetilde{\mathcal{M}}, g, f)$ be any oriented matroid program, and let ν be the number of order ideals in its poset* VSC *of very strong components. Then $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ is homotopy equivalent to a discrete set of ν points.*

Proof. Let $\mathcal{J}(\texttt{VSC})$ denote the discrete set of order ideals of VSC. By Lemma 3.6(ii) there is a map $i : \mathcal{E}(\widetilde{\mathcal{M}}, g, f) \to \mathcal{J}(\texttt{VSC})$ which associates with every localization $\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ the order ideal $i(\sigma) = \{\mathbf{c} \in \texttt{VSC} : \sigma(\mathbf{c}) = -\}$.

Consider any order ideal K in VSC, and let I' denote the order ideal it generates in SC. Let I be the set of all $\mathbf{c}$ in SC such that whenever $\mathbf{c}' \in \texttt{VSC}$ and $\mathbf{c}' \leq \mathbf{c}$ in SC then $\mathbf{c}' \in K$. Note that I and I' are order ideals in SC with $I' \subseteq I$. It follows from the definitions that

$$\mathcal{E}(I', I) \quad = \quad \{\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f) : i(\sigma) = K\} = i^{-1}(K). \tag{3.7}$$

Moreover, no element $\mathbf{c}_0$ in $I \setminus I'$ lies in VSC. Thus by repeated application of Lemma 3.10 we find $\mathcal{E}(I', I) \simeq \mathcal{E}(I', I') = \{\text{single point}\}$.

We have shown that the inverse image under i of each $K \in \mathcal{J}(\texttt{VSC})$ is a contractible space. This completes the proof of Theorem 3.11. □

Corollary 3.12. *The space $\mathcal{E}(I',I)$ is contractible if and only if $I\backslash I'$ does not contain a very strong component. In particular, the space $\mathcal{E}(\widetilde{\mathcal{M}},g,f)$ is contractible if and only if* $\texttt{VSC} = \emptyset$*, that is, if and only if the oriented matroid program $(\widetilde{\mathcal{M}},g,f)$ is euclidean.*

Proof. The first part follows from Lemma 3.10, analogous to the proof of Theorem 3.11. The second part is obtained by specializing to $I' = \emptyset$ and $I = \texttt{SC}$, using the characterization of euclideanness in Proposition 3.9. □

We close this section by showing that the extension space $\mathcal{E}(\widetilde{\mathcal{M}},g,f)$ of an oriented matroid program can have an arbitrary number of connected components.

Theorem 3.13. *For every $t \geq 1$ there exists a rank 4 oriented matroid program on $n = 3t+5$ elements whose poset* VSC *is a chain of t elements.*

Proof. We construct a series of examples whose first case ($t = 1$) is the program $(\widetilde{\mathcal{M}'},g,f)$ in Example 3.5. Again, we apply Mnëv's technique for generating cycles in G_f.

For $t \geq 1$ and $\epsilon < \frac{1}{4t}$ we define an arrangement of $3t+3$ affine halfspaces in $\mathbb{R}^3$ by

$$\begin{array}{rrrcll}
 & y & & \geq & 0 & (1)\\
-x & & & \geq & -1 & (2)\\
 & -y & +z & \geq & 4k & (3_k)\\
x & & -z & \geq & -4k & (4_k)\\
x & -y & & \leq & 1+\epsilon & (5)\\
(1+\epsilon)x & -(1+2\epsilon^2)y & +\epsilon^2 z & \leq & 1+2\epsilon+(4k+1)\epsilon^2+\epsilon^3 & (6_k)
\end{array}$$

where $k = 0,1,\ldots,t-1$. By adding the plane at infinity g and a suitable level plane of the linear functional $f(x,y,z) := x+y+z$, we get an oriented matroid program of rank 4 on the set $\{f,g\} \cup \{1,2,3_k,4_k,5,6_k : 0 \leq k < t\}$.

The quadruples of halfspaces $[1,2,3_k,4_k]$ describe tetrahedra in the affine arrangement. All vertices in the affine arrangement are simple, i.e., determined by three halfspaces. For each $k \in \{0,1,\ldots,t-1\}$, we have the following sequence of vertices (cocircuits). Each vertex v_i^k is listed with a triple of halfspaces defining it and with the value of the linear functional $f = x+y+z$ on v_i^k:

	vertex	basis	$f =$
$v_0^k =$	$(0,0,4k)$	$[1,3_k,4_k]$	$4k$
$v_1^k =$	$(1,0,4k)$	$[1,3_k,2]$	$4k+1$
$v_2^k =$	$(1+\epsilon,0,4k)$	$[1,3_k,5]$	$4k+1+\epsilon$
$v_3^k =$	$(1+\epsilon,0,4k+\epsilon)$	$[1,6_k,5]$	$4k+1+2\epsilon$
$v_4^k =$	$(1,-\epsilon,4k+1-\epsilon)$	$[2,6_k,5]$	$4k+2-2\epsilon$
$v_5^k =$	$(1,-\epsilon,4k+1)$	$[2,4_k,5]$	$4k+2-\epsilon$
$v_6^k =$	$(1,0,4k+1)$	$[2,4_k,1]$	$4k+2$
$v_7^k =$	$(1,1,4k+1)$	$[2,4_k,3_k]$	$4k+3$

This sequence is a directed path in G_f and hence an increasing chain in SC. The first two and the last two vertices are on the tetrahedron $[1,2,3_k,4_k]$, the intermediate ones are not.

We can define a new oriented matroid program by inverting the orientation of the t bases $[123_k4_k]$. This operation introduces t directed cycles in the graph of the program, because it reverses the directed edge from v_1^k to v_6^k, but preserves the other edges. This directed cycle gives rise to a very strong component $\mathbf{c}_k$.

If we perform the inversions by local surgery on the pseudoarrangement, then the local deformations for the k-th mutation occur within the level set $4k-\frac{1}{2} < f(x,y,z) < 4k+3+\frac{1}{2}$. So there is a directed path from $\mathbf{c}_k$ to $\mathbf{c}_{k+1}$, namely, along the pseudoline determined by $\{1,2\}$. There is no directed path backwards, which means the very strong components $\mathbf{c}_k$ and $\mathbf{c}_{k+1}$ are really distinct. This completes the proof. □

We remark that all oriented matroids in this construction can be perturbed into general position, so Theorem 3.13 remains valid for uniform programs.

4. Sphericity results

Theorem 1.2 states that the extension space $\mathcal{E}(\mathcal{M})$ of every strongly euclidean rank r oriented matroid has the homotopy type of the $(r-1)$-sphere. Here we prove this theorem, and we derive combinatorial criteria for strong euclideanness. In particular, we show that each oriented matroid of rank $r \leq 3$ or corank $n-r \leq 2$ is strongly euclidean and hence has spherical extension space.

The extension posets in Definition 3.4 satisfy the inclusions $\mathcal{E}(\widetilde{\mathcal{M}},g,f) \subseteq \mathcal{E}(\mathcal{M},g) \subseteq \mathcal{E}(\mathcal{M})$. Let $[g^+]$ denote the (localization of $\mathcal{M}$ defining the) extension parallel to g, and let $[g^-]$ denote the extension antiparallel to g. Note that $\mathcal{E}(\mathcal{M})\setminus\mathcal{E}(\mathcal{M},g) = \{[g^+],[g^-]\}$. We will consider the subposet $\mathcal{E}(\mathcal{M})_{>[g^+]} = \{\sigma \in \mathcal{E}(\mathcal{M}) : \sigma > [g^+]\} \subseteq \mathcal{E}(\mathcal{M},g)$. In the interpretation of $(\mathcal{M},g)$ as an affine pseudoarrangement, $\mathcal{E}(\mathcal{M})_{>[g^+]}$ is the set of "extensions at the horizon" for which all the affine vertices are on the positive side. Similarly, we consider $\mathcal{E}(\mathcal{M})_{>[g^-]} = \{\sigma \in \mathcal{E}(\mathcal{M}) : \sigma > [g^-]\}$.

Lemma 4.1. *The posets $\mathcal{E}(\mathcal{M})_{>[g^+]}$ and $\mathcal{E}(\mathcal{M})_{>[g^-]}$ are isomorphic to $\mathcal{E}(\mathcal{M}/g)$.*

Proof. We define injective maps $\kappa^+,\kappa^- : \mathcal{E}(\mathcal{M}/g) \hookrightarrow \mathcal{E}(\mathcal{M})$ by

$$\kappa^{\pm}(\rho)(Y) \quad := \quad \begin{cases} \pm Y_g & \text{if } Y_g \neq 0, \\ \rho(Y) & \text{otherwise.} \end{cases}$$

By definition, $\mathcal{E}(\mathcal{M})_{>[g^+]}$ is the image of κ^+ and $\mathcal{E}(\mathcal{M})_{>[g^-]}$ is the image of κ^-. □

Proposition 4.2.

(1) *The contraction of g in $\mathcal{M}$ induces surjective, order-preserving maps*

$$\begin{array}{llcl} \pi: & \mathcal{E}(\mathcal{M},g) & \to & \mathcal{E}(\mathcal{M}/g) \\ \Pi: & \mathcal{E}(\mathcal{M}) & \to & \{[g^+],[g^-]\} \oplus \mathcal{E}(\mathcal{M}/g) \simeq \textit{susp}\, \mathcal{E}(\mathcal{M}/g). \end{array}$$

(2) *Suppose that $(\widetilde{\mathcal{M}},g,f)$ is euclidean for all extensions $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$. Then the maps π and Π in (1) are homotopy equivalences.*

Here "$\oplus$" denotes the ordinal sum of posets, which means that $[g^+]$ and $[g^-]$ are two incomparable elements placed below all elements of $\mathcal{E}(\mathcal{M}/g)$. Topologically this corresponds to forming the suspension over the space of $\mathcal{E}(\mathcal{M}/g)$.

Proof. The maps in (1) are defined as follows. Given any localization ρ of $\mathcal{M}$, then the localization $\pi(\rho)$ of $\mathcal{M}/g$ is the restriction of ρ to the set of cocircuits Y with $Y_g = 0$. The localization $\pi(\rho)$ is non-trivial (i.e., does not define the extension by a loop) if and only if $\rho \in \mathcal{E}(\mathcal{M}, g)$. In this case $\Pi(\rho) := \pi(\rho)$; in addition, we set $\Pi([g^+]) := [g^+]$ and $\Pi([g^-]) := [g^-]$. The maps π and Π are order preserving, and they are surjective by Lemma 4.1.

For the proof of (2) we apply Babson's Lemma 3.2. In order to verify hypothesis (i) of Lemma 3.2, we let $\rho \in \mathcal{E}(\mathcal{M}/g)$ and consider the extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ defined by $\kappa^+(\rho)$. Then the fiber $\pi^{-1}(\rho) = \{\tau \in \mathcal{E}(\mathcal{M}/g) : \pi(\tau) = \rho\}$ is the extension space $\mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ of the program $(\widetilde{\mathcal{M}}, g, f)$. This program is assumed to be euclidean, and hence $\Pi^{-1}(\rho) = \mathcal{E}(\widetilde{\mathcal{M}}, g, f)$ is contractible by Corollary 3.12.

In order to verify hypothesis (ii), we consider $\rho \in \mathcal{E}(\mathcal{M}/g)$ and $\tau \in \mathcal{E}(\mathcal{M}, g)$ with $\pi(\tau) < \rho$. Let $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ as in the previous paragraph, and let SC be the poset of strong components of $(\widetilde{\mathcal{M}}, g, f)$. By Lemma 3.6(ii), the sets

$$I' := \{\mathbf{c} \in \mathrm{SC} : \tau(\mathbf{c}) = -\} \quad \text{and} \quad I := \{\mathbf{c} \in \mathrm{SC} : \tau(\mathbf{c}) \neq +\}$$

are order ideals in SC satisfying $I' \subseteq I$. We find that

$$\begin{aligned} \pi^{-1}(\rho) \cap \mathcal{E}(\mathcal{M}, g)_{\geq \tau} &= \{\sigma \in \mathcal{E}(\mathcal{M}, g) : \pi(\sigma) = \rho,\ \sigma \geq \tau\} \\ &= \{\sigma \in \mathcal{E}(\widetilde{\mathcal{M}}, g, f) : \sigma \geq \tau\} \quad = \quad \mathcal{E}(I', I). \end{aligned}$$

This poset is contractible by Corollary 3.12. This proves that π is a homotopy equivalence.

In order to show the same result for Π, we note that the hypothesis (i) of Lemma 3.2 follows from hypothesis (i) for π and $\Pi^{-1}([g^+]) = \{[g^+]\}$ and $\Pi^{-1}([g^-]) = \{[g^-]\}$. For hypothesis (ii), we only need to check the case $\rho > \pi(\tau) = [g^{\pm}]$. In this case we have $\tau = [g^{\pm}]$ and $\rho \in \mathcal{E}(\mathcal{M})_{>[g^{\pm}]}$, and therefore

$$\pi^{-1}(\rho) \cap \mathcal{E}(\mathcal{M})_{\geq [g^{\pm}]} = \{\kappa^{\pm}(\rho)\}$$

is a single point, by Lemma 4.1. □

Proof of Theorem 1.2. Let $\mathcal{M}$ be a strongly euclidean oriented matroid of rank r. In order to show $\mathcal{E}(\mathcal{M}) \simeq S^{r-1}$, we proceed by induction on r. If $r \leq 2$, then $\mathcal{E}(\mathcal{M})$ is homeomorphic to the $(r-1)$-sphere. Suppose that $r \geq 3$. By strong euclideanness there exists an element $g \in E$ such that

(a) $\mathcal{M}/g$ is strongly euclidean, and

(b) for every extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$, the program $(\widetilde{\mathcal{M}}, g, f)$ is euclidean.

By (b) and Proposition 4.2(2), the map Π is a homotopy equivalence between $\mathcal{E}(\mathcal{M})$ and the suspension of $\mathcal{E}(\mathcal{M}/g)$. By (a) and the induction hypothesis, the extension space of

$\mathcal{M}/g$ satisfies $\mathcal{E}(\mathcal{M}/g) \simeq S^{r-2}$. Hence the suspension of $\mathcal{E}(\mathcal{M}/g)$ is homotopy equivalent to the suspension of S^{r-2}, which is the $(r-1)$-sphere S^{r-1}. □

The remainder of this section deals with explicit classes of oriented matroids to which Theorem 1.2 can be applied. We first collect some facts about the structure of minor-minimal non-euclidean programs. Recall that a *minor* of an oriented matroid program $(\mathcal{M}, g, f)$ is a program $(\mathcal{M}', g, f)$, where $\mathcal{M}'$ is a minor of $\mathcal{M}$ in which g is not a loop and f is not a coloop. The *dual* of $(\mathcal{M}, g, f)$ is the program $(\mathcal{M}^*, f, g)$.

The class of euclidean oriented matroid programs is closed under minors and duality [10, p. 276], [9, Thm. 13.14], [6, Sect. 10.5]. In particular, every non-euclidean program has a minor $(\mathcal{M}, g, f)$ which is *minimally non-euclidean.* This means that $(\mathcal{M}, g, f)$ is not euclidean, but every proper minor of $(\mathcal{M}, g, f)$ is euclidean. A program $(\mathcal{M}, g, f)$ is minimally non-euclidean if and only if its dual program $(\mathcal{M}^*, f, g)$ is minimally non-euclidean.

Lemma 4.3. *Let $(\mathcal{M}, g, f)$ be a minimally non-euclidean program. Then*
(i) every circuit or cocircuit of $\mathcal{M}$ has size at least 3.
(ii) every circuit or cocircuit that contains both f and g has size at least 4.

Proof. We only have to treat the case of circuits, by duality. Clearly a minimally non-euclidean program cannot contain a loop. The set $\{f, g\}$ cannot be a circuit, otherwise all edges in G_f are undirected and $(\mathcal{M}, g, f)$ is euclidean. If $h \notin \{f, g\}$ is contained in a 2-element circuit, then the programs $(\mathcal{M}, g, f)$ and $(\mathcal{M}\backslash h, g, f)$ have the same graph G_f, and hence $(\mathcal{M}\backslash h, g, f)$ is also non-euclidean. If $\{f, g, h\}$ is a circuit, then $(\mathcal{M}\backslash h, g, f)$ is non-euclidean as well by [6, Prop. 10.5.10]. □

Recall that a pair of elements $e, e' \in E$ forms an *inseparable pair* in $\mathcal{M}$ if they are either *covariant* (that is, have the same sign in every circuit that contains both) or *contravariant* (that is, have the same sign in every cocircuit that contains both). An inseparable pair is inseparable also in the dual and in every minor that contains it. The *inseparability graph* has the vertex set E and an edge for every inseparable pair of elements [6, Sect. 7.8]. The inseparability graph is invariant under dualization and reorientation. An element of $\mathcal{M}$ is *isolated* if it is not contained in any inseparable pair.

Proposition 4.4. *Let $(\mathcal{M}, g, f)$ be a minimally non-euclidean oriented matroid program such that the contraction $\mathcal{M}/\{g, f\}$ is uniform. Then f is isolated in $\mathcal{M}/g$.*

Proof. Let $\mathbf{C}$ be a directed cycle in G_f and suppose that f and h are contravariant in $\mathcal{M}/g$ for some element h. If all vertices of $\mathbf{C}$ lie in the pseudohyperplane S_h (that is, if $Y_h = 0$ for all $Y \in \mathbf{C}$), then we can restrict the program to S_h, and in this case $(\mathcal{M}/h, g, f)$ is non-euclidean. If no vertex of $\mathbf{C}$ lies in S_h (if $Y_h \neq 0$ for all $Y \in \mathbf{C}$), then we can delete S_h from the program, and in this case $(\mathcal{M}\backslash h, g, f)$ is non-euclidean. Hence the cycle $\mathbf{C}$ contains vertices both with $Y_h = 0$ and with $Y_h \neq 0$. We claim that the sign Y_h increases weakly along the directed cycle $\mathbf{C}$ in the linear order "$- \prec 0 \prec +$", which leads to a contradiction. To prove this, we consider edges $W = (Y, Y')$ on which Y_h is not constant, that is, $Y_h = 0$ and $Y'_h \neq 0$ or $Y'_h = 0$ and $Y_h \neq 0$.

Given W, choose a basis $e_1, \dots, e_{r-2}$ of $E \backslash (\underline{Y} \cup \underline{Y'})$ such that $S_{e_1} \cap \dots \cap S_{e_{r-2}}$ is the pseudoline containing the edge W. The *direction determined by W* is the unique cocircuit at infinity $Z \in S_g$ which satisfies $Z_g = 0$ and $Y \circ Z = Y \circ Y'$ [6, Lemma 10.1.15]. In the arrangement we have $S_g \cap S_{e_1} \cap \dots \cap S_{e_{r-2}} = \{Z, -Z\}$.

Assume that W is undirected, i.e., $Z_f = 0$. Since $\mathcal{M}/\{g, f\}$ is uniform of rank $r-2$, every cocircuit Z with $Z_f = Z_g = 0$ satisfies $|E \backslash \underline{Z}| = r - 1$. This implies $f \in \{e_1, \dots, e_{r-2}, g\}$, where $f \neq g$, and hence $Y_f = Y'_f = 0$. This is impossible since the cycle $\mathbf{C}$ can involve only cocircuits Y with $Y_f \neq 0$.

Now assume that $W = (Y, Y')$ is directed from Y to Y'. If $Y_h = 0$ and $Y'_h \neq 0$ then $Y \circ Z = Y \circ Y'$ implies $Z_h \neq 0$, and hence $Y'_h = Z_h = Z_f = +$ since f and h are contravariant in $\mathcal{M}/g$ and $Z_g = 0$. Similarly, if $Y_h \neq 0$, $Y'_h = 0$ then $Y' \circ (-Z) = Y' \circ Y$ implies $Z_h \neq 0$, and hence $Y_h = -Z_h = -Z_f = -$ since f and h are contravariant in $\mathcal{M}/g$ and $Z_g = 0$. In both cases we have $Y_h \prec Y'_h$ in the order "$- \prec 0 \prec +$". □

We can now derive strong euclideanness results in small rank and small corank.

Corollary 4.5. *Every oriented matroid of rank at most* 3 *is strongly euclidean.*

Proof. If not, then there exists a minimally non-euclidean program $(\mathcal{M}, g, f)$ of rank at most 3. By Lemma 4.3, there is no 3-circuit $\{g, f, h\}$. This implies that $\mathcal{M}/\{f, g\}$ is uniform of rank 1. Now Proposition 4.4 implies that f is not contained in any inseparable pair of $\mathcal{M}/g$. But this is impossible since the rank of $\mathcal{M}/g$ is 2, so the inseparability graph has no isolated vertices. □

Corollary 4.5 is equivalent to the *Levi Enlargement Lemma* [17], [6, Prop. 10.5.7] with a new proof. With Theorem 2.1 it implies that the extension space of a rank 3 oriented matroid is homotopy equivalent to the 2-sphere. The same result has been obtained independently by Babson [1].

Corollary 4.6. *Every rank r oriented matroid on* $n \leq r+2$ *elements is strongly euclidean.*

Proof. In this case $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ has at most $r + 3$ elements. Thus the dual program $((\widetilde{\mathcal{M}})^*, f, g)$ has at most rank 3 and is therefore euclidean by Corollary 4.5. □

Proposition 4.7. *Let* $(\widetilde{\mathcal{M}}, g, f)$ *be a minimally non-euclidean program (with* $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$*) such that* $\mathcal{M} \backslash g$ *is uniform. Then* g *is isolated in* $\mathcal{M}$.

Proof. We use that $(\widetilde{\mathcal{M}}, g, f)$ is minimally non-euclidean if and only if its dual $((\widetilde{\mathcal{M}})^*, f, g)$ is minimally non-euclidean. Also, $\mathcal{M} \backslash g$ is uniform if and only if $(\mathcal{M} \backslash g)^* = (\widetilde{\mathcal{M}} \backslash \{g, f\})^* = (\widetilde{\mathcal{M}})^* / \{g, f\}$ is uniform. Thus we can apply Proposition 4.4 to $((\widetilde{\mathcal{M}})^*, f, g)$, which yields that g is not in an inseparable pair of $(\widetilde{\mathcal{M}})^*/f$, or, equivalently, of $((\widetilde{\mathcal{M}})^*/f)^* = \mathcal{M}$. □

Corollary 4.8. *Let $\mathcal{M}$ be a uniform oriented matroid so that no minor (on at least two elements) has an isolated element. Then $\mathcal{M}$ is strongly euclidean.*

Proof. We proceed by induction on the rank and cardinality of $\mathcal{M}$. Suppose $\mathcal{M}$ is not strongly euclidean and let g by any element of $\mathcal{M}$. By the induction hypothesis, the contraction $\mathcal{M}/g$ is strongly euclidean. Hence there exists an extension $\widetilde{\mathcal{M}} = \mathcal{M} \cup f$ such that the program $(\widetilde{\mathcal{M}}, g, f)$ is non-euclidean. Applying the induction hypothesis again, we see that $(\widetilde{\mathcal{M}}, g, f)$ must be minimally non-euclidean. Now Proposition 4.7 implies that g is isolated in $\mathcal{M}$, which is a contradiction. □

In view of Corollaries 4.5 and 4.6, rank 4 oriented matroids on 7 elements are the next smallest case of interest.

Proposition 4.9 *All uniform rank 4 oriented matroids on 7 elements are strongly euclidean, and their extension spaces are homotopy equivalent to the 3-sphere.*

Proof. By the classification of Grünbaum [14, p. 395], there are eleven isomorphism types of simple arrangements of seven lines in the projective plane, or, equivalently, eleven reorientation classes of uniform rank 3 oriented matroids on $[7] = \{1, \ldots, 7\}$. Among the eleven types there is only one arrangement $\mathcal{A}$ whose inseparability graph is empty. The arrangement $\mathcal{A}$ is characterized by the property that it has 10 triangles (mutations). We let $\mathcal{M}^*$ denote the corresponding uniform rank 3 oriented matroid on [7].

Suppose there exists a rank 4 oriented matroid on [7] which is not strongly euclidean. Then its inseparability graph must be empty by Proposition 4.7 (every uniform oriented matroid on at most seven elements is euclidean), and hence it must be reorientation equivalent to the dual $\mathcal{M}$ of $\mathcal{M}^*$. Since strong euclideanness is invariant under reorientation (both the extension space and euclideanness are invariant), it will be sufficient to prove that $\mathcal{M}$ is strongly euclidean. This is equivalent to the existence of an element $g \in [7]$ such that for all extensions $\mathcal{M} \cup f$, the program $(\mathcal{M} \cup f, g, f)$ is euclidean, because all rank 3 oriented matroid programs are strongly euclidean. Lemma 4.10 below implies that we need to consider only uniform extensions $\widetilde{\mathcal{M}}$.

By a complete enumeration using the computer algebra system MAPLE, we found that $\mathcal{M}$ has 7690 distinct uniform extensions $\widetilde{\mathcal{M}}$. A total number of 7648 extensions $\widetilde{\mathcal{M}}$ has the property that $(\widetilde{\mathcal{M}}, g, f)$ is euclidean for all $g \in [7]$. This number includes all realizable extensions of $\mathcal{M}$. All remaining 42 extensions $\widetilde{\mathcal{M}}$ have the property that $(\widetilde{\mathcal{M}}, g, f)$ is euclidean if and only if $g \in \{1, 2, 3, 4, 5, 6\}$. These computational results have been confirmed independently by J. Richter-Gebert, using the classification in [8]. We conclude that for all extensions $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, the program $(\widetilde{\mathcal{M}}, 1, f)$ is euclidean. Hence $\mathcal{M}$ is strongly euclidean. □

For the proof of Proposition 4.9 used the following perturbation lemma.

Lemma 4.10. *Let $(\widetilde{\mathcal{M}}, g, f)$ be a non-euclidean oriented matroid program of rank 4 such that $\mathcal{M} = \widetilde{\mathcal{M}} \backslash f$ is uniform. Then there exists a non-euclidean program $(\widetilde{\mathcal{M}}', g, f)$ such that $\widetilde{\mathcal{M}}'$ is uniform and $\widetilde{\mathcal{M}}$ is a weak image of $\mathcal{M}'$.*

Proof. We denote by σ_f the localization of f on the cocircuits of $\mathcal{M}$. By replacing σ_f by $\sigma_f \circ \sigma_g$ we may assume that we have a non-euclidean extension that satisfies $\sigma_f(Y) \neq 0$ whenever $Y_g \neq 0$.

Now consider a cycle $\mathbf{C}$ in G_f. If $\mathbf{C}$ contains no undirected edge, then we can arbitrarily perturb f into general position. Also, by considering a "shortest" directed cycle we can assume that $\mathbf{C}$ does not traverse any undirected pseudoline in both directions. Hence, we can perturb to a uniform non-euclidean program if we can arbitrarily direct all the undirected affine pseudolines, that is, if we can independently alter σ_f on all pairs of cocircuits $\pm Y$ with $Y_g = \sigma_f(Y) = 0$.

To see this, we use reduction to corank 2 [16], [6, Thm. 7.1.8]: it suffices to show that the sign pattern on every pseudoline of the arrangement representing $\mathcal{M}$ is consistent. Since we only switch zeroes of σ_f to non-zeroes, problems can only occur on a pseudoline for which previously all cocircuits had sign 0, i.e., on a pseudoline of Type I (cf. [6, Example 7.1.7]). From the above perturbation we know that the pseudoline is contained in the sphere at infinity S_g, so it is determined by a pair $\{g, h\}$. Since the pseudoline is of Type I we get that $\{g, h, f\}$ is a 3-circuit in $\widetilde{\mathcal{M}}$. In particular, h and f are parallel in $\widetilde{\mathcal{M}}/g$, and hence we get that $(\mathcal{M}, g, h)$ is also non-euclidean, containing the same cycle $\mathbf{C}$ in its graph as $(\widetilde{\mathcal{M}}, g, f)$. However, since $\mathcal{M}$ is uniform, we get that the graph G_h of $(\mathcal{M}, g, h)$ has no undirected edges. Hence the cycle $\mathbf{C}$ has no undirected edges. □

We next show that for any rank r and for any number n of elements there exists an oriented matroid which is strongly euclidean, so that Theorem 1.2 is applicable. The *alternating matroid* $\mathrm{C}^{n,r}$ is the uniform rank r oriented matroid of linear dependencies on the columns of the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ t_1 & t_2 & t_3 & \dots & t_n \\ t_1^2 & t_2^2 & t_3^2 & \dots & t_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1^{r-1} & t_2^{r-1} & t_3^{r-1} & \dots & t_n^{r-1} \end{pmatrix}$$

where $t_1 < t_2 < t_3 < \ldots < t_n$ are any n distinct real numbers. Thus $\mathrm{C}^{n,r}$ is the oriented matroid associated with the cyclic $(r-1)$-polytope with n vertices. We will need the following two properties of $\mathrm{C}^{n,r}$. For a detailed discussion of alternating matroids and a proof of Lemma 4.11 see [6, Sect. 9.4].

Lemma 4.11.

(a) *The inseparability graph of* $\mathrm{C}^{n,r}$ *is the* n*-cycle* 1—2—3—$\cdots$—n—1, *for* $2 \leq r \leq n-2$, *and a complete graph otherwise.*

(b) *Every minor of* $\mathrm{C}^{n,r}$ *is isomorphic to an alternating matroid.*

The following theorem is the main application of our results on the structure of minimally non-euclidean programs.

Theorem 4.12. *The alternating matroid* $C^{n,r}$ *is strongly euclidean. Hence its extension space* $\mathcal{E}(C^{n,r})$ *is homotopy equivalent to the* $(r-1)$*-sphere.*

Proof. By Lemma 4.11, every minor of $C^{n,r}$ on at least two elements has no isolated elements. Corollary 4.8 implies that $C^{n,r}$ is strongly euclidean. □

5. Remarks

A potential application of extension spaces lies in the interplay of combinatorics and (differential) topology. This interplay is highlighted by the recent combinatorial construction of Pontrjagin classes due to Gelfand & MacPherson [13]. Their work suggests in particular that the *MacPhersonian* $Mc(n,r)$ of all oriented matroids of rank r on $[n]$, ordered by weak maps, could serve as a discrete model for the real Grassmannian $G(n,r)$.

The crucial question about MacPhersonians is whether $Mc(n,r)$ and $G(n,r)$ have isomorphic cohomology. An inductive approach is to consider the projection $G(n,r)\backslash G(n-1,r-1) \to G(n-1,r)$ whose fibers are $(r-1)$-spheres. The combinatorial analogue is the deletion map $Mc(n,r)\backslash Mc(n-1,r-1) \to Mc(n-1,r)$ on the poset of rank r oriented matroids on $[n]$ for which n is not a coloop. The fibers of this map are the extension spaces $\mathcal{E}(\mathcal{M})$. A different approach to the McPhersonian problem is currently taken in Babson's work on the homotopy type and the cohomology of the MacPhersonian [1]. In this context our extension spaces $\mathcal{E}(\mathcal{M})$ appear as *flag spaces* $Mc(\mathcal{M}, r-1)$.

In spite of the results in this article, it remains an open problem whether the extension space $\mathcal{E}(\mathcal{M})$ is connected for every oriented matroid $\mathcal{M}$. We only know that $\mathcal{E}(\mathcal{M})$ always contains a large connected subspace $\mathcal{E}(\mathcal{M})_{real}$ by Theorem 1.1, and that it is spherical (and hence connected) for $r \leq 3$ by Corollary 4.5. It would be worthwhile to investigate the scope of the methods developed in this paper. For instance, a challenging problem is to construct an explicit oriented matroid which is not strongly euclidean.

The extension poset $\mathcal{E}(C^{n,r})$ of the alternating matroid is closely related to the *higher Bruhat order* $B(n, n-r)$, which was introduced by Manin & Schechtman [18], and further studied by Kapranov & Voevodsky [15, Sect. 4]. It turns out that $B(n, n-r)$ is isomorphic to a natural partial order on the subset $\mathcal{E}(C^{n,r})_{uni}$ of uniform extensions of $C^{n,r}$. This connection will be developed in a forthcoming paper.

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