BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 4, July 1975

EXTENSION THEOREMS FOR REDUCTIVE GROUP ACTIONS ON COMPACT KAEHLER MANIFOLDS

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Communicated by Stephen S. Shatz, March 13, 1975

Let G be a connected complex reductive Lie group. Noting [3], [7], [8] that G has the structure of a linear algebraic group, let $\bar{G}$ be any projective manifold in which G is Zariski open and which induces the above algebraic structure on G. The purpose of the present note is to announce

PROPOSITION I. *Let G be as above and act holomorphically on a compact Kaehler manifold X. Assume that the Lie algebra of holomorphic vector fields on X generated by G is annihilated by every holomorphic one form. Let $\Phi: Y \longrightarrow X$ be a holomorphic map where Y is a normal reduced analytic space. Consider the equivariant map $\Phi': G \times Y \longrightarrow X$; Φ' extends meromorphically (in the sense of Remmert) to $\bar{G} \times Y$.*

REMARKS. The condition on vector fields annihilated by one forms is automatically satisfied if (cf. [12]–[14]) $H^1(X, \mathbf{Q}) = 0$, or G is semisimple, or if every generator of the solvable radical of G has a fixed point, or if G is a linear algebraic group acting algebraically on a projective X. Taking Y to be a point, one gets the orbits of G to be Zariski open in their closures which are analytic sets. A simple corollary is the classical result that there is only one structure of a linear algebraic group on G (cf. [7]), and in fact any reductive connected subgroup of an algebraic group over $\mathbf{C}$ is an algebraic subgroup.

As a further application of the techniques used, a new proof of an improved form of a fixed point theorem (cf. [12], [13], [14]) of the author is given:

PROPOSITION II. *Let S be a connected solvable Lie group acting holomorphically on a compact Kaehler manifold X. The following are equivalent:*

(a) *S has a fixed point on X.*

AMS (MOS) subject classifications (1970). Primary 32M05, 32J25, 53C55; Secondary 22E10.

Key words and phrases. Reductive group actions, Kaehler manifolds, linear algebraic groups.

(b) *S leaves a compact set in a fibre of the Albanese map invariant.*

(c) *S has a fixed point within any compact set K on X that S leaves invariant.*

(d) *The Lie algebra of vector-fields that S generates on X is annihilated by every holomorphic one form on X.*

REMARK. The assertion (c) where K is any compact set is new; the method of proof allows one to relax the compactness of X and show if, in addition, $H^1(X, \mathcal{O}_X) = 0$, then (c) is true.

The following is the fundamental observation on which everything rests.

LEMMA. *Let X be a compact Kaehler manifold and* $\rho\colon \mathbf{C}^* \longrightarrow \mathrm{Aut}(X)$ *a holomorphic* $\mathbf{C}^*$ *action that has at least one fixed point. Let* $A\colon \mathbf{C}^* \longrightarrow X$ *be a holomorphic equivariant map onto an orbit: then A extends to a homomorphic equivariant map* $\widetilde{A}$ *of* $\mathbf{C}P^1$ *to X.*

PROOF. Assume without loss of generality that $A(\mathbf{C}^*)$ is not a point. Let μ be a Kaehler metric on X and ω the associated Kaehler form. Assume that μ has been averaged with respect to the circle subgroup $S^1 \subseteq \mathbf{C}^*$. Let χ be the holomorphic vector-field on X associated to $\rho\colon \mathbf{C}^* \longrightarrow \mathrm{Aut}(X)$.

Because of equivariance, the Jacobian, dA, of A, maps some constant multiple of $z(\partial/\partial z)$ onto the restriction of the vector-field χ to $A(\mathbf{C}^*)$. Without loss of generality this constant is assumed to be one.

Let $A^*\mu = a(r)\, dz \otimes d\bar{z}$ where $a(r)$ is positive and depends only on r due to the S^1 averaging of μ. $A^*\omega = (i/2)a(r)\, dz \wedge d\bar{z}$.

$$\mu(\chi, \chi) = \mu\left(dA\left(z\frac{\partial}{\partial z}\right), dA\left(z\frac{\partial}{\partial z}\right)\right) = A^*\mu\left(z\frac{\partial}{\partial z}, z\frac{\partial}{\partial z}\right)$$

$$= a(r)|z|^2 \leqslant M < \infty$$

where $\sup_X \mu(\chi, \chi) = M < \infty$.

Now by Lichnerowicz [5] there exists a C^∞ function ϕ on X such that $\bar{\partial}\phi = \omega(\chi)$. Pulling back and, without confusion, letting ϕ stand for $A^*\phi = \phi(A(z))$, one has

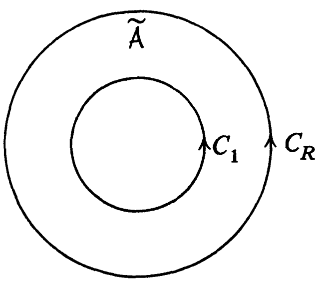
$$\frac{i}{2} z\, a(r)\, d\bar{z} = \frac{\partial\phi}{\partial\bar{z}}\, d\bar{z} \quad \text{or} \quad \frac{i}{2} z\, a(r) = \frac{\partial\phi}{\partial\bar{z}}.$$

Now fix one circle, say the unit circle $C_1 \subset \mathbf{C}^*$ and let $C_R = \{z \in \mathbf{C}^* \mid |z| = R\}$. Assume $R > 1$; C_1 and C_R bound an annulus $\widetilde{A}$ with $\partial\widetilde{A} = C_R - C_1$. Now

$$\iint_{\tilde{A}} A^*\mu = \iint_{\tilde{A}} \frac{i}{2} a(r)\, dz \wedge d\bar{z} = -\iint_{\tilde{A}} \frac{\partial\phi}{\partial\bar{z}} \frac{d\bar{z} \wedge dz}{z}$$

$$= -\int_{C_R} \phi \frac{dz}{z} + \int_{C_1} \phi \frac{dz}{z} = \frac{1}{i}\int_0^{2\pi} \phi(Re^{i\theta})\, d\theta - C$$

with C a constant. Now $|\int_0^{2\pi} \phi(Re^{i\theta})\, d\theta| \leq M' < \infty$ since ϕ is the pullback of a bounded function on X.



Therefore $\iint_{\tilde{A}} A^*\mu \leq M'' < \infty$ where M'' is a positive constant independent of R. Thus by Bishop's extension theorem (cf. [1], [2]), A extends holomorphically over ∞. An identical argument gives extension at 0. Q.E.D.

Using the above Lemma and the Levi-Griffiths-Shiffman-Siu extension theorem (cf. [2], [9], [10], [11]) repeatedly, one proves the result for SL(2, **C**) and groups of the form $(\mathbf{C}^*)^n$ that have a fixed point on X. Then one proves it for one parameter unipotent subgroup of G by using the above SL(2, **C**) result on an SL(2, **C**) in G containing the subgroup; this can be done by Jacobson-Morosow (cf. [4]). One now proves it for a Borel subgroup of G and uses an argument depending on the fact that one has a locally trivial fibring of G over G/B which is compact.

In the very interesting paper [6] of Lieberman, related matters are discussed.

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