EXTENSION THEORY FOR CONNECTED HOPF ALGEBRAS

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1. **Introduction.** Let K be a fixed commutative ring with unit. We will deal with graded algebras, coalgebras, and Hopf algebras over K as defined in Milnor-Moore [4], but we assume that the underlying K-modules are connected.

Suppose A, B are Hopf algebras; A commutative and B cocommutative. By an extension of B by A we mean a diagram of Hopf algebras and Hopf maps

$$A \xrightarrow{\alpha} C \xrightarrow{\beta} B \tag{1.1}$$

in which C is isomorphic to $A \otimes B$ simultaneously as a left A-module and right B-comodule. In this paper we announce results which describe and classify all extensions by B by A. Proofs will appear in [5].

2. **Matched pairs.** If B is an algebra we write

$$\eta : K \to B, \qquad \mu_B : B \otimes B \to B$$

for the unit and multiplication, respectively. If A is a coalgebra we write

$$\epsilon : A \to K, \qquad \psi_A : A \to A \otimes A$$

for the counit and comultiplication.

As the first step in classifying extensions, we will show in [5] how a diagram (1.1) determines a pair of K-linear maps

$$\sigma_A : B \otimes A \to A, \qquad \rho_B : B \to A.$$

σ_A is the "action" of base on fiber that one expects in an extension problem; ρ_B is its dual. We prove:

(a) σ_A gives A the structure of a left B-module algebra;

(b) ρ_B gives B the structure of a right A-comodule coalgebra;

(c) the diagram commutes:

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$$
\text{(2.1)}\qquad
\begin{array}{ccc}
B \otimes A & \xrightarrow{\sigma_A} A \xrightarrow{\psi_A} & A \otimes A \\
\downarrow{\scriptstyle \psi_B \otimes \psi_A} & & \uparrow \\
B \otimes B \otimes A \otimes A & & \\
\downarrow{\scriptstyle \rho_B \otimes B \otimes \mathrm{A} \otimes \mathrm{A}} & & {\scriptstyle A \otimes \mu_A} \\
B \otimes A \otimes B \otimes A \otimes A & & \\
\downarrow{\scriptstyle (1,4,2,3,5)} & & \\
B \otimes A \otimes A \otimes B \otimes A & \xrightarrow{\sigma_A \otimes \mathrm{A} \otimes \sigma_A} & A \otimes A \otimes A
\end{array}
$$

(d) the diagram commutes:

$$
\text{(2.2)}\qquad
\begin{array}{ccc}
B \otimes B & \xrightarrow{\mu_B} B \xrightarrow{\rho_B} & B \otimes A \\
\downarrow{\scriptstyle \psi_B \otimes B} & & \uparrow \\
B \otimes B \otimes B & & \\
\downarrow{\scriptstyle \rho_B \otimes \mathrm{B} \otimes \rho_B} & & {\scriptstyle \mu_B \otimes \mu_A} \\
B \otimes A \otimes B \otimes B \otimes A & & \\
\downarrow{\scriptstyle (1,4,2,3,5)} & & \\
B \otimes B \otimes A \otimes B \otimes A & \xrightarrow{B \otimes B \otimes A \otimes \sigma_A} & B \otimes B \otimes A \otimes A
\end{array}
$$

Conversely, suppose given a pair of Hopf algebras (A, B) with A commutative and B cocommutative, and suppose given K-linear maps σ_A, ρ_B satisfying conditions (a)–(d). Then we call (A, B) a "matched pair".

3. Bimodules over a matched pair. Suppose (A, B) is a matched pair. Suppose N is simultaneously a left B-module under $\sigma_N: B \otimes N \to N$, and a right A-comodule under

$$\rho_N: N \to N \otimes A.$$

Then we call N an (A, B)-bimodule if the diagram commutes:

$$
\begin{array}{ccccc}
B \otimes N & \xrightarrow{\sigma_N} & N & \xrightarrow{\rho_N} & N \otimes A \\
\downarrow{\scriptstyle \psi_B \otimes N} & & & & \\
B \otimes B \otimes N & & & & \uparrow{\scriptstyle N \otimes \mu_A} \\
\downarrow{\scriptstyle \rho_B \otimes B \otimes \rho_N} & & & & \\
B \otimes A \otimes B \otimes N \otimes A & & & & \\
\downarrow{\scriptstyle (1,4,2,3,5)} & & & & \\
B \otimes N \otimes A \otimes B \otimes A & & \xrightarrow{\sigma_N \otimes A \otimes \sigma_A} & & N \otimes A \otimes A
\end{array}
\tag{3.1}
$$

For example, if (A, B) is a matched pair we can give A the structure of an (A, B)-bimodule, with left B-action $\sigma_A : B \otimes A \to A$, and right A-coaction $\psi_A : A \to A \otimes A$. A dual construction makes B into an (A, B)-bimodule. We say $f: M \to N$ is a map of (A, B)-bimodules if f is simultaneously a map of left B-modules and right A-comodules.

The interpretation of diagrams (2.1), (2.2), (3.1) is found in:

THEOREM 3.1. *Let N be a bimodule over the matched pair (A, B). Let the map $\tilde{\sigma}_{N \otimes A} : B \otimes N \otimes A \to N \otimes A$ be the composition*

$$(N \otimes \mu_A)(\sigma_N \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5)(\rho_B \otimes B \otimes N \otimes A)(\psi_B \otimes N \otimes A).$$

Let the map $\tilde{\rho}_{N \otimes A} : N \otimes A \to N \otimes A \otimes A$ be $N \otimes \psi_A$. Then:

(a) *$\tilde{\sigma}_{N \otimes A}$ gives $N \otimes A$ the structure of a left B-module;*

(b) *$\tilde{\rho}_{N \otimes A}$ gives $N \otimes A$ the structure of a right A-comodule;*

(c) *with these structure maps $N \otimes A$ is in fact an (A, B) bimodule which we denote $N \tilde{\otimes} A$;*

(d) *$\rho_N : N \to N \tilde{\otimes} A$ is a map of (A, B)-bimodules.*

Theorem 3.1 has a dual. Given an (A, B)-bimodule M, Theorem (3.1)* tells how to give $B \otimes M$ the structure of an (A, B)-bimodule, denoted $B \underset{\sim}{\otimes} M$, in such a way that $\sigma_M : B \underset{\sim}{\otimes} M \to M$ is a map of (A, B)-bimodules.

4. (A, B)-**algebras and** (A, B)-**coalgebras.** By an (A, B)-algebra we mean an (A, B)-bimodule N that is also a commutative algebra, in such a way that $\mu_N : N \otimes N \to N$ is both a map of left B-modules and right A-comodules. (A, B)-coalgebras are defined similarly. For example if (A, B) is a matched pair, then A itself is an (A, B)-algebra, and B is an (A, B)-coalgebra.

THEOREM 4.1. *Let N be an (A, B)-algebra. Let $N \tilde{\otimes} A$ have the algebra structure of the tensor product $N \otimes A$. Then $N \tilde{\otimes} A$ is an (A, B)-algebra, and $\rho_N : N \to N \tilde{\otimes} A$ is a map of (A, B)-algebras.*

Theorem 4.1 can be interpreted in the language of "triples" [1], [2]. Let Γ be the category of (A, B)-algebras. Let $S:\Gamma\to\Gamma$ be the functor which carries N to $N\tilde{\otimes}A$, and let $I:\Gamma\to\Gamma$ be the identity functor. Define functor transforms $\delta:I\to S$, $\sigma:S^2\to S$ by

$$\delta(N)=\rho_N:N\to N\tilde{\otimes}A;\quad \sigma(N)=N\otimes\epsilon\otimes A:N\tilde{\otimes}A\tilde{\otimes}A\to N\tilde{\otimes}A.$$

Then $V\equiv(S, \delta, \sigma)$ is a triple on the category Γ.

Similarly, the dual of Theorem 4.1 gives a cotriple $W\equiv(T, d, s)$ on the category Δ of (A, B)-coalgebras. Here $T(M)=B\underset{\sim}{\otimes}M$, $d(M)=\sigma_M:B\underset{\sim}{\otimes}M\to M$, and $S(M)=B\otimes\eta\otimes M:B\underset{\sim}{\otimes}M\to B\underset{\sim}{\otimes}B\underset{\sim}{\otimes}M$.

5. **The cohomology of matched pairs.** If ϕ is any category, write $S^*(\phi)$ for the category of cosimplicial objects over ϕ, and $S_*(\phi)$ for the category of simplicial objects over ϕ. Then the triple V of §4 gives rise in the usual way [1], [2] to a functor $V:\Gamma\to S^*\Gamma$; the cotriple W gives rise to a functor $W:\Delta\to S_*\Delta$. For example, if (A, B) is a matched pair, then $V(K)$ is the acyclic cobar construction on A, but it has some structure not present in the classical case . . . an action of B compatible with the coface operators. Similarly, $W(K)$ is the acyclic bar construction on B, with an A-coaction added.

If M is an (A, B)-coalgebra and N an (A, B)-algebra, let $\mathrm{Hom}_{(A,B)}(M, N)$ denote the set of maps of (A, B)-modules $f:M\to N$ for which $f_0:M_0\to N_0$ is the identity on K. Then $\mathrm{Hom}_{(A,B)}(M, N)$ is an abelian group under the composition law $f+g=\mu_N(f\otimes g)\psi_M$.

Now to any matched pair (A, B) we associate a double cosimplicial abelian group $X(B, A)$ by setting:

$$X^{p,q}(B, A) = \underset{(A,B)}{\mathrm{Hom}}\,(W(K)_p, V(K)^q). \tag{5.1}$$

Let $\overline{X}(B, A)$ be the associated "total" cochain complex. Then we define the cohomology of the matched pair (A, B) by:

$$\overline{H}^*(B, A) = \overline{H}^*(\overline{X}(B, A)). \tag{5.2}$$

6. **Classification of extensions.** Let (A, B) be a matched pair under (σ_A, ρ_B). Denote by $\mathrm{Opext}(B, A)$ the set of equivalence classes of extensions (1.1) which give rise to the given "matching." Our main result is:

THEOREM 6.1. *There is a natural isomorphism:*

$$\overline{H}^3(B, A) = \mathrm{Opext}\,(B, A).$$

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