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Extension theory of infinite symmetric products

by

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Abstract. We present an approach to cohomological dimension theory based on infinite symmetric products and on the general theory of dimension called the extension dimension. The notion of the extension dimension ext-dim(X) was introduced by A. N. Dranishnikov [9] in the context of compact spaces and CW complexes. This paper investigates extension types of infinite symmetric products $\mathrm{SP}(L)$. One of the main ideas of the paper is to treat ext-dim$(X) \leq \mathrm{SP}(L)$ as the fundamental concept of cohomological dimension theory instead of $\dim_G(X) \leq n$. In a subsequent paper [18] we show how properties of infinite symmetric products lead naturally to a calculus of graded groups which implies most of the classical results on the cohomological dimension. The basic notion in [18] is that of homological dimension of a graded group which allows for simultaneous treatment of cohomological dimension of compacta and extension properties of CW complexes.

We introduce cohomology of X with respect to L (defined as homotopy groups of the function space $\mathrm{SP}(L)^X$). As an application of our results we characterize all countable groups G so that the Moore space $M(G,n)$ is of the same extension type as the Eilenberg–MacLane space $K(G,n)$. Another application is a characterization of infinite symmetric products of the same extension type as a compact (or finite-dimensional and countable) CW complex.

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1. Introduction

Notation 1.1. Throughout the paper K, L, and M are reserved for CW complexes. X and Y are general topological spaces (quite often compact or compact metrizable). We will frequently omit coefficients in the case of integral homology and cohomology. Thus, $H_n(K;\mathbb{Z})$ will be shortened to $H_n(K)$, and $H^n(X;\mathbb{Z})$ to $H^n(X)$.

Recall that K is an *absolute extensor* of X (denoted by $K \in \mathrm{AE}(X)$) if any map $f : A \to K$, A closed in X, extends over X. That concept creates a partial order on the class of CW complexes. Namely, $K \le L$ if $K \in \mathrm{AE}(X)$ implies $L \in \mathrm{AE}(X)$ for every compact space X. The partial order induces an equivalence relation on the class of all CW complexes. The equivalence class $[K]$ of K is called its *extension type*.

Definition 1.2. A CW complex K is called the *extension dimension* of a compact space X (notation: $K = \text{ext-dim}(X)$) if K is a minimum of the class $\{L \mid L \in \mathrm{AE}(X)\}$.

Theorem 1.3 (Dranishnikov Duality Theorem [9]). *Extension dimension of compact spaces exists and for each CW complex K there exists a compact space X such that $K = \text{ext-dim}(X)$.*

The concept of extension dimension generalizes both the covering dimension $\dim(X)$ and the cohomological dimension $\dim_G(X)$ with respect to an Abelian group G. Indeed, $\dim(X) \le n$ is equivalent to $\text{ext-dim}(X) \le S^n$, and $\dim_G(X) \le n$ is equivalent to $\text{ext-dim}(X) \le K(G,n)$.

The theory of extension dimension is mostly geometric in nature (see Section 2). We introduce algebra to it following the basic idea of [1], where algebraic topology is outlined via properties of infinite symmetric products $\mathrm{SP}(K)$. Thus, in this paper we show that the relation $\mathrm{SP}(K) \le \mathrm{SP}(L)$ is of purely algebraic nature. We analyze it by generalizing the connectivity index of Shchepin [28] to the concept of homological dimension of CW complexes. To analyze the relation $\text{ext-dim}(X) \le \mathrm{SP}(L)$ we introduce the concept of cohomology groups $H^*(X;L)$ of X with coefficients in a CW complex L (see Section 4). Those cohomology groups have natural formulae facilitating proofs and applications. We show in Section 6 that the class $\{\mathrm{SP}(L) \mid \text{ext-dim}(X) \le \mathrm{SP}(L)\}$ has a minimum which should be interpreted as the coefficient-free cohomological dimension of X.

In Section 8 we dualize the connectivity index and use it to derive algebraic implications of $\text{ext-dim}(X) = \text{ext-dim}(Y)$.

In a subsequent paper [18] we will explain that Bockstein theory plays the role of homological algebra in algebraic topology. In the present paper we use Bockstein theory to give necessary and sufficient conditions for $\mathrm{SP}(L)$ to have the same extension type as an Eilenberg–MacLane space $K(G,n)$ (see

Section 7). Later on (Section 9) we characterize extension types $[\mathrm{SP}(L)]$ containing compact (respectively, countable and finite-dimensional) CW complexes. That characterization generalizes all the previously known results about different extension types.

The last part of the paper (Section 9) is devoted to comparison of extensional properties of $M(G,n)$ and $K(G,n)$.

DEFINITION 1.4. Suppose G is an Abelian group and $n \geq 1$ is an integer. By $M(G,n)$ we will denote a *Moore space*, i.e. a CW complex K so that $\widetilde{H}_n(K;\mathbb{Z}) = G$ and $\widetilde{H}_m(K;\mathbb{Z}) = 0$ for $m \neq n$.

More precisely, Moore spaces discussed in this paper are constructed as follows. Choose a short exact sequence $0 \to F_1 \to F \to G \to 0$ so that F is free Abelian. Let L be the wedge of n-spheres enumerated by generators of F. Attach $(n+1)$-cells to L enumerated by generators of F_1 via characteristic maps corresponding to $F_1 \to F$. In particular, such Moore spaces are finite-dimensional and one has a map $M(\pi_n(K),n)) \to K$ (provided $\pi_1(K)$ is Abelian if $n = 1$) inducing isomorphism of nth homotopy groups for any space K.

We will use the following generalization of the relation $K \leq L$ (see [11]):

DEFINITION 1.5. Suppose K and L are CW complexes and $\mathcal{C}$ is a class of spaces. $K \leq_{\mathcal{C}} L$ means that if $X \in \mathcal{C}$ and $K \in \mathrm{AE}(X)$, then $L \in \mathrm{AE}(X)$. In particular $K \leq_X L$ means $K \leq_{\{X\}} L$.

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Some of the results of this paper were circulated in unpublished notes [17].

2. Geometry of extension theory. The purpose of this section is to list results of extension theory which involve no algebraic computations.

PROPOSITION 2.1 (see [11], [5]). *Suppose K is a CW complex and X is metrizable. If $K \in \mathrm{AE}(X)$, then every map $f : X \to \Sigma(K)$ from X to the suspension of K is null-homotopic.*

THEOREM 2.2 (see [8]). *If $X = \bigcup_{n=1}^{\infty} X_n$, X_n is a closed subset of X and $K \in \mathrm{AE}(X_n)$ for all n, then $K \in \mathrm{AE}(X)$ provided X is normal and K is an absolute neighborhood extensor of X ($K \in \mathrm{ANE}(X)$).*

THEOREM 2.3 (see [31] and [28]). *Suppose K is a CW complex. If $K \in \mathrm{AE}(X)$, then $K \in \mathrm{AE}(Y)$ for every $Y \subset X$ if X is metrizable.*

THEOREM 2.4 (see [14]). *If $X = A \cup B$ is metrizable, and $K \in \mathrm{AE}(A)$ and $L \in \mathrm{AE}(B)$ are CW complexes, then $K * L \in \mathrm{AE}(X)$.*

THEOREM 2.5 (see [8]). *Suppose K and L are countable CW complexes. If $K * L \in \mathrm{AE}(X)$ and X is a compactum, then there is a G_δ-subset A of X such that $K \in \mathrm{AE}(A)$ and $L \in \mathrm{AE}(X - A)$.*

THEOREM 2.6 (see [27]). *Suppose K is a countable CW complex. If $K \in \mathrm{AE}(X)$ and X is a subset of a metric separable space X', then there is a G_δ-subset A of X' containing X such that $K \in \mathrm{AE}(A)$.*

THEOREM 2.7 (see [11]). *Suppose X is compact or metrizable and K is a pointed connected CW complex. The following conditions are equivalent*:

(1) $K \in \mathrm{AE}(X \times I)$.
(2) $\Omega(K) \in \mathrm{AE}(X)$, *where $\Omega(K)$ is the loop space of K.*
(3) $[X/A, K] = 0$ *for all closed subsets $A \neq \emptyset$ of X.*
(4) $K \in \mathrm{AE}(\Sigma(X))$.

THEOREM 2.8 (see [7]). *Suppose K is a CW complex. If X is finite-dimensional and $\prod_{i=1}^{\infty} K(\pi_i(K), i) \in \mathrm{AE}(X)$, then $K \in \mathrm{AE}(X)$.*

3. Transition to algebra in extension theory. Given a space X and $k > 0$, the kth *symmetric product* $\mathrm{SP}^k(X)$ of X is the space of orbits of the action of the symmetric group S_k on X^k. Points of $\mathrm{SP}^k(X)$ can be written in the form $\sum_{i=1}^{k} x_i$. The set $\mathrm{SP}^k(X)$ is equipped with the quotient topology given by the natural map $\pi : X^k \to \mathrm{SP}^k(X)$. If X is metrizable, then $\pi : X^k \to \mathrm{SP}^k(X)$ is both open and closed (see p. 255 of [4]), so $\mathrm{SP}^k(X)$ is also metrizable (use 4.4.18 of [21]).

If X has a base point a, then $\mathrm{SP}^k(X)$ has $\sum_{i=1}^{k} a$ as its base point. Notice that there is a natural inclusion $i : \mathrm{SP}^n(X) \to \mathrm{SP}^k(X)$ for all $n < k$. It is given by the formula

$$i\Big(\sum_{i=1}^{n} x_i\Big) = \sum_{i=1}^{n} x_i + (k-n)a.$$

In this way, points of the form $\sum_{i=1}^{n} x_i$, $n < k$, can be considered as belonging to $\mathrm{SP}^k(X)$.

The direct limit of $\mathrm{SP}^2(X) \to \cdots \to \mathrm{SP}^n(X) \to \cdots$ is denoted by $\mathrm{SP}(X)$ and called the *infinite symmetric product* (see [4] or [1, p. 168]).

The main property of the infinite symmetric product is expressed in the following result.

Theorem 3.1 (Dold–Thom Theorem [4]). *If K is a pointed CW complex, then the natural inclusion $i : K \to \mathrm{SP}(K)$ induces an isomorphism $\widetilde{H}_i(K) \to \pi_i(\mathrm{SP}(K))$.*

The meaning of the Dold–Thom Theorem is that one can define singular homology groups geometrically, without the apparatus of homological algebra, as homotopy groups of infinite symmetric products (see [1]).

Of major importance to us is the following result of Dranishnikov:

Theorem 3.2 (see [9]). *Suppose K is a CW complex. If X is compact and $K \in \mathrm{AE}(X)$, then $\mathrm{SP}(K) \in \mathrm{AE}(X)$.*

Since $\mathrm{SP}(K)$ is homotopy equivalent to the weak product of Eilenberg–MacLane spaces $K(\widetilde{H}_i(K), i)$ (see [1, Corollary 6.4.17 on p. 223]) one has the following:

Theorem 3.3 (see [9]). *Suppose K is a CW complex. If X is compact, then $\mathrm{SP}(K) \in \mathrm{AE}(X)$ is equivalent to $K(\widetilde{H}_i(K), i) \in \mathrm{AE}(X)$ for all $i \geq 0$.*

The following result of Dranishnikov uses a high level of algebraic arguments and can be viewed as an analog of the Hurewicz Theorem in extension theory.

Theorem 3.4 (see [9]). *Suppose K is a CW complex. If X is compact finite-dimensional, K is simply connected, and $\mathrm{SP}(K) \in \mathrm{AE}(X)$, then $K \in \mathrm{AE}(X)$.*

Let us move to algebraic concepts associated with extension theory by employing the connectivity index introduced by E. Shchepin [28].

Definition 3.5 ([28, p. 985]). Suppose K is a CW complex. Its *connectivity index* $\mathrm{cin}(K)$ is either ∞ or a non-negative integer. $\mathrm{cin}(K) = \infty$ means that all reduced integral homology groups of K are trivial. $\mathrm{cin}(K) = n$ means that $\widetilde{H}_n(K; \mathbb{Z}) \neq 0$ and $\widetilde{H}_k(K; \mathbb{Z}) = 0$ for all $k < n$.

Proposition 3.6. *Suppose K is a simply connected CW complex. The following numbers are equal:*

(1) $\mathrm{cin}(K)$,
(2) *the supremum of all $n \geq 0$ so that $K \in \mathrm{AE}(S^n)$*,
(3) *the supremum of all $n \geq 0$ so that K is homotopy k-connected for all $k < n$.*

Proof. If $K \in \mathrm{AE}(S^n)$, then any map $S^j \to K$, $j < n$, extends over S^n and must be null-homotopic. Hence $H_j(K) = 0$ for $j < n$. If $H_j(K) = 0$ for $j < n$, then $\pi_j(K) = 0$ for $j < n$ and any map $f : A \to K$, with A closed in S^n, can be extended over S^n as follows: first extend it over a polyhedral neighborhood N of A, then keep extending over simplices which are not contained in N using induction on the dimension of the simplices. ∎

PROPOSITION 3.7. *Suppose K and L are pointed CW complexes. If K is countable and $K \le_X L$ for all finite-dimensional compacta X, then* $\operatorname{cin}(K \wedge M) \le \operatorname{cin}(L \wedge M)$ *for every CW complex M.*

Proof.

CASE 1: *M is countable.* Since suspending a CW complex pushes its connectivity index up by 1, it suffices to prove 3.7 for $M = \Sigma^2 M'$, in which case $K \wedge M$ and $L \wedge M$ are simply connected and we may use 3.6. Notice that $\operatorname{cin}(K \wedge M) = \operatorname{cin}(K * M) - 1$, where $K * M$ is the join of K and M. Suppose $K * M \in \mathrm{AE}(S^m)$. Express S^m as $A \cup B$ so that $K \in \mathrm{AE}(A)$, $M \in \mathrm{AE}(B)$, and A is F_σ (see 2.5). Now, $L \in \mathrm{AE}(A)$ (see 2.2), which implies $L * M \in \mathrm{AE}(S^m)$ by 2.4.

CASE 2: *M is arbitrary.* Suppose $m = \operatorname{cin}(K \wedge M) > \operatorname{cin}(L \wedge M) = n$. There is an integral cycle c in $H_n(L \wedge M) - \{0\}$. That cycle lies in $L \wedge M_1$ for some finite subcomplex M_1 of M. Given a countable subcomplex M_i of M, construct a countable subcomplex $M_{i+1} \supset M_i$ of M so that $H_k(K \wedge M_i) \to H_k(K \wedge M_{i+1})$ is trivial for $k < m$. Let M' be the union of all M_i. By Case 1, $m \le \operatorname{cin}(K \wedge M') \le \operatorname{cin}(L \wedge M')$, which means that c becomes 0 in $H_n(L \wedge M')$, a contradiction. ∎

4. Cohomology with coefficients in a complex. With the hindsight of homological algebra one has the pairing

$$H_*(K) \wedge H_*(L) \to \pi_*(\mathrm{SP}(K \wedge L))$$

for any two pointed CW complexes K and L. It corresponds to the well known Künneth formula for homology and we will give it a slightly non-traditional form.

THEOREM 4.1. *If K and L are pointed CW complexes, then*

$$H_n(K \wedge L) \equiv \bigoplus_{i+j=n} H_i(K; H_j(L)).$$

REMARK 4.2. Theorem 4.1 is a direct consequence of the Universal Coefficient Theorem (see [29, p. 222] and the classical Künneth Theorem (see [29, p. 235]).

Notice that

$$\bigoplus_{i+j=n} H_i(K; H_j(L)) \equiv \bigoplus_{i+j=n} H_i(L; H_j(K))$$

as both groups are isomorphic to $H_n(K \wedge L)$. A natural question to ask is if

$$\bigoplus_i H^i(K; H_{n-i}(L)) \quad \text{or} \quad \bigoplus_i H_i(L; H^{n-i}(K))$$

have similar geometrical interpretation, and if they are isomorphic. Knowing that the smash product $K \wedge L$ is adjoint to the function space functor one can speculate that:

IDEA 4.3. *There ought to be a dual pairing*

$$H^*(X) \wedge H_*(L) \equiv \pi_*(\mathrm{SP}(L)^X).$$

The above pairing should correspond to the Künneth formula for cohomology. It turns out that such a pairing exists for pointed compact spaces.

Our first idea is to introduce cohomology of a pointed space X via homotopy groups of function spaces $\mathrm{SP}(K)^X$.

DEFINITION 4.4. Suppose K is a pointed CW complex and X is a pointed space. The *cohomology* $H^n(X;K)$ of X *with coefficients* in K is defined as follows:

$$H^n(X;K) = \begin{cases} [X, \mathrm{SP}(\Sigma^n(K))] & \text{if } n \geq 0, \\ [\Sigma^{-n}X, \mathrm{SP}(K)] & \text{if } n \leq 0. \end{cases}$$

PROPOSITION 4.5. *If* $K = M(G,n)$ *is a Moore space*, *then* $H^k(X;K) = H^{n+k}(X;G)$.

Proof. Notice that $\Sigma^r(K) = M(G, n+r)$ and $\mathrm{SP}(M(G,r)) = K(G,r)$ for $r \geq 1$ (see 3.1). As $H^m(X;G) = [X, K(G,m)]$ one gets 4.5 immediately from the definition of $H^m(X;K)$. ∎

It is shown in [4] that $\mathrm{SP}(K)$ is homotopy equivalent to the union $\bigcup_{m=1}^{\infty}\prod_{n=1}^{m} K(H_n(K),n)$ for every connected CW complex K (see also [1, Corollary 6.4.17 on p. 223]). We will need a more general result.

PROPOSITION 4.6. *Suppose K is a pointed CW complex and X is a pointed k-space. There is a weak homotopy equivalence*

$$i : \bigcup_{m=1}^{\infty}\prod_{n=1}^{m} K(\pi_n(\mathrm{SP}(K)^X), n) \to \mathrm{SP}(K)^X.$$

Proof. Notice any map $f : L \to \mathrm{SP}(K)^X$ extends to $F : \mathrm{SP}(L) \to \mathrm{SP}(K)^X$ if L is a CW complex. Indeed, one can define F as follows:

$$F\Big(\sum_{i=1}^{n} a_i\Big)(x) = \sum_{i=1}^{n} f(a_i)(x) \quad \text{for } \sum_{i=1}^{n} a_i \in \mathrm{SP}(L) \text{ and } x \in X.$$

Let L be the wedge of $M(\pi_n(\mathrm{SP}(K)^X), n)$, $n \geq 1$. There is a map $f : L \to \mathrm{SP}(K)^X$ so that $\pi_n(f|M(\pi_n(\mathrm{SP}(K)^X), n))$ is an isomorphism for each n (see the discussion after 1.4). In particular, $H_k(L) \to \pi_k(\mathrm{SP}(K)^X)$ is an isomorphism for all $k \geq 1$. Pick an extension $F : \mathrm{SP}(L) \to \mathrm{SP}(K)^X$ of f. Since $H_k(L) \to \pi_k(\mathrm{SP}(L))$ and $H_k(L) \to \pi_k(\mathrm{SP}(K)^X)$ are isomorphisms for all $k \geq 1$, it follows that F is a weak homotopy equivalence. As shown in [4], $\mathrm{SP}(L)$ is homotopy equivalent to $\bigcup_{m=1}^{\infty}\prod_{n=1}^{m} K(H_n(L),n)$, which is exactly the space $\bigcup_{m=1}^{\infty}\prod_{n=1}^{m} K(\pi_n(\mathrm{SP}(K)^X), n)$. ∎

Corollary 4.7. *Suppose K is a pointed connected CW complex and X is a pointed k-space. Then* $\mathrm{SP}(K)$ *and the spaces* $\Omega^r(\mathrm{SP}(\Sigma^r(K)))$ *are homotopy equivalent for $r \geq 1$. In particular*, $H^n(X,\Sigma^r(K)) \equiv H^{n+r}(X;K)$ *for all $n \in \mathbb{Z}$ and all $r \geq 1$.*

Proof. Notice that

$$\pi_k(\Omega^r(\mathrm{SP}(\Sigma^r(K)))) = \pi_{k+r}(\mathrm{SP}(\Sigma^r(K))) = H_{k+r}(\Sigma^r(K)) = H_k(K)$$

for each $k \geq 0$. By 4.6 (applied to $X = S^r$) there is a weak homotopy equivalence $\bigcup_{m=1}^{\infty}\prod_{n=1}^{m} K(H_n(K),n) \to \Omega^r(\mathrm{SP}(\Sigma^r(K)))$. This map is a homotopy equivalence as both spaces are homotopy equivalent to CW complexes. As in [4], $\bigcup_{m=1}^{\infty}\prod_{n=1}^{m} K(H_n(K),n)$ is homotopically equivalent to $\mathrm{SP}(K)$.

If $n \geq 0$, then

$$H^n(X;\Sigma^r(K)) = [X,\mathrm{SP}(\Sigma^{n+r}(K))] = H^{n+r}(X;K).$$

If $n < 0$, then $H^n(X;\Sigma^r(K)) = [\Sigma^{-n}(X),\mathrm{SP}(\Sigma^r(K))]$. If $n+r \geq 0$, then

$$\begin{aligned}[\Sigma^{-n}(X),\mathrm{SP}(\Sigma^r(K))] &= [X,\Omega^{-n}(\mathrm{SP}(\Sigma^r(K)))]\\ &= [X,\mathrm{SP}(\Sigma^{n+r}(K))] = H^{n+r}(X;K).\end{aligned}$$

If $n+r < 0$, then

$$\begin{aligned}[\Sigma^{-n}(X),\mathrm{SP}(\Sigma^r(K))] &= [\Sigma^{-n-r}(X),\Omega^r(\mathrm{SP}(\Sigma^r(K)))]\\ &= [\Sigma^{-n-r}(X),\mathrm{SP}(K)] = H^{n+r}(X;K). \ ■\end{aligned}$$

The purpose of the next result is to generalize the well known theorem of Cohen [3].

Theorem 4.8. *Suppose K is a pointed CW complex and X is compact or metrizable. The following conditions are equivalent*:

(1) $\mathrm{SP}(K) \in \mathrm{AE}(X)$.
(2) $H^n(X/A;K) = 0$ *for all $n > 0$ and all closed subsets A of X.*
(3) $H^1(X/A;K) = 0$ *for all closed subsets A of X.*

Proof. $H^n(X/A;K)$ was defined as $[X/A,\mathrm{SP}(\Sigma^n(K))]$ for $n \geq 1$ (see 4.4). Now 2.7 says that $[X/A,\mathrm{SP}(\Sigma(K))] = 0$ for all closed subsets A of X if and only if $\Omega(\mathrm{SP}(\Sigma(K))) \in \mathrm{AE}(X)$. By 4.7, $\Omega(\mathrm{SP}(\Sigma(K)))$ is homotopy equivalent to $\mathrm{SP}(K)$, which proves (1)$\Leftrightarrow$(3).

Clearly, (2) is stronger than (3).

Suppose (3) and (1) hold. If $n > 1$, then (see 2.7) $\mathrm{SP}(\Sigma^n(K)) \in \mathrm{AE}(X)$ and $H^1(X/A;\Sigma^{n-1}(K)) = 0$ for all closed subsets A of X (as (1) is equivalent to (3) for all CW complexes K). Now, 4.7 says that $H^n(X/A;K) = H^1(X/A;\Sigma^{n-1}(K))$, which proves that (2) holds. ■

Notice that in 4.4 the homotopy groups of $\mathrm{SP}(K)^X$ correspond to negative cohomology groups of X with coefficients in K. This seems unnatural but we chose it that way in order to adhere to common practice. However, it is time to break with tradition and achieve a better theory.

DEFINITION 4.9. Suppose h^* is a cohomology theory. By ${}^r h^*$ we will denote the *reversed cohomology* defined via ${}^r h^m(X) = h^{-m}(X)$.

The following result shows that using reversed cohomology one can achieve similarity between homology and cohomology (compare 4.10 with 4.1).

THEOREM 4.10. *If K is a pointed CW complex and X is a pointed compact space, then*

$$ {}^r H^n(X;K) \equiv \bigoplus_i {}^r H^i(X;H_{n-i}(K)), $$

$$ {}^r H^n(X;K) \equiv \bigoplus_i H_i(K;{}^r H^{n-i}(X)). $$

Proof. If Y is a pointed compact space and L is a pointed CW complex, then $[Y, \mathrm{SP}(L)]$ is the direct sum $\bigoplus_i [Y, K(H_i(L), i)]$. Indeed, $\mathrm{SP}(L)$ is homotopically equivalent to $\bigcup_{m=1}^{\infty} \prod_{n=0}^{m} K(H_n(L), n)$ (see [4]) and any map from Y (or any homotopy from $Y \times I$) to $\bigcup_{m=1}^{\infty} \prod_{n=0}^{m} K(H_n(L), n)$ has image contained in $\prod_{n=0}^{m} K(H_n(L), n)$ for some m.

If $n \geq 0$, then

$$ {}^r H^{-n}(X;K) = [X, \mathrm{SP}(\Sigma^n(K)] = \bigoplus_i [X, K(H_{i-n}(K), i)] $$

$$ = \bigoplus_i {}^r H^{-i}(X; H_{i-n}(K)). $$

If $n < 0$, then ${}^r H^{-n}(X;K) = [\Sigma^{-n} X, \mathrm{SP}(K)]$ is $\bigoplus_i [\Sigma^{-n}(X), K(H_i(K), i)]$, which is the same as $\bigoplus_i H^i(X; H_{i-n}(K)) = \bigoplus_i {}^r H^{-i}(X; H_{i-n}(K))$.

By the Universal Coefficient Theorem for homology (see [29, Theorem 14, p. 226], $\bigoplus_i H_i(K; {}^r H^{n-i}(X))$ is isomorphic to

$$ G_1 = \bigoplus_i (H_i(K) \otimes {}^r H^{n-i}(X) \oplus H_{i-1}(K) * {}^r H^{n-i}(X)) $$

($G * G'$ is the torsion product of G and G'). By the Universal Coefficient Theorem for cohomology (see [22, Statement 5 on p. 4], $\bigoplus_i {}^r H^i(X; H_{n-i}(K))$ is isomorphic to

$$ G_2 = \bigoplus_i (H_{n-i}(K) \otimes {}^r H^i(X) \oplus H_{n-i}(K) * {}^r H^{i-1}(X)). $$

Notice that G_1 is isomorphic to G_2 (change i to $n-i$ in the first summand of G_1 and change $i-1$ to $n-i$ in the second summand of G_1). ∎

As a simple consequence of the fact that K^Y is homotopy equivalent to a CW complex if K is a CW complex and Y is compact, we get the following version of the Künneth formula.

THEOREM 4.11. *Suppose K is a pointed CW complex. If X and Y are pointed compact spaces, then*

$$ {}^rH^n(X \wedge Y; K) \equiv \bigoplus_i {}^rH^i(X; {}^rH^{n-i}(Y; K)). $$

Proof. Suppose $n \geq 0$. We have ${}^rH^n(X \wedge Y; K) = [\Sigma^n(X \wedge Y), \mathrm{SP}(K)] = [\Sigma^n(X), \mathrm{SP}(K)^Y]$. Since $\mathrm{SP}(K)^Y$ is homotopy equivalent to a CW complex, 4.6 says that it is homotopy equivalent to $\bigcup_{m=1}^{\infty} \prod_{i=0}^{m} K({}^rH^i(Y; K), i)$. Thus,

$$ \begin{aligned} {}^rH^n(X \wedge Y; K) &= \bigoplus_i H^i(\Sigma^n(X); {}^rH^i(Y; K)) \\ &= \bigoplus_i H^{i-n}(X; {}^rH^i(Y; K)) = \bigoplus_i {}^rH^{n-i}(X; {}^r H^i(Y; K)). \end{aligned} $$

The case $n < 0$ reduces to the case $n = 0$ by observing that ${}^rH^n(Z; K) = {}^rH^0(Z; \Sigma^{-n}(K))$ for every pointed k-space Z (see 4.7). ∎

COROLLARY 4.12. *If K is a pointed CW complex and X is a pointed compact space, then the following conditions are equivalent* (m *is an integer*):

(1) $H^n(X; K) = 0$ *for all* $n \geq m$.
(2) $H_i(K; H^n(X)) = 0$ *for all* $i \leq n - m$.
(3) $H^i(X; H_n(K)) = 0$ *for all* $i \geq n + m$.

Proof. Notice that $H^n(X; K) = {}^rH^{-n}(X; K) \equiv \bigoplus_i H_i(K; {}^rH^{-n-i}(X))$ (see 4.10), which is the same as $\bigoplus_i H_i(K; H^{n+i}(X))$. That means $H^n(X; K) = 0$ for all $n \geq m$ is equivalent to $H_i(K; H^{n+i}(X)) = 0$ for all $n \geq m$, which is the same as saying that $H_i(K; H^n(X)) = 0$ for all $i \leq n - m$. Similarly, $H^n(X; K) = {}^rH^{-n}(X; K)$ is isomorphic to $\bigoplus_i {}^rH^i(X; H_{-n-i}(K))$ (see 4.10), which is the same as $\bigoplus_i H^{-i}(X; H_{-n-i}(K))$. That means $H^n(X; K) = 0$ for all $n \geq m$ is equivalent to $H^{-i}(X; H_{-n-i}(K)) = 0$ for all $n \geq m$, which is the same as saying $H^i(X; H_n(K)) = 0$ for all $i \geq n + m$. ∎

COROLLARY 4.13. *Suppose K is a pointed CW complex and X is a compact space. The following conditions are equivalent*:

(1) $\mathrm{SP}(K) \in \mathrm{AE}(X)$.
(2) $H_i(K; H^n(X; A)) = 0$ *for all* $i < n$ *and all closed subsets* A *of* X.
(3) $H^i(X/A; H_n(K)) = 0$ *for all* $n < i$ *and all closed subsets* A *of* X.

Proof. By 4.8, $\mathrm{SP}(K) \in \mathrm{AE}(X)$ if and only if $H^n(X/A; K) = 0$ for all $n \geq 1$ and all closed subsets A of X. Using 4.12 one finds that $\mathrm{SP}(K) \in \mathrm{AE}(X)$ if and only if $H_i(K; H^n(X, A)) = 0$ for all $i \leq n - 1$. That proves (1)⇔(2); and (2)⇔(3) follows from 4.12 applied to $m = 1$. ∎

5. Homological dimension of CW complexes. 4.13 suggests that $K \leq L$ should have some algebraic implications. The next result specifies the nature of those implications.

Theorem 5.1. *Suppose* $n > 0$, G *is an Abelian group*, K, L *are CW complexes*, *and* $\mathcal{C}$ *is a class of spaces containing all finite-dimensional compacta. If* K *is countable*, $K \le_{\mathcal{C}} L$, *and* $\widetilde{H}_i(K;G) = 0$ *for all* $i \le n$, *then* $\widetilde{H}_i(L;G) = 0$ *for all* $i \le n$.

Proof. Make K and L pointed CW complexes and switch from the reduced homology to ordinary homology of pointed CW complexes. By 4.1, $H_i(A \wedge M(G,1)) = H_{i-1}(A;G)$ for any pointed CW complex A, where $M(G,1)$ is the Moore space. Thus $\mathrm{cin}(K \wedge M(G,1)) \ge n+2$. By 3.7, $\mathrm{cin}(L \wedge M(G,1)) \ge n+2$, which is the same as $\widetilde{H}_i(L;G) = 0$ for all $i \le n$. ■

Theorem 5.1 suggests a new concept of *homological dimension* $\dim_G(K)$ of a pointed CW complex.

Definition 5.2. Suppose K is a pointed CW complex and G is an Abelian group. Then $\dim_G(K) = n < \infty$ means that $H_i(K;G) = 0$ for all $i < n$ and $H_n(K;G) \ne 0$; and $\dim_G(K) = \infty$ means that $H_i(K;G) = 0$ for all i.

Remark 5.3. Notice that the above concept generalizes the concept of connectivity index. Indeed, $\mathrm{cin}(K) = \dim_{\mathbb{Z}}(K)$ for all pointed CW complexes K.

Definition 5.2 suggests a new partial order on the class of CW complexes:

Definition 5.4. Suppose K and L are CW complexes and G is an Abelian group. Then $K \le_G L$ means $\dim_G(K) \le \dim_G(L)$.

If $\mathcal{G}$ is a class of Abelian groups, then $K \le_{\mathcal{G}} L$ means that $K \le_G L$ for all $G \in \mathcal{G}$; and $K \sim_{\mathcal{G}} L$ means that $K \le_{\mathcal{G}} L$ and $L \le_{\mathcal{G}} K$.

$K \le_{\mathrm{Gr}} L$ means that $K \le_G L$ for all Abelian groups G; and $K \sim_{\mathrm{Gr}} L$ means that $K \le_{\mathrm{Gr}} L$ and $L \le_{\mathrm{Gr}} K$.

Corollary 5.5. $K \sim_{\mathrm{Gr}} \mathrm{SP}(K)$ *for each pointed CW complex* K.

Proof. Notice that $H_n(K)$ is a direct summand of $H_n(\mathrm{SP}(K))$ for each n. Indeed, $\mathrm{SP}(K)$ is homotopy equivalent to $\bigcup_{m=1}^{\infty} \prod_{n=1}^{m} K(H_n(K), n)$ and each $K(H_n(K), n)$ (whose nth homology group is $H_n(K)$) is a retract of $\mathrm{SP}(K)$. If $H_k(\mathrm{SP}(K);G) = 0$ for $k < n$, then it amounts to $H_k(\mathrm{SP}(K)) \otimes G = H_{k-1}(\mathrm{SP}(K)) * G = 0$ for $k < n$ (in view of the Universal Coefficient Theorem). Therefore $H_k(K) \otimes G = H_{k-1}(K) * G = 0$ for $k < n$ and $H_k(K;G) = 0$ for $k < n$. This proves $\mathrm{SP}(K) \le_{\mathrm{Gr}} K$.

To prove $K \le_{\mathrm{Gr}} \mathrm{SP}(K)$ notice that, for K countable, this follows from 5.1 and 3.2. Suppose $H_k(K;G) = 0$ for $k \le n$ and there is $c \in H_n(\mathrm{SP}(K);G) - \{0\}$. Find a finite subcomplex K_1 of K and a countable subgroup G_1 of G such that c belongs to the image of $H_n(\mathrm{SP}(K_1);G_1) \to H_n(\mathrm{SP}(K);G)$. By induction find countable subcomplexes $K_1 \subset K_2 \subset \cdots$ of K and countable subgroups $G_1 \subset G_2 \subset \cdots$ of G so that $H_k(K_i;G_i) \to H_k(K_{i+1};G_{i+1})$ is trivial for all i and all $k \le n$. Let K' be the union of all K_i and let G'

be the union of all G_i. Notice that $H_k(K';G')=0$ for $k\le n$. Therefore $H_n(\mathrm{SP}(K');G')=0$, contradicting $c\ne 0$. ∎

THEOREM 5.6. *Suppose K and L are connected CW complexes and K is countable. Consider the following conditions*:

(1) $K\le_{\mathrm{Gr}} L$.
(2) $\mathrm{SP}(K)\le_X \mathrm{SP}(L)$ *for all compact X.*
(3) $\mathrm{cin}(K\wedge M)\le \mathrm{cin}(L\wedge M)$ *for each complex M.*
(4) $K\le_X L$ *for all finite-dimensional compacta X.*

Condition (4) *implies* (1). *Conditions* (1)–(3) *are equivalent. If L is simply connected*, *then* (1)–(4) *are equivalent.*

Proof. (4)⇒(1) follows from 5.1.

(3)⇒(1). Suppose $H_k(K;G)=0$ for $k<n$. Use $M=M(G,1)$ and 4.1 to conclude that $H_k(K\wedge M)=0$ for $k\le n$. Therefore $H_k(L\wedge M)=0$ for $k\le n$, which means $H_k(L;G)=0$ for $k<n$.

(1)⇒(3). Suppose $H_k(K\wedge M)=0$ for $k<n$. That means (see 4.1) $H_i(K;H_j(M))=0$ if $i+j<n$. Since $K\le_{\mathrm{Gr}} L$, $H_i(L;H_j(M))=0$ if $i+j<n$, i.e. $H_k(L\wedge M)=0$ for $k<n$.

(2)⇒(1) follows from 5.1 and 5.5.

(1)⇒(2). Suppose $\mathrm{SP}(K)\in\mathrm{AE}(X)$. 4.13 says that $H_n(K;H^p(X;A))=0$ for all $n<p$ and all closed subsets A of X. Thus, $H_n(L;H^p(X;A))=0$ for all $n<p$ and all closed subsets A of X. Applying 4.13 again we get $\mathrm{SP}(L)\in\mathrm{AE}(X)$.

If L is simply connected, then (2) implies (4) by 3.4. ∎

COROLLARY 5.7. *If K and L are countable*, *pointed*, *connected CW complexes*, *then the following conditions are equivalent*:

(1) $\dim_G(K)=\dim_G(L)$ *for all Abelian groups G.*
(2) $\dim_{\mathbb{Z}}(K\wedge M)=\dim_{\mathbb{Z}}(L\wedge M)$ *for all pointed CW complexes M.*

Proof. In view of 5.5, (1) is equivalent to $\mathrm{SP}(K)\sim_{\mathrm{Gr}}\mathrm{SP}(L)$, and that is equivalent (see 5.6) to (2). ∎

REMARK 5.8. 5.7 is dual to the well known characterization of $\dim_G(X)=\dim_G(Y)$ for all Abelian groups G (X and Y are compact). Namely, $\dim_G(X)=\dim_G(Y)$ is equivalent to $\dim_{\mathbb{Z}}(X\times T)=\dim_{\mathbb{Z}}(Y\times T)$ for all compact spaces T (see [22]).

6. Cohomological dimension of compact spaces. The purpose of this section is to show that, given a compact space X, the class $\{\mathrm{SP}(L)\mid \text{ext-dim}(X)\le\mathrm{SP}(L)\}$ has a minimum which we call the *cohomological dimension* of X. To do so we will need the basics of Bockstein theory (see [2] or [22]).

DEFINITION 6.1 (Bockstein groups). The set $\mathcal{B}_{\mathcal{G}}$ of *Bockstein groups* is

$$\{\mathbb{Q}\} \cup \bigcup_{\mathbf{p}\,\mathrm{prime}} \{\mathbb{Z}/\mathbf{p}, \mathbb{Z}/\mathbf{p}^{\infty}, \mathbb{Z}_{(\mathbf{p})}\},$$

where $\mathbb{Z}/\mathbf{p}^{\infty}$ is the $\mathbf{p}$-torsion of $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{Z}_{(\mathbf{p})}$ are the rationals whose denominator is not divisible by $\mathbf{p}$.

DEFINITION 6.2 (Bockstein basis). Given an abelian group G its *Bockstein basis* $\sigma(G)$ is the subset of $\mathcal{B}_{\mathcal{G}}$ defined as follows:

(1) $\mathbb{Q} \in \sigma(G)$ iff $\mathbb{Q} \otimes G \neq 0$,
(2) $\mathbb{Z}/\mathbf{p} \in \sigma(G)$ iff $(\mathbb{Z}/\mathbf{p}) \otimes G \neq 0$,
(3) $\mathbb{Z}_{(\mathbf{p})} \in \sigma(G)$ iff $(\mathbb{Z}/\mathbf{p}^{\infty}) \otimes G \neq 0$,
(4) $\mathbb{Z}/\mathbf{p}^{\infty} \in \sigma(G)$ iff $(\mathbb{Z}/\mathbf{p}^{\infty}) * G \neq 0$ (here $H * G$ is the torsion product of groups H and G) or $(\mathbb{Z}/\mathbf{p}) \otimes G \neq 0$.

REMARK 6.3. Our definition of Bockstein basis is slightly different from the standard ones (see [22] or [10]). Namely, if $\mathbb{Z}_{(\mathbf{p})} \in \sigma(G)$ (respectively, $\mathbb{Z}/\mathbf{p} \in \sigma(G)$), then $\mathbb{Z}/\mathbf{p} \in \sigma(G)$ (respectively, $\mathbb{Z}/\mathbf{p}^{\infty} \in \sigma(G)$). In traditional definitions of Bockstein basis only one group among $\{\mathbb{Z}_{(\mathbf{p})}, \mathbb{Z}/\mathbf{p}, \mathbb{Z}/\mathbf{p}^{\infty}\}$ is admitted for any $\mathbf{p}$. Since our only application of Bockstein basis is 6.4, the change of the definition will not cause any problems as $\dim_{\mathbb{Z}_{(\mathbf{p})}}(X) \geq \dim_{\mathbb{Z}/\mathbf{p}}(X) \geq \dim_{\mathbb{Z}/\mathbf{p}^{\infty}}(X)$ for all primes $\mathbf{p}$ (see [22]).

THEOREM 6.4 (First Bockstein Theorem). *If X is compact, then*

$$\dim_G(X) = \max\{\dim_H(X) \mid H \in \sigma(G)\}.$$

Here is the existence of cohomological dimension:

THEOREM 6.5. *If X is compact, then there is a countable CW complex K_X such that $\mathrm{SP}(K_X) \in \mathrm{AE}(X)$ and $\mathrm{SP}(K_X) \leq \mathrm{SP}(L)$ for every CW complex L satisfying $\mathrm{SP}(L) \in \mathrm{AE}(X)$.*

Proof. Consider the set $\mathcal{B}_{\mathcal{X}}$ of all Bockstein groups H such that $\dim_H(X) < \infty$. Put $K_X = \bigvee_{H \in \mathcal{B}_{\mathcal{X}}} K(H, \dim_H(X))$. Notice that $K_X \in \mathrm{AE}(X)$, hence $\mathrm{SP}(K_X) \in \mathrm{AE}(X)$ (see 3.2). Suppose $\mathrm{SP}(L) \in \mathrm{AE}(X)$ for some CW complex L and $\mathrm{SP}(K_X) \in \mathrm{AE}(Y)$ for some compact space Y. We need to show that $\mathrm{SP}(L) \in \mathrm{AE}(Y)$.

Since $\mathrm{SP}(K_X) \in \mathrm{AE}(Y)$ is equivalent to $H_i(K_X; H^n(Y, A)) = 0$ for all $i < n$ and all closed subsets A of Y (see 4.13), one has

$$H_i(K(H, \dim_H(X)); H^n(Y, A)) = 0 \quad \text{for all } i < n$$

and all closed subsets A of Y. This, in turn, implies $K(H, \dim_H(X)) \in \mathrm{AE}(Y)$ (see 5.5 and 4.13) for all $H \in \mathcal{B}_{\mathcal{X}}$. Thus $\dim_H(Y) \leq \dim_H(X)$ for all Bockstein groups H and, in view of 6.4, $\dim_G(Y) \leq \dim_G(X)$ for all Abelian groups G. Since, for any compact space T, $\mathrm{SP}(L) \in \mathrm{AE}(T)$ is equivalent to

$\dim_{H_i(L)}(T) \le i$ for all i, one has $\dim_{H_i(L)}(Y) \le \dim_{H_i(L)}(X) \le i$ for all i, i.e. $\mathrm{SP}(L) \in \mathrm{AE}(Y)$. ∎

7. Extension types of Eilenberg–MacLane spaces. The following problem was posed in [11].

PROBLEM 7.1. Find necessary and sufficient conditions for two CW complexes K and L to be of the same extension type.

In this section we consider 7.1 for K being the infinite symmetric product and L being an Eilenberg–MacLane space. Our characterization of infinite symmetric products having extension type of an Eilenberg–MacLane space involves an enlargement of Bockstein basis.

DEFINITION 7.2. Let G be an abelian group. Then $\tau(G)$ is the subset of Bockstein groups containing $\sigma(G)$ and satisfying the following conditions for each prime $\mathbf{p}$:

(1) $\mathbb{Z}/\mathbf{p} \in \tau(G)$ iff $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$,
(2) $\mathbb{Z}_{(\mathbf{p})} \in \tau(G)$ iff $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$ and $\mathbb{Q} \in \sigma(G)$.

LEMMA 7.3. *Suppose G and F are non-trivial Abelian groups and $m \ge 1$. If $\sigma(F)\setminus\tau(G) \ne \emptyset$, then there is a compact space X such that $\dim_G(X) = m$ and $\dim_F(X) = \infty$.*

Proof. Define $\alpha : \mathcal{B}_\mathcal{G} \to \mathcal{N}$ (natural numbers plus infinity) as follows: $\alpha(H) = \infty$ iff $H \in \mathcal{B}_\mathcal{G} \setminus \tau(\mathcal{G})$, $\alpha(H) = m+1$ iff $H \in \tau(G) \setminus \sigma(G)$, and $\alpha(H) = m$ iff $H \in \sigma(G)$.

Let us show that α is a Bockstein function, i.e. the following inequalities hold for all primes $\mathbf{p}$:

(1) $\alpha(\mathbb{Z}/\mathbf{p}^\infty) \le \alpha(\mathbb{Z}/\mathbf{p}) \le \alpha(\mathbb{Z}/\mathbf{p}^\infty) + 1$,
(2) $\alpha(\mathbb{Z}/\mathbf{p}) \le \alpha(\mathbb{Z}_{(\mathbf{p})})$,
(3) $\alpha(\mathbb{Q}) \le \alpha(\mathbb{Z}_{(\mathbf{p})})$,
(4) $\alpha(\mathbb{Z}_{(\mathbf{p})}) \le \max(\alpha(\mathbb{Q}), \alpha(\mathbb{Z}/\mathbf{p}^\infty) + 1)$,
(5) $\alpha(\mathbb{Z}/\mathbf{p}^\infty) \le \max(\alpha(\mathbb{Q}), \alpha(\mathbb{Z}_{(\mathbf{p})}) - 1)$.

Inequalities (1) can fail only if $\mathbb{Z}/\mathbf{p} \in \tau(G)$ and $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$. In that case $\alpha(\mathbb{Z}/\mathbf{p}^\infty) = m$ and $m \le \alpha(\mathbb{Z}/\mathbf{p}) \le m+1$, so inequalities (1) hold. Inequality (2) can fail only if $\mathbb{Z}_{(\mathbf{p})} \in \tau(G)$. In that case, however, $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$, implying $\mathbb{Z}/\mathbf{p} \in \tau(G)$. Since $\mathbb{Z}_{(\mathbf{p})} \in \sigma(G)$ implies $\mathbb{Z}/\mathbf{p} \in \sigma(G)$, inequality (2) holds. Inequality (3) can fail only if $\mathbb{Z}_{(\mathbf{p})} \in \tau(G)$. In that case, however, $\mathbb{Q} \in \sigma(G)$, so inequality (3) holds. Inequality (4) can fail only if $\mathbb{Z}/\mathbf{p}^\infty \in \tau(G)$ and $\mathbb{Q} \in \tau(G)$. That however implies $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$ and $\mathbb{Q} \in \sigma(G)$. Consequently, $\mathbb{Z}_{(\mathbf{p})} \in \tau(G)$ and $\alpha(\mathbb{Z}_{(\mathbf{p})}) \le m+1 \le \max(\alpha(\mathbb{Q}), \alpha(\mathbb{Z}/\mathbf{p}^\infty)+1)$, i.e. (4) holds. Inequality (5) can fail only if $\mathbb{Z}_{(\mathbf{p})} \in \tau(G)$. Therefore $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G)$ and $\mathbb{Q} \in \sigma(G)$. Hence $\alpha(\mathbb{Z}/\mathbf{p}^\infty) = m \le \max(\alpha(\mathbb{Q}), \alpha(\mathbb{Z}_{(\mathbf{p})}) - 1)$.

By Dranishnikov's Realization Theorem (see [5] or [10]) there is a compactum X such that $\dim_H(X) = \alpha(H)$ for all $H \in \mathcal{B}_{\mathcal{G}}$. It is clear, in view of Bockstein's First Theorem 6.4, that X satisfies the desired conditions. ∎

LEMMA 7.4. *Suppose G and F are non-trivial Abelian groups and $m \geq 1$. If $\sigma(F) \subset \tau(G)$ and $\sigma(F) \setminus \sigma(G) \neq \emptyset$, then there is a compact space X such that* $\dim_F(X) = \dim(X) = m+1$ *and* $\dim_G(X) = m$.

Proof. There is a prime $\mathbf{q}$ such that $\mathbb{Z}_{(\mathbf{q})} \in \sigma(F) \setminus \sigma(G)$ ($\mathbb{Z}/\mathbf{q} \in \sigma(F) \setminus \sigma(G)$, respectively). Define $\alpha : \mathcal{B}_{\mathcal{G}} \to \mathcal{N}$ by sending $\mathbb{Z}_{(\mathbf{q})}$ to $m+1$ (respectively, $\mathbb{Z}/\mathbf{q}$ and $\mathbb{Z}_{(\mathbf{q})}$ to $m+1$), and sending all the other groups to m. Notice that α is a Bockstein function. By Dranishnikov's Realization Theorem (see [5] or [10]) there is a compactum X such that $\dim_H(X) = \alpha(H)$ for all $H \in \mathcal{B}_{\mathcal{G}}$ and $\dim(X) = \max(\alpha)$. It is clear, in view of Bockstein's First Theorem, that X satisfies the desired conditions. ∎

THEOREM 7.5. *Suppose L is a pointed countable CW complex, G is an Abelian group, and $n \geq 1$. The space* $\mathrm{SP}(L)$ *is of the same extension type as $K(G,n)$ if and only if the following conditions are satisfied*:

(a) $H_i(L) = 0$ *for* $i < n$.
(b) $\sigma(H_n(L)) = \sigma(G)$.
(c) $\sigma(H_i(L)) \subset \tau(G)$ *for all* $i \geq n$.

Proof. We can reduce the general case to that of $K(G,n)$ being a countable CW complex. Indeed, 6.4 implies that any $K(G,n)$ has the extension type of $K(G',n)$ such that G' is countable and $\sigma(G) = \sigma(G')$.

Assume $\mathrm{SP}(L)$ is of the same extension type as $K(G,n)$. Since $L \sim_{\mathrm{Gr}} K(G,n)$ (see 5.5 and 5.6), (a) follows. Pick $i \geq n$ and denote $H_i(L)$ by F. Suppose $\sigma(F) \setminus \tau(G) \neq \emptyset$. By 7.3 there is a compactum X such that $\dim_G(X) = n$ and $\dim_F(X) = \infty$. Since $\mathrm{SP}(L)$ is of the same extension type as $K(G,n)$, $\dim_G(X) = n$ implies $\mathrm{SP}(L) \in \mathrm{AE}(X)$. Consequently, $\dim_F(X) = \dim_{H_i(L)}(X) \leq i$ (see 3.3), a contradiction. Thus (c) holds.

Denote $H_n(L)$ by F. Thus $\sigma(F) \subset \tau(G)$. Suppose $\sigma(F) \setminus \sigma(G) \neq \emptyset$. By 7.4 there is a compact space X such that $\dim_G(X) = \dim(X) = n+1$ and $\dim_F(X) = n$. Since $\dim(X) = n+1$, $\dim_{H_i(L)}(X) \leq i$ for all $i \geq n$. Consequently (see 3.3), $\mathrm{SP}(L) \in \mathrm{AE}(X)$, which implies $K(G,n) \in \mathrm{AE}(X)$, contradicting $\dim_G(X) = n+1$. That proves $\sigma(F) \subset \sigma(G)$.

Suppose $\sigma(G) \setminus \sigma(F) \neq \emptyset$. By 7.4 there is a compact space X such that $\dim_F(X) = \dim(X) = n+1$ and $\dim_G(X) = n$. Since $\mathrm{SP}(L)$ is of the same extension type as $K(G,n)$, $\dim_G(X) = n$ implies $\mathrm{SP}(L) \in \mathrm{AE}(X)$. Consequently, $\dim_F(X) = \dim_{H_n(L)}(X) \leq n$ (see 3.3), a contradiction. That proves (b).

Suppose (a), (b), and (c) hold. If $K(G,n) \in \mathrm{AE}(X)$ (i.e., $\dim_G(X) \leq n$), then $\dim_F(X) \leq n+1$ for all $F \in \tau(G)$ in view of Bockstein's Inequalities.

Hence, $\dim_{H_i(L)}(X) \le i$ for all $i \ge n$, which implies $\mathrm{SP}(L) \in \mathrm{AE}(X)$ (see 3.3). That shows $K(G,n) \le \mathrm{SP}(L)$.

If $\mathrm{SP}(L) \in \mathrm{AE}(X)$, then $K(H_n(L),n) \in \mathrm{AE}(X)$ (see 3.3), which is equivalent to $K(G,n) \in \mathrm{AE}(X)$ in view of $\sigma(G) = \sigma(H_n(L))$ and the First Bockstein Theorem. ∎

8. Dimension types and the connectivity index. Shchepin's connectivity index is of homological nature. Similarly, one can introduce the *homotopy connectivity index* $\mathrm{hcin}(K)$.

DEFINITION 8.1. Suppose X is a pointed space. Then $\mathrm{hcin}(X)$ is a non-negative integer defined as follows:

(a) $\mathrm{hcin}(X) = 0$ means that X is not path-connected.
(b) If $0 < r < \infty$, then $\mathrm{hcin}(X) = r$ means that $\pi_r(X) \ne 0$ and $\pi_k(X) = 0$ for all $0 \le k < r$.
(c) $\mathrm{hcin}(X) = \infty$ means that $\pi_k(X) = 0$ for all $0 \le k < \infty$.

Homotopy connectivity index can be easily dualized. Following G. Whitehead [33, pp. 421–423] we introduce the *anticonnectivity index* $\mathrm{acin}(X)$ as follows.

DEFINITION 8.2. Suppose X is a pointed space. Then $\mathrm{acin}(X)$ is an integer greater than or equal to -1, or infinity, defined as follows:

(a) $\mathrm{acin}(X) = -1$ means that X is path-connected and all its homotopy groups are trivial.
(b) If $0 \le r < \infty$, then $\mathrm{acin}(X) = r$ means that $\pi_r(X) \ne 0$ and $\pi_k(X) = 0$ for all $k > r$.
(c) $\mathrm{acin}(X) = \infty$ means that infinitely many homotopy groups of X are non-trivial.

We start with the concept of the total function space which is related to Shchepin's [28] concept of the total cohomology of a space.

DEFINITION 8.3. Suppose X is a pointed compact space and P is a pointed CW complex. The *total function space* $\mathrm{Tot}(P^X)$ is the wedge of all function spaces $P^{X/A}$, where A is a closed subspace of X.

Using the homotopy connectivity index and the concept of the total function space one can introduce a new relation on the class of compact spaces: $X \sim Y$ iff $\mathrm{hcin}(\mathrm{Tot}(P^X)) = \mathrm{hcin}(\mathrm{Tot}(P^Y))$ for all pointed CW complexes P. It turns out that this relation means exactly that ΣX and ΣY are of the same extension dimension. The first part of this section is devoted to that fact and culminates in 8.7.

In the case of pointed CW complexes one can define a relation $K \sim L$ to mean that $\mathrm{cin}(K \wedge M) = \mathrm{cin}(L \wedge M)$ for all M. We saw in 5.6 and 5.7

that, in the case of countable CW complexes, that relation is identical with equality of extension types of $\mathrm{SP}(K)$ and $\mathrm{SP}(L)$. Dualizing that relation leads to $K \sim L$ iff $\mathrm{acin}(P^K) = \mathrm{acin}(P^L)$ for all P. In the last part of this section we investigate how extension types are connected to that relation.

LEMMA 8.4. *Let X be a compact connected space. If P is a pointed CW complex, then*

$$\mathrm{hcin}(\mathrm{Tot}(\Omega P)^X) \geq \mathrm{hcin}(\mathrm{Tot}(P)^X) - 1.$$

Moreover, if $\mathrm{hcin}(\mathrm{Tot}(P)^X) \geq 1$, *then*

$$\mathrm{hcin}(\mathrm{Tot}(\Omega P)^X) = \mathrm{hcin}(\mathrm{Tot}(P)^X) - 1.$$

Proof. If $\mathrm{hcin}(\mathrm{Tot}(P)^X) = 0$, then the inequality obviously holds, so assume $\mathrm{hcin}(\mathrm{Tot}(P)^X)$ is at least 1. Now, 8.4 follows from the equality $[S^{n-1}, (\Omega P)^{X/A}] = [S^n, P^{X/A}]$ for all non-empty closed subsets A of X and all $n \geq 1$. ■

REMARK 8.5. If $\mathrm{hcin}(\mathrm{Tot}(P)^X) = 0$, then $\mathrm{hcin}(\mathrm{Tot}(\Omega P)^X)$ may be arbitrarily high. For example, if $P = X = S^1$, then $\mathrm{Tot}(\Omega P)^X$ is contractible but $\mathrm{Tot}(P)^X$ is not path-connected.

LEMMA 8.6. *Let X be a compact connected space and $k > 0$. If P is a pointed CW complex, then the following conditions are equivalent*:

(a) $\mathrm{hcin}(\mathrm{Tot}(P)^X) \geq k$.
(b) $\Omega^k(P) \in \mathrm{AE}(X)$.

Proof. Notice that both $\mathrm{hcin}(\mathrm{Tot}(P)^X)$ and $\Omega^k(P)$ depend only on the component of the base point of P, so we may reduce 8.6 to the case of P being connected.

SPECIAL CASE: $k = 1$. Notice that $\mathrm{hcin}(\mathrm{Tot}(P^X)) \geq 1$ is equivalent to $[X/A, P] = *$ for all non-empty closed subsets A of X. That statement, in view of 2.7, is equivalent to $\Omega P \in \mathrm{AE}(X)$.

If $k > 1$, then $\mathrm{hcin}(\mathrm{Tot}(P^X)) \geq k$ is equivalent (in view of 8.4) to

$$\mathrm{hcin}(\mathrm{Tot}((\Omega^{k-1} P)^X)) \geq 1,$$

which is equivalent, by the Special Case, to $\Omega^k(P) \in \mathrm{AE}(X)$. ■

THEOREM 8.7. *If X and Y are non-empty connected compact spaces, the following conditions are equivalent*:

(a) *ΣX and ΣY are of the same extension dimension.*
(b) *$\Omega(P) \in \mathrm{AE}(X)$ is equivalent to $\Omega(P) \in \mathrm{AE}(Y)$ for all pointed CW complexes P.*
(c) $\mathrm{hcin}(\mathrm{Tot}(P^X)) = \mathrm{hcin}(\mathrm{Tot}(P^Y))$ *for all pointed CW complexes P.*

Proof. (a)⇒(b). Suppose $\Omega P \in \mathrm{AE}(X)$ for some pointed CW complex P. Since $\Omega P = \Omega P_0$, where P_0 is the component of the base point of P, we may assume P is connected. By 2.7, $P \in \mathrm{AE}(\Sigma X)$, which implies $P \in \mathrm{AE}(\Sigma Y)$, as ΣY is of the same extension dimension as ΣX. Again, by 2.7, $\Omega P \in \mathrm{AE}(X)$. The same argument shows that $\Omega P \in \mathrm{AE}(Y)$ implies $\Omega P \in \mathrm{AE}(X)$ for any pointed CW complex P.

(b)⇒(c). Suppose $\mathrm{hcin}(\mathrm{Tot}(P^X)) \geq k$. It suffices to show (by symmetry) that $\mathrm{hcin}(\mathrm{Tot}(P^Y)) \geq k$. If $k = 0$ this is obviously true, so assume $k \geq 1$. However, in that case, $\mathrm{hcin}(\mathrm{Tot}(P^T)) \geq k$ is equivalent (see 8.6) to $\Omega^k(P) \in \mathrm{AE}(T)$ for any compact connected space T, so (b)⇒(c) follows from 8.6 as $\Omega^k(P) \in \mathrm{AE}(X)$ is equivalent to $\Omega^k(P) \in \mathrm{AE}(Y)$.

(c)⇒(a). Suppose $P \in \mathrm{AE}(\Sigma X)$. It suffices to show $P \in \mathrm{AE}(\Sigma Y)$. Since ΣX is connected and contains at least two points, P must be connected (otherwise we pick $x_1 \neq x_2 \in \Sigma X$, map them to two different components of P, and that map cannot be extended over ΣX), so (see 2.7) $P \in \mathrm{AE}(\Sigma X)$ is equivalent to $\Omega P \in \mathrm{AE}(X)$. Hence $\Omega P \in \mathrm{AE}(X)$ and, by 2.7, $P \in \mathrm{AE}(\Sigma Y)$. ∎

8.7 implies that if two compacta X and Y are of the same extension dimension, then the total function spaces $\mathrm{Tot}(P^X)$ and $\mathrm{Tot}(P^Y)$ have the same homotopy connectivity index. The rest of this section is devoted to the dual result: if two countable complexes K and L are of the same extension type, then the function spaces P^K and P^L have the same anticonnectivity index for certain P.

Since some of the techniques of this section are well known (see [20] and [10]), we will only outline how one translates known results in terms of truncated cohomology to results in terms of function spaces.

The following result corresponds to the Combinatorial Vietoris–Begle Theorem of [20] (see Lemma 2 there or Lemma 5.9 in [10]) and the proof is similar.

PROPOSITION 8.8. *Suppose P is a CW complex and $f : K \to L$ is a combinatorial map from a CW complex K to a finite simplicial complex L. If $P^{f^{-1}(\Delta)}$ is weakly contractible for each simplex Δ of L, then $f^* : P^L \to P^K$ is a weak homotopy equivalence.*

The next two results isolate essential parts of Lemma 3 in [20] with proofs being similar.

LEMMA 8.9. *Suppose P is a CW complex. The following conditions are equivalent*:

(1) $[S^k, P]$ *is finite for each* $k \geq 0$ (*unpointed spheres*).
(2) $[K, P]$ *is finite for each finite CW complex* K.
(3) *For each pair* (K, L) *of finite CW complexes and each map* $f : L \to P$, *the set of all homotopy classes rel.* L *of maps* $g : K \to P$ *such that* $g|L = f$, *is finite.*

LEMMA 8.10. *Suppose P is a CW complex such that $[S^k, P]$ is finite for each $k \geq 0$ (unpointed spheres). If K is a countable CW complex and $f : K \to P$ is a map such that $f|L \sim 0$ for each finite subcomplex L of K, then $f \sim$* const.

THEOREM 8.11. *Suppose P is a CW complex and K is a countable CW complex such that P^K is weakly contractible. Suppose L is a countable complex and $f : L \to P$ is a homotopically non-trivial map. There is a compactum X and a map $g : X \to L$ so that $f \circ g$ is homotopically non-trivial and $K \in \mathrm{AE}(X)$ if one of the following conditions is satisfied*:

(1) *K is compact and L is finitely dominated*,
(2) *$[S^k, P]$ is finite for each $k \geq 0$ (unpointed spheres).*

Proof. In the case of (2), one follows the same technique as described in [10] (see 5.5–5.7 there). We will outline how to use that technique for both K and L being compact. The inductive step consists in constructing a homotopically non-trivial map $f' : L' \to P$ and a map $h' : L_0 \to K$, where L' is a compact simplicial complex and L_0 is a subcomplex of L'. As in Lemma 5.6 of [10] one constructs a combinatorial map $\pi : L'' \to L'$ so that $\pi^{-1}(\Delta)$ is either contractible or homotopy equivalent to K for each simplex Δ of L'. Moreover, the composition $\pi^{-1}(L_0) \to L_0 \to K$ extends over L''. The construction in [15] produces L'' as a subcomplex of $L' \times K$, while [10] constructs L'' as the pull-back of a certain diagram. 8.8 says that the composition $L'' \to L' \to P$ is homotopically non-trivial. Starting with $f : L \to P$ one can construct inductively finite simplicial complexes L_n and maps $f_n : L_{n+1} \to L_n$ so that the inverse limit X of the sequence $\cdots \to L_{n+1} \to L_n \to \cdots$ satisfies $K \in \mathrm{AE}(X)$, L_1 is homotopically equivalent to L, and the composition $L_n \to \cdots \to L_1 \to P$ is homotopically non-trivial for all n (see [10, 5.5 and 5.9] for details).

If L is finitely dominated, we pick L' finite and maps $u : L \to L'$, $d : L' \to L$ so that $d \circ u \approx \mathrm{id}_L$. Let $f' = f \circ d : L' \to P$. Notice that f' is homotopically non-trivial. By the previous case, there is a compactum X and a map $g' : X \to L'$ so that $K \in \mathrm{AE}(X)$ and $f' \circ g'$ is homotopically non-trivial. Let $g = d \circ g'$. Since $f \circ g = f' \circ g'$, it is homotopically non-trivial. ∎

COROLLARY 8.12. *Suppose $K \leq L$ are countable pointed complexes. Suppose P is a pointed CW complex so that $\pi_i(P^K) = 0$ for all $i \geq r$, where $r \geq 1$. Then the homotopy groups $\pi_i(P^L)$ are trivial for all $i \geq r$ if one of the following conditions hold*:

(1) *$\pi_i(P)$ is finite for all $i \geq r$*,
(2) *K is compact and L is finitely dominated*,
(3) *both K and L are finitely dominated and $r \geq 2$.*

Proof. Suppose $\pi_m(P^L) \neq 0$ for some $m \geq r$. Put $L' = \Sigma(L)$ in cases (1) and (2), and $L' = \Sigma^2(L)$ in case (3). Put $P' = \Omega^{m-1}(P)$ in cases (1) and (2), and $P' = \Omega^{m-2}(P)$ in case (3). Put $K' = K$ in cases (1) and (2), and $K' = \Sigma(K)$ in case (3). Notice that K' and L' are homotopically equivalent to compact CW complexes in cases (2) and (3) (see [32]). In all cases there is a homotopically non-trivial map $f : L' \to P'$, so there is a map $g : X \to L'$ such that $K' \in \mathrm{AE}(X)$ and $f \circ g$ is homotopically non-trivial (see 8.11). This contradicts 2.1. Indeed, $K \leq L$ implies $\Sigma(K) \leq \Sigma(L)$ (see [11]). That means $K' \in \mathrm{AE}(X)$ implies that any map $X \to L'$ is null-homotopic by 2.1. ∎

COROLLARY 8.13. *Suppose $K \leq L$ are countable pointed complexes and $\mathbf{p}$ is a prime. If $H^*(K; \mathbb{Z}/\mathbf{p}) = 0$, then $H^*(L; \mathbb{Z}/\mathbf{p}) = 0$.*

Proof. Suppose $H^*(K; \mathbb{Z}/\mathbf{p}) = 0$ and $H^n(L; \mathbb{Z}/\mathbf{p}) \neq 0$ for some $n \geq 0$. If $n = 0$, then L must be disconnected and K is connected. Hence $K \in \mathrm{AE}(S^1)$ and $L \notin \mathrm{AE}(S^1)$, a contradiction. Thus, $n \geq 1$. Put $P = K(\mathbb{Z}/\mathbf{p}, n+1)$. Notice that $\pi_1(P^L) = H^n(L; \mathbb{Z}/\mathbf{p}) \neq 0$ and $\pi_i(P^K) = H^{n+1-i}(K; \mathbb{Z}/\mathbf{p}) = 0$ for all $i \geq 1$, a contradiction in view of 8.12. ∎

THEOREM 8.14. *Suppose L is a countable, finite-dimensional CW complex and $\mathbf{p}$ is a prime number. If $H^n(L; \mathbb{Z}/\mathbf{p}) \neq 0$ for some $n \geq 1$, then there is a non-trivial map $f : X \to \Sigma^{n+2}(L)$ from a compactum X so that $\dim_{\mathbb{Z}[1/\mathbf{p}]} X = 1 = \dim_{\mathbb{Z}/\mathbf{p}} X$.*

Proof. Let
$$M = M(\mathbb{Z}/\mathbf{p}, 1), \qquad L' = \Sigma^{n+2}(L).$$
In particular, $H^{2n+2}(L'; \mathbb{Z}/\mathbf{p}) \neq 0$. Since $K(\mathbb{Z}, 2n+3)^M$ is a $K(\mathbb{Z}/\mathbf{p}, 2n+2)$, there is a non-trivial map $g : L' \to K(\mathbb{Z}, 2n+3)^M$. Its adjoint $g' : L' \wedge M \to K(\mathbb{Z}, 2n+3)$ is non-trivial and we may assume that its image is contained in a finite subcomplex A of $K(\mathbb{Z}, 2n+3)$ which is $(2n+2)$-connected. Notice that $P = A^M$ is simply connected and its homotopy groups are finite. Indeed, the homotopy groups of P coincide with groups of homotopy classes of maps from suspensions of M to A. Those sets are the same as mod $\mathbf{p}$ homotopy groups of A (see [26]). In view of Proposition 1.4 on p. 3 in [26] one has an exact sequence
$$0 \to \pi_m(A) \otimes \mathbb{Z}/\mathbf{p} \to \pi_m(A; \mathbb{Z}/\mathbf{p}) \to \pi_{m-1}(A) * \mathbb{Z}/\mathbf{p} \to 0,$$
and since the homotopy groups of A are finitely generated, the homotopy groups of P are finite. By Miller's Theorem (Sullivan Conjecture—see [25]), $A^{K(\mathbb{Z}/\mathbf{p},1)}$ is weakly contractible as A is finite-dimensional. Therefore, $P^{K(\mathbb{Z}/\mathbf{p},1)} = (A^{K(\mathbb{Z}/\mathbf{p},1)})^M$ is weakly contractible. Since $K(\mathbb{Z}[1/\mathbf{p}], 1) \wedge M$ is contractible (compute its homology groups), $P^{K(\mathbb{Z}[1/\mathbf{p}],1)}$ is weakly contractible. Let $K = K(\mathbb{Z}/\mathbf{p}, 1) \vee K(\mathbb{Z}[1/\mathbf{p}], 1)$. Notice that P^K is weakly

contractible. Applying 8.11 one gets a non-trivial map $f : X \to L'$ so that $K \in \mathrm{AE}(X)$. ∎

8.14 and 2.1 imply the following.

Corollary 8.15. *Suppose L is a countable, finite-dimensional CW complex and $\mathbf{p}$ is a prime number. If $H^n(L;\mathbb{Z}/\mathbf{p}) \neq 0$ for some $n \geq 1$, then there is a compactum X so that $\dim_{\mathbb{Z}[1/\mathbf{p}]} X = 1 = \dim_{\mathbb{Z}/\mathbf{p}} X$ and L is not an absolute extensor of X.*

9. Extension types of infinite symmetric products. In this section we consider Problem 7.1 in the case of K being an infinite symmetric space and L being a compact CW complex (respectively, a countable, finite-dimensional CW complex).

Lemma 9.1. *Suppose $\mathbf{p}$ is a prime. The following conditions are equivalent for any countable connected pointed CW complex K:*

(1) *$H^*(\mathrm{SP}(K);\mathbb{Z}/\mathbf{p}) = 0$.*
(2) *$\mathbb{Z}/\mathbf{p}^\infty \notin \sigma(H_s(K))$ for all $s \geq 1$.*

Proof. (1)⇒(2). Let $G_i = H_i(K)$ for $i \geq 1$. If $\mathbf{p} \cdot G_i \neq G_i$ for some i, then there is a non-trivial map $K(G_i, i) \to K(G_i/\mathbf{p} \cdot G_i, i)$, a contradiction as $G_i/\mathbf{p} \cdot G_i$ is a direct sum of copies of $\mathbb{Z}/\mathbf{p}$. Thus $\mathbf{p} \cdot G_i = G_i$ for all i. If $\mathbf{p}$-$\mathrm{Tor}(G_i) \neq 0$ for some i, then $K(G_i, i)$ dominates $K(\mathbb{Z}/\mathbf{p}^\infty, i)$. To complete the proof of (1)⇒(2) it suffices to show that $K = K(\mathbb{Z}/\mathbf{p}^\infty, i)$ has non-trivial $\mathbb{Z}/\mathbf{p}$-cohomology. Put $L = K(\mathbb{Z}/\mathbf{p}, i+2)$ and notice that $K \leq L$ in view of Bockstein's inequalities. Put $P = K(\mathbb{Z}/\mathbf{p}, i+4)$ and notice that the triviality of $\mathbb{Z}/\mathbf{p}$-cohomology of K means $\pi_n(P^K) = 0$ for all $n \geq 1$. Now, 8.13 says that $\pi_n(P^L) = 0$ for all $n \geq 1$, a contradiction as $\pi_2(P^K) = \mathbb{Z}/\mathbf{p}$.

(2)⇒(1). Notice that $M(\mathbb{Z}[1/\mathbf{p}], 1) \leq_{\mathrm{Gr}} K$. Indeed, $\mathbb{Z}/\mathbf{p}^\infty \notin \sigma(H_s(K))$ for all $s \geq 1$ means that $H_*(K)$ has no $\mathbf{p}$-torsion and is divisible by $\mathbf{p}$. If $F \otimes \mathbb{Z}[1/\mathbf{p}] = 0$ and F is a Bockstein group, then F must be either $\mathbb{Z}/\mathbf{p}$ or $\mathbb{Z}/\mathbf{p}^\infty$. Therefore $H_*(K;F) = 0$, which completes the proof of $M(\mathbb{Z}[1/\mathbf{p}], 1) \leq_{\mathrm{Gr}} K$. Also, $H^*(K(\mathbb{Z}[1/\mathbf{p}], 1);\mathbb{Z}/\mathbf{p}) = 0$. If $H^*(\mathrm{SP}(K);\mathbb{Z}/\mathbf{p}) \neq 0$, then there is a compactum X so that $K(\mathbb{Z}[1/\mathbf{p}], 1) \in \mathrm{AE}(X)$ but $\mathrm{SP}(K) \notin \mathrm{AE}(X)$ (see 8.15), which contradicts

$$K(\mathbb{Z}[1/\mathbf{p}], 1) \sim_{\mathrm{Gr}} M(\mathbb{Z}[1/\mathbf{p}], 1) \leq_{\mathrm{Gr}} K \sim_{\mathrm{Gr}} \mathrm{SP}(K)$$

(see 5.5 and 5.6). ∎

Theorem 9.2. *Suppose L is a connected countable CW complex and $n \geq 0$. If $\Sigma^n(\mathrm{SP}(L))$ has the extension type of a countable, finite-dimensional and non-trivial CW complex, then either $\mathrm{SP}(L)$ is of the same extension type as $K(\mathbb{Z}_{(l)}, 1)$ for some subset l of primes or it is of the same extension type as $K(\mathbb{Q}, m)$ for some $m \geq 1$.*

Proof. Let $G_i = H_i(L)$ for $i \geq 1$.

Suppose $\mathrm{SP}(L)$ is not of the same extension type as $K(\mathbb{Q}, m)$ for any m. There is the smallest r with $\sigma(G_r) \neq \sigma(\mathbb{Q})$ (see 7.5). There must be a prime $\mathbf{p}$ with $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G_r)$, which means $H^*(\mathrm{SP}(L); \mathbb{Z}/\mathbf{p}) \neq 0$ (see 9.1). Suppose $\Sigma^n(\mathrm{SP}(L)) \sim K$, where K is finite-dimensional. That means $H^*(K; \mathbb{Z}/\mathbf{p}) \neq 0$ (see 8.13) and there is a compactum $X_{\mathbf{p}}$ with $\dim_{\mathbb{Z}[1/\mathbf{p}]}(X_{\mathbf{p}}) = 1 = \dim_{\mathbb{Z}/\mathbf{p}}(X_{\mathbf{p}})$ so that K is not an absolute extensor of $X_{\mathbf{p}}$ (see 8.15). If $r \geq 2$, then $\dim_{G_i}(X_{\mathbf{p}}) \leq i$ for all i as $\dim_{\mathbb{Z}}(X_{\mathbf{p}}) \leq 2$. Hence $\mathrm{SP}(L) \in \mathrm{AE}(X_{\mathbf{p}})$, which implies $K \in \mathrm{AE}(X_{\mathbf{p}})$, a contradiction. Thus, $r = 1$. If $\sigma(G_1) \neq \sigma(\mathbb{Z}_{(l)})$ for all sets l of primes, then the $\mathbf{p}$ above may be chosen so that $\mathbb{Z}_{(\mathbf{p})} \notin \sigma(G_1)$, which implies $\dim_{G_1}(X_{\mathbf{p}}) = 1$. Again, $\mathrm{SP}(L) \in \mathrm{AE}(X_{\mathbf{p}})$, which implies $K \in \mathrm{AE}(X_{\mathbf{p}})$ (see 4.4), a contradiction. Assume $\sigma(G_1) = \sigma(\mathbb{Z}_{(l)})$. If $\sigma(G_s) \subset \sigma(\mathbb{Z}_{(l)})$ for each $s > 1$, then we are done by 7.5. Suppose $\sigma(G_s) \subset \sigma(\mathbb{Z}_{(l)})$ does not hold for some $s > 1$. There must be a prime $\mathbf{p} \notin l$ so that $\mathbb{Z}/\mathbf{p}^\infty \in \sigma(G_s)$. Again, there is a compactum $X_{\mathbf{p}}$ with $\dim_{\mathbb{Z}[1/\mathbf{p}]}(X_{\mathbf{p}}) = 1 = \dim_{\mathbb{Z}/\mathbf{p}}(X_{\mathbf{p}})$ so that K is not an absolute extensor of $X_{\mathbf{p}}$. This implies $\dim_{G_i} X_{\mathbf{p}} \leq 1$ for all i and $\mathrm{SP}(L) \in \mathrm{AE}(X_p)$. Again, $K \in \mathrm{AE}(X_{\mathbf{p}})$, a contradiction. ∎

THEOREM 9.3. *Suppose L is a connected countable CW complex and $n \geq 0$. If $\Sigma^n(\mathrm{SP}(L))$ is of a compact non-trivial extension type (i.e., there is a compact CW complex K of the same extension type as $\Sigma^n(\mathrm{SP}(L))$), then $\mathrm{SP}(L)$ is of the same extension type as S^1.*

Proof. Let K be a compact CW complex of the same extension type as $\Sigma^n(\mathrm{SP}(L))$. By 9.2 either $\mathrm{SP}(L) \sim K(\mathbb{Z}_{(l)}, 1)$ or $\mathrm{SP}(L) \sim K(\mathbb{Q}, m)$ for some $m \geq 1$. I l is not the set of all primes or $\mathrm{SP}(L) \sim K(\mathbb{Q}, m)$ for some $m \geq 1$, then there is a prime $\mathbf{p}$ such that $H^*(\mathrm{SP}(L); \mathbb{Z}/\mathbf{p}) = 0$ (choose $\mathbf{p} \notin l$ if $\mathrm{SP}(L) \sim K(\mathbb{Z}_{(l)}, 1)$ or any $\mathbf{p}$ if $\mathrm{SP}(L) \sim K(\mathbb{Q}, m)$). That implies $H^*(K; \mathbb{Z}/\mathbf{p}) = 0$ by 8.13. Since K is finite, $\widetilde{H}_*(K)$ must be a torsion graded group and $\widetilde{H}_*(K; \mathbb{Q}) = 0$. Thus, $\widetilde{H}_*(\mathrm{SP}(L); \mathbb{Q}) = 0$ (see 5.6), a contradiction as $\mathrm{SP}(L) \sim K(\mathbb{Z}_{(l)}, 1)$ or $\mathrm{SP}(L) \sim K(\mathbb{Q}, m)$ for some $m \geq 1$. ∎

Theorem 9.3 generalizes all the known theorems related to the difference between extension types:

(1) S^n and $K(\mathbb{Z}, n)$ are of different extension types for $n \geq 3$ ([6]).
(2) S^n and $K(\mathbb{Z}, n)$ are of different extension types for $n \geq 2$ ([20]).
(3) $M(\mathbb{Z}/\mathbf{p}, n)$ and $K(\mathbb{Z}/\mathbf{p}, n)$ are of different extension types for $n \geq 1$ ([24]).
(4) $M(\mathbb{Z}/2, 1)$ and $K(\mathbb{Z}/2, 1)$ are of different extension types ([23]).
(5) $\mathbb{R}P^n$ and $\mathbb{R}P^\infty$ are of different extension types for $n \geq 1$ ([12]).

Besides generalizing the above-mentioned results, a major reason for 9.3 was our interest in pursuing ways of proving/disproving existence of universal spaces of given cohomological dimension. In [14], the author generalized a result of Shvedov which states that, for any compact CW complex K, the class of compacta X such that K is an absolute extensor of X has a universal space. That generalization deals with K homotopy dominated by a compact CW complex. Thus a natural way to see if there is a universal space of a given cohomological dimension is to verify if a particular CW complex has extension type of a compact CW complex. Theorem 9.3 closes that route of proving existence of universal spaces for any infinite symmetric product but S^1.

LEMMA 9.4. *$M(\mathbb{Q}, n)$ and $K(\mathbb{Q}, n)$ are of the same extension type for all $n \geq 2$.*

Proof. $M = M(\mathbb{Q}, n)$ can be realized as the telescope of $f_m : S^n \to S^n$, where f_m is of degree $m!$ for $m \geq 1$. In particular, the homotopy groups of $M(\mathbb{Q}, n)$ are torsion groups for $i \geq 2n$. By Sullivan's Theorem [30] there is a map $f : M(\mathbb{Q}, n) \to K$ such that $H_i(K) = H_i(M) \otimes \mathbb{Q}$, $\pi_i(K) = \pi_i(M) \otimes \mathbb{Q}$ for all $i \geq 1$ and $f_* : H_i(M) \to H_i(K)$ corresponds to $H_i(M) \otimes \mathbb{Z} \to H_i(M) \otimes \mathbb{Q}$ for all $i \geq 1$. Thus, f is a homotopy equivalence. In particular, $\pi_i(M) = 0$ for $i \geq 2n$. Thus, $M(\mathbb{Q}, n) \sim K(\mathbb{Q}, n)$ (see [13]). ∎

THEOREM 9.5. *Suppose G is a countable Abelian group. The following conditions are equivalent*:

(1) *There is a Moore space $M(G, n)$ of the same extension type as $K(G, n)$.*
(2) *Either $n = 1$ and there is a subset l of primes such that $K(G, 1)$ is of the same extension type as $K(\mathbb{Z}_{(l)}, 1)$, or $n \geq 2$ and $K(G, n)$ is of the same extension type as $K(\mathbb{Q}, n)$.*

Proof. (2)⇒(1) follows from 9.4 and the fact that one can choose $M(\mathbb{Q}, 1)$ to be $K(\mathbb{Q}, 1)$.

(1)⇒(2). Notice that $\Sigma(M(G, n))$ is homotopy equivalent to a finite-dimensional CW complex and has the same extension type as $\Sigma(K(G, n))$. Use 9.2. ∎

REMARK 9.6. Theorem 9.5 solves the following problem posed by Dranishnikov (see [28, p. 983]): Is it true that for any compactum X and any countable group G the conditions $M(G, 1) \in \mathrm{AE}(X)$ and $K(G, 1) \in \mathrm{AE}(X)$ are equivalent?

PROBLEM 9.7. Suppose the extension type of a countable CW complex K is at most the extension type of a compact non-contractible CW complex L. Is there a finite-dimensional, countable CW complex M of the same extension type as K?

REMARK 9.8. One cannot replace compactness of L by finite-dimensionality of L. Indeed, $K(\mathbb{Z}, 2) \leq M(\mathbb{Q}, 2)$ but $K(\mathbb{Z}, 2)$ does not have the extension type of a countable finite-dimensional CW complex (see 9.2 and 7.5).

Problem 9.9. Suppose the extension type of a countable CW complex K is at most the extension type of a compact non-contractible CW complex L. Is there a universal space among all compacta X so that $K \in \mathrm{AE}(X)$?

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