EXTENSIONS OF A PROPERTY OF THE HEAT EQUATION TO LINEAR THERMOELASTICITY AND OTHER THEORIES*

By

W. A. Day

Hertford College, Oxford

Introduction. The heat equation

$$\theta_{xx} = \theta_t,$$

which describes the behavior of the temperature $\theta(x, t)$ in a rigid conductor, has the following property.

(P) Suppose that $\theta(x, t) \in C^2(R)$, where R is the closed rectangle $0 \leq x \leq 1, 0 \leq t \leq T$, and that $\theta(x, t)$ satisfies the heat equation on R and the boundary conditions $\theta(0, t) = \theta(1, t) = 0$ on the interval $0 \leq t \leq T$. Then the maximum modulus

$$\max_{0 \leqslant x \leqslant 1} |\theta(x, t)|$$

is a decreasing function of t for $0 \leq t \leq T$.

It should be noted that (P) holds whatever initial values the temperature may take on $t = 0$.

The adjective 'decreasing' is to be understood in the wide sense as meaning 'monotone nonincreasing'.

The condition $\theta(x, t) \in C^2(R)$ can easily be replaced by one that is weaker, but to insist upon the weakest smoothness hypotheses for our theorems would overburden them and obscure their main point and instead we shall habitually assume more then is really necessary.

Property (P) was formulated and proved first by Pólya and Szegö [1]. It is in fact a straightforward deduction from the maximum principle but a different method of proof, based upon convexity arguments, was discovered by Bellman [2] and it turns out that it is Bellman's method which is the more suited to proving the extensions we have in mind.

The heat equation describes the conduction of heat with considerable success. From the point of view of continuum mechanics, though, it rests upon highly restrictive assumptions, and it is interesting to ask if (P), or some suitably modified form of (P), continues to hold in other theories which reflect more nearly the behavior of real bodies.

The object of this paper is to show that modified forms of (P) remain true within both the quasi-static and the dynamic theories of coupled linear thermoelasticity—these theories remove the rigidity requirement—and also within certain theories which replace the parabolic heat equation by a linear hyperbolic equation and thereby ensure that temperature disturbances propagate at finite speed.

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The equations of one-dimensional linear thermoelasticity. We shall consider a slab $0 \leq x \leq a$ which is made from a homogeneous and isotropic material and which undergoes a motion in which the displacement vector is parallel to the x-axis of a system of rectangular cartesian coordinates. The temperature $\theta(x, t)$, the displacement $u(x, t)$, the entropy per unit volume $\eta(x, t)$, and the stress $\sigma(x, t)$ are functions of the coordinate x and the time t only and they satisfy the thermoelastic equations

$$k \frac{\partial^2 \theta}{\partial x^2} = \theta_0 \frac{\partial \eta}{\partial t},$$

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2},$$

$$\eta = \frac{c}{\theta_0}(\theta - \theta_0) + (3\lambda + 2\mu)\alpha \frac{\partial u}{\partial x},$$

$$\sigma = (\lambda + 2\mu) \frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha(\theta - \theta_0),$$

in which θ_0 is the (absolute) uniform reference temperature, k is the thermal conductivity, ρ is the density, c is the specific heat per unit volume, α is the coefficient of thermal expansion, and λ and μ are the elastic moduli. The reader may consult Boley and Weiner [3], Chadwick [4], or Carlson [5] for the derivation of these equations.

On substituting for η and σ in the first two equations we arrive at the temperature-displacement equations

$$k \frac{\partial^2 \theta}{\partial x^2} = c \frac{\partial \theta}{\partial t} + (3\lambda + 2\mu)\alpha\theta_0 \frac{\partial^2 u}{\partial x\, \partial t},$$

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} = (3\lambda + 2\mu)\alpha \frac{\partial \theta}{\partial x} + \rho \frac{\partial^2 u}{\partial t^2}.$$

We shall suppose, for the sake of comparison with (P), that the faces $x = 0$ and $x = a$ are maintained at the reference temperature and are clamped in an attempt to hold the slab as nearly rigid as can be, that is to say

$$\theta(0, t) = \theta(a, t) = \theta_0, \qquad u(a, t) = 0.$$

The abundance of constants is tiresome but we can reduce their number by making the change of variables

$$x \to x/a, \qquad t \to kt/ca^2$$

$$\theta \to (\theta - \theta_0)/\theta_0, \qquad u \to (\lambda + 2\mu)^{1/2}u/(\theta_0 c)^{1/2}a,$$

$$\eta \to \eta/c, \qquad \sigma \to \sigma/(\theta_0 c(\lambda + 2\mu))^{1/2},$$

whose effect is to transform the temperature-displacement equations to

$$\theta_{xx} = \theta_t + \delta u_{xt}, \tag{1}$$

$$u_{xx} = \delta\theta_x + \varepsilon u_{tt}, \tag{2}$$

the definitions of the entropy and the stress to

$$\eta = \theta + \delta u_x, \tag{3}$$

$$\sigma = u_x - \delta\theta, \tag{4}$$

and the boundary conditions to

$$\theta(0, t) = \theta(1, t) = u(0, t) = u(1, t) = 0, \tag{5}$$

where δ, ε are the dimensionless constants

$$\delta = (3\lambda + 2\mu)a\left(\frac{\theta_0}{(\lambda + 2\mu)c}\right)^{1/2}, \qquad \varepsilon = \frac{\rho k^2}{(\lambda + 2\mu)c^2a^2}.$$

We notice that δ depends upon the properties of the material, as measured by c, α, λ, μ, and the reference temperature, but not upon the thickness a. We shall impose the restriction $0 < \delta \leq 1$, which appears to be realistic under normal conditions. Indeed, Boley and Weiner [3, section 2.2], whose constant δ is our δ^2, show that the appropriate values for aluminium and steel at the reference temperature $\theta_0 = 366°\mathrm{K}$ are $\delta = 0.17$ and $\delta = 0.12$, approximately.

The constant ε decreases to zero as the thickness increases and if we set $\varepsilon = 0$ in (1), (2) or, what is the same thing, if we neglect inertia we get the quasi-static equations

$$\theta_{xx} = \theta_t + \delta u_{xt}, \tag{6}$$

$$u_{xx} = \delta\theta_x, \tag{7}$$

with which we shall be concerned in the next two sections. We get the classical theory, of course, if we take both $\delta = 0$ and $\varepsilon = 0$. In that case (6) reduces to the heat equation and (7) reduces to the equation $u_{xx} = 0$, which, in the light of the boundary conditions $u(0, t) = u(1, t) = 0$, implies that $u(x, t) \equiv 0$ or, in other words, that the slab is rigid.

An extension to quasi-static thermoelasticity. Our first extension of (P) says that, according to the quasi-static theory, the maximum modulus of the entropy decreases.

THEOREM 1. Suppose that $0 < \delta \leq 1$ and that the functions $\theta(x, t) \in C^2(R)$, $u(x, t) \in C^3(R)$ satisfy the quasi-static equations (6), (7) on R and the boundary conditions (5) on the interval $0 \leq t \leq T$. Then

$$\max_{0 \leq x \leq 1} |\eta(x, t)|$$

is a decreasing function of t for $0 \leq t \leq T$.

The key to the theorem is the observation that the entropy satisfies the heat equation

$$\eta_{xx} = (1 + \delta^2)\eta_t \text{ on } R, \tag{8}$$

and the conditions

$$\eta(0, t) = \eta(1, t) = -\delta^2 \int_0^1 \eta(x, t)\, dx \text{ for } 0 \leq t \leq T. \tag{9}$$

Eq. (8) is an immediate consequence of the definition of η and the quasi-static equations

(6), (7). To verify (9) we use the fact that, as (7) tells us, the stress $\sigma = u_x - \delta\theta$ is independent of x. Since

$$\eta = \theta + \delta u_x = \left(\delta + \frac{1}{\delta}\right)u_x - \frac{1}{\delta}\sigma$$

we see with the help of the boundary conditions $u(0, t) = u(1, t) = 0$ that

$$\sigma(t) = -\delta \int_0^1 \eta(x, t)\, dx \tag{10}$$

and, since

$$\eta = \theta + \delta u_x = (1 + \delta^2)\theta + \delta\sigma$$

and $\theta(0, t) = \theta(1, t) = 0$, we have arrived at the equations

$$\eta(0, t) = \eta(1, t) = \delta\sigma(t) = -\delta^2 \int_0^1 \eta(x, t)\, dx$$

which are the required conditions (9).

In order to prove the theorem it is enough to prove that the maximum modulus of the entropy is the pointwise limit of a sequence of functions, each of which decreases on $0 \le t \le T$. To this end we consider the functions

$$I_n(t) = \int_0^1 \eta(x, t)^{2n}\, dx + \delta^{4n-2}\left(\int_0^1 \eta(x, t)\, dx\right)^{2n},$$

n being any positive integer. If we differentiate $I_n(t)$ and appeal to the heat equation (8) we find that

$$\begin{aligned}
\frac{dI_n}{dt} &= 2n \int_0^1 \eta^{2n-1}\eta_t\, dx + 2n\, \delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1} \int_0^1 \eta_t\, dx \\
&= \frac{2n}{1+\delta^2} \int_1^2 \eta^{2n-1}\eta_{xx}\, dx + 2n\delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1} \int_0^1 \eta_t\, dx \\
&= \frac{2n}{1+\delta^2} \int_0^1 \left(\frac{\partial}{\partial x}(\eta^{2n-1}\eta_x) - (2n-1)\eta^{2n-2}\eta_x^2\right) dx \\
&\quad + 2n\delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1} \int_0^1 \eta_t\, dx \\
&= \frac{2n}{1+\delta^2} [\eta(1, t)^{2n-1}\eta_x(1, t) - \eta(0, t)^{2n-1}\eta_x(0, t)] \\
&\quad + 2n\delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1} \int_0^1 \eta_t\, dx \\
&\quad - \frac{2n(2n-1)}{1+\delta^2} \int_0^1 \eta^{2n-2}\eta_x^2\, dx.
\end{aligned}$$

However, the conditions (9) tell us that the expression enclosed within square brackets equals

$$-\delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1}(\eta_x(1,t)-\eta_x(0,t)) = -\delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1}\int_0^1 \eta_{xx}\, dx$$

$$= -(1+\delta^2)\delta^{4n-2}\left(\int_0^1 \eta\, dx\right)^{2n-1}\int_0^1 \eta_t\, dx$$

and, thus, that the derivate of $I_n(t)$ is

$$\frac{dI_n}{dt} = \frac{-2n(2n-1)}{1+\delta^2}\int_0^1 \eta^{2n-2}\eta_x^2\, dx \le 0,$$

whence $I_n(t)$ decreases for $0 \le t \le T$.

If we can show that

$$I_n(t)^{1/2n} \to \max_{0\le x\le 1} |\eta(x,t)| \text{ as } n \to \infty \tag{11}$$

we shall have proved the theorem. In view of the way $I_n(t)$ is defined, we have the estimates

$$\left(\int_0^1 \eta^{2n}\, dx\right)^{1/2n} \le I_n(t)^{1/2n}$$

$$\le (1+\delta^{4n-2})^{1/2n}\left(\int_0^1 \eta^{2n}\, dx\right)^{1/2n}$$

$$\le 2^{1/2n}\left(\int_0^1 \eta^{2n}\, dx\right)^{1/2n},$$

which follow on appealing to the restriction $0 < \delta \le 1$ and to the inequality

$$\left(\int_0^1 \eta\, dx\right)^{2n} \le \int_0^1 \eta^{2n}\, dx,$$

this last being an instance of Jensen's inequality. Since, as a lemma of M. Riesz assures us (see Beckenbach and Bellman [6], chapter 4, section 24),

$$\left(\int_0^1 \eta(x,t)^{2n}\, dx\right)^{1/2n} \to \max_{0\le x\le 1} |\eta(x,t)| \text{ as } n \to \infty,$$

and since $2^{1/2n} \to 1$ as $n \to \infty$, we conclude that (11) is correct, and the proof is complete.

Some additional consequences for quasi-static thermoelasticity. The hypotheses of Theorem 1 remain in force throughout this section. That theorem asserts the decreasing property of

$$\max_{0\le x\le 1} |\eta(x,t)|,$$

that is of

$$\max\{-\min_{0\le x\le 1} \eta(x,t),\ \max_{0\le x\le 1} \eta(x,t)\}.$$

In fact, we can deduce something about the behavior of the individual functions

$$\min_{0\leqslant x\leqslant 1} \eta(x, t), \qquad \max_{0\leqslant x\leqslant 1} \eta(x, t)$$

from what we have proved already.

We recall that, according to the quasi-static equation (7) the stress $\sigma = u_x - \delta\theta$ is independent of x. We recall too that the slab is subject to tension at an instant t at which $\sigma(t) \geq 0$, and is subject to compression at an instant at which $\sigma(t) \leq 0$. Of course we do not know in advance when $\sigma(t)$ is negative and when it is positive; we should have to solve the quasi-static equations in order to discover this information.

THEOREM 2. The function

$$\min_{0\leqslant x\leqslant 1} \eta(x, t)$$

is nonpositive for $0 \leq t \leq T$, and it increases on any subinterval of $0 \leq t \leq T$ on which $\sigma(t) \geq 0$.

The function

$$\max_{0\leqslant x\leqslant 1} \eta(x, t)$$

is nonnegative for $0 \leq t \leq T$, and it decreases on any subinterval of $0 \leq t \leq T$ on which $\sigma(t) \leq 0$.

We need prove only the first half of the theorem, for the second follows on making obvious changes.

The facts required for the proof are the heat equation (8), the formula (10) for the stress, and the conditions (9) which, in the light of (10), can be rewritten as

$$\eta(0, t) = \eta(1, t) = \delta\sigma(t) \qquad \text{for} \qquad 0 \leq t \leq T. \tag{12}$$

Suppose, contrary to what is claimed, that it were the case that

$$\min_{0\leqslant x\leqslant 1} \eta(x, t') > 0$$

for some t' in $0 \leq t' \leq T$. It would follow that $\eta(x, t') > 0$ for every x in $0 \leq x \leq 1$, which would conflict with (9) when $t = t'$, and so it must be that

$$\min_{0\leqslant x\leqslant 1} \eta(x, t) \leq 0 \qquad \text{for} \qquad 0 \leq t \leq T.$$

Next, suppose that $0 \leq t_0 < t_1 \leq T$ and that $\sigma(t) \geq 0$ on the interval $t_0 \leq t \leq t_1$. Let t', t'' be any numbers which satisfy $t_0 \leq t' < t'' \leq t_1$. Then, the conditions (12) and the maximum principle for the heat equation, applied to the rectangle $0 \leq x \leq 1$, $t' \leq t \leq t''$, tell us that

$$\min_{0\leqslant x\leqslant 1} (x, t'') \geq \min\{\min_{0\leqslant x\leqslant 1} \eta(x, t'), \delta \min_{t'\leqslant t\leqslant t''} \sigma(t)\}. \tag{13}$$

However, we know that

$$\min_{0\leqslant x\leqslant 1} \eta(x, t') \leq 0,$$

while the hypothesis on $\sigma(t)$ ensures that

$$\min_{t'\leqslant t\leqslant t''} \sigma(t) \geq 0.$$

Thus, the right-hand side of (13) reduces to

$$\min_{0\leqslant x\leqslant 1} \eta(x, t')$$

and we have confirmed that

$$\min_{0\leqslant x\leqslant 1} \eta(x, t'') \geq \min_{0\leqslant x\leqslant 1} \eta(x, t')$$

and, therefore, that

$$\min_{0\leqslant x\leqslant 1} \eta(x, t)$$

increases on the interval $t_0 \leq t \leq t_1$.

Our next theorem says that the minimum entropy must increase and the maximum entropy must decrease whenever the temperature is of constant sign.

THEOREM 3. Suppose that $0 \leq t_0 < t_1 \leq T$ and that $\theta(x, t)$ is either nonnegative or nonpositive throughout the rectangle $0 \leq x \leq 1, t_0 \leq t \leq t_1$. Then

$$\min_{0\leqslant x\leqslant 1} \eta(x, t)$$

is an increasing function of t, and

$$\max_{0\leqslant x\leqslant 1} \eta(x, t)$$

is a decreasing function of t, for $t_0 \leq t \leq t_1$.

We need consider only the case when $\theta(x, t) \geq 0$ throughout the rectangle $0 \leq x \leq 1$, $t_0 \leq t \leq t_1$, the proof when $\theta(x, t) \leq 0$ being similar. Since eq. (10) and the boundary conditions $u(0, t) = u(1, t) = 0$ imply that

$$\sigma = -\delta \int_0^1 \eta \, dx = -\delta \int_0^1 (\theta + \delta u_x) \, dx = -\delta \int_0^1 \theta \, dx,$$

we see that in this case $\sigma(t) \leq 0$ for $t_0 \leq t \leq t_1$ and we deduce immediately from the second half of Theorem 2 that

$$\max_{0\leqslant x\leqslant 1} \eta(x, t)$$

decreases on $t_0 \leq t \leq t_1$.

To prove that

$$\min_{0\leqslant x\leqslant 1} \eta(x, t)$$

increases on $t_0 \leq t \leq t_1$ we start with the observation that if $\theta(x, t) \geq 0$ throughout the rectangle $0 \leq x \leq 1$, $t_0 \leq t \leq t_1$ then, since $\theta(0, t) = \theta(1, t) = 0$ for $t_0 \leq t \leq t_1$, it must be that $\theta_x(0, t) \geq 0$ and $\theta_x(1, t) \leq 0$ for $t_0 \leq t \leq t_1$. It follows from eqs. (3), (6), (10) that

$$\frac{d\sigma}{dt} = -\delta \int_0^1 \eta_t \, dx = -\delta \int_0^1 \theta_{xx} \, dx = \delta(\theta_x(0, t) - \theta_x(1, t)) \geq 0$$

for every t in $t_0 \leq t \leq t_1$ and, hence, that $\sigma(t)$ must increase on the interval $t_0 \leq t \leq t_1$. Now let t', t'' be any numbers which satisfy $t_0 \leq t' \leq t'' \leq t_1$. The maximum principle tells us once again that the inequality (13) holds. However, the fact that $\sigma(t)$ increases implies that

$$\delta \min_{t'\leqslant t\leqslant t''} \sigma(t) = \delta\sigma(t') = \eta(0, t') \geq \min_{0\leqslant x\leqslant 1} \eta(x, t')$$

and, therefore, that (13) reduces to the inequality

$$\min_{0 \leq x \leq 1} \eta(x, t'') \geq \min_{0 \leq x \leq 1} \eta(x, t').$$

Thus we have proved that

$$\min_{0 \leq x \leq 1} \eta(x, t)$$

increases on the interval $t_0 \leq t \leq t_1$ and the proof of Theorem 3 is complete.

An extension to dynamic thermoelasticity. We turn to considering an initial and boundary value problem for the dynamic equations (1), (2), in which ε is a small positive constant. The obvious way to proceed is to attempt to show that solutions of the dynamic equations can be approximated closely by solutions of the quasi-static equations and then to make use of what we have proved already. It is clear, though, that for this procedure to work we shall have to impose restrictions upon the initial data, for in the dynamic case ($\varepsilon > 0$) we expect to be able to prescribe at will the initial temperature $\theta(x, 0)$, the initial displacement $u(x, 0)$, and the initial velocity $u_t(x, 0)$, whereas in the quasi-static case ($\varepsilon = 0$) we expect to be able to prescribe only the initial temperature. It turns out that the quasi-static equations do provide adequate approximations if the initial temperature and displacement are such as to make the initial stress homogeneous, that is to say independent of x, and if the initial velocity vanishes. Our proof of this fact is based to some extent upon Bobisud's discussion [7] of the behavior of the telegraph equation for large speeds.

We shall compare the solution of a dynamic problem (I) with the solution of a corresponding quasi-static problem (II).

(I) The functions $\theta(x, t)$, $u(x, t) \in C^2(R)$ satisfy the dynamic equations (1), (2) on R and the boundary conditions (5) on the interval $0 \leq t \leq T$. Furthermore

(i) $\theta(x, 0) = \phi(x)$ for $0 \leq x \leq 1$,

(ii) $\sigma(x, 0)$ is independent of x for $0 \leq x \leq 1$,

(iii) $u_t(x, 0) = 0$ for $0 \leq x \leq 1$.

(II) The functions $\theta^*(x, t) \in C^2(R)$, $u^*(x, t) \in C^3(R)$ satisfy the quasi-static equations (6), (7) on R, the boundary conditions (5) on the interval $0 \leq t \leq T$, and the initial condition

$$\theta^*(x, 0) = \phi(x) \qquad \text{for} \qquad 0 \leq x \leq 1.$$

It should be noted that $\theta^*(x, t)$ and $u^*(x, t)$ are determined uniquely by $\phi(x)$. Of course, $\theta(x, t)$, $u(x, t)$, $\eta(x, t)$ depend upon ε but $\theta^*(x, t)$, $u^*(x, t)$, and the corresponding entropy $\eta^*(x, t) = \theta^*(x, t) + \delta u_x^*(x, t)$, are independent of ε. We shall show that $\eta(x, t)$ converges to $\eta^*(x, t)$ in mean square as ε tends to zero at fixed δ, T and fixed initial temperature $\phi(x)$.

THEOREM 4. Suppose that $0 < \delta \leq 1$, $\varepsilon > 0$ and that $\theta(x, t)$, $u(x, t)$ are as in (I), while $\theta^*(x, t)$, $u^*(x, t)$ are as in (II). Then we can write the entropy as

$$\eta(x, t) = \eta^*(x, t) + \xi(x, t)$$

where (i) $\max_{0 \leq x \leq 1} |\eta^*(x, t)|$ decreases for $0 \leq t \leq T$,

(ii) $\int_0^1 \xi(x, t)^2 \, dx = O(\varepsilon)$ uniformly in t for $0 \leq t \leq T$.

Theorem 1 tells us that (i) holds and, thus, we have to prove (ii) only. Since the initial stress $\sigma(x, 0) = \sigma_0$, say, is independent of x we have

$$u_x(x, 0) - \delta\phi(x) = \sigma_0$$

for $0 \leq x \leq 1$ and, in view of the conditions $u(0, 0) = u(1, 0) = 0$, it must be that

$$\sigma_0 = -\delta \int_0^1 \phi(y)\, dy$$

and that

$$u_x(x, 0) = \delta\phi(x) - \delta \int_0^1 \phi(y)\, dy.$$

On the other hand, if we argue from the quasi-static equation (7), with t set equal to zero, the initial condition $\theta^*(x, 0) = \phi(x)$, and the conditions $u^*(0, 0) = u^*(1, 0) = 0$ we deduce exactly the same expression for $u_x^*(x, 0)$, that is to say

$$u_x(x, 0) = u_x^*(x, 0) \qquad \text{for} \qquad 0 \leq x \leq 1. \tag{14}$$

Now let us write

$$\psi(x, t) = \theta(x, t) - \theta^*(x, t),\ w(x, t) = u(x, t) - u^*(x, t)$$

and

$$E = \int_0^1 (\psi^2 + w_x^2 + \varepsilon w_t^2)\, dx.$$

Since $\theta(x, t)$, $u(x, t)$ satisfy eqs. (1), (2) and $\theta^*(x, t)$, $u^*(x, t)$ satisfy eqs. (6), (7) we have

$$\psi_{xx} = \psi_t + \delta w_{xt}, \tag{15}$$

$$w_{xx} = \delta\psi_x + \varepsilon w_{tt} + \varepsilon u_{tt}^* \tag{16}$$

on R. Moreover, the boundary conditions imply that

$$\psi(0, t) = \psi(1, t) = w(0, t) = w(1, t) = 0 \qquad \text{for} \qquad 0 \leq t \leq T, \tag{17}$$

and the initial conditions $\theta(x, 0) = \phi(x) = \theta^*(x, 0)$, $u_t(x, 0) = 0$ and Eq. (14) that

$$\psi(x, 0) = 0, \qquad w_x(x, 0) = 0, \qquad w_t(x, 0) = -u_t^*(x, 0) \qquad \text{for} \qquad 0 \leq x \leq 1,$$

and, hence, that

$$E(0) = \varepsilon \int_0^1 u_t^*(x, 0)^2\, dx. \tag{18}$$

If we differentiate $E(t)$ and appeal to (15), (16), (17) we find that its derivative is

$$\begin{aligned}\frac{dE}{dt} &= 2 \int_0^1 (\psi\psi_t + w_x w_{xt} + \varepsilon w_t w_{tt})\, dx \\ &= 2 \int_0^1 [\psi(\psi_{xx} - \delta w_{xt}) + w_x w_{xt} + w_t(w_{xx} - \delta\psi_x - \varepsilon u_{tt}^*)\, dx \\ &= 2 \int_0^1 \left[\frac{\partial}{\partial x}(\psi\psi_x - \delta\psi w_t + w_x w_t) - \psi_x^2 - \varepsilon w_t u_{tt}^*\right] dx\end{aligned}$$

$$= -2\int_0^1 (\psi_x^2 + \varepsilon w_t u_{tt}^*)\,dx$$

$$\leq -2\varepsilon\int_0^1 w_t u_{tt}^*\,dx$$

$$\leq 2\varepsilon\left(\int_0^1 w_t^2\,dx\right)^{1/2}\left(\int_0^1 u_{tt}^{*2}\,dx\right)^{1/2}$$

and, in this way, we arrive at the differential inequality

$$\frac{dE}{dt} \leq 2\varepsilon^{1/2}E^{1/2}\left(\int_0^1 u_{tt}^{*2}\,dx\right)^{1/2}$$

which implies the inequality

$$E(t)^{1/2} \leq E(0)^{1/2} + \varepsilon^{1/2}\int_0^t\left(\int_0^1 u_{tt}^*(x, s)^2\,dx\right)^{1/2} ds$$

upon integration. When we combine this last inequality with (18) we arrive at the uniform estimate

$$E(t) \leq \varepsilon A^2 \qquad \text{for} \qquad 0 \leq t \leq T,$$

where the constant

$$A = \left(\int_0^1 u_{tt}^*(x, 0)^2\,dx\right)^{1/2} + \int_0^T\left(\int_0^1 u_{tt}^*(x, s)^2\,dx\right)^{1/2} ds$$

depends upon δ, T and $\phi(x)$ only.

Lastly, since

$$\xi = \eta - \eta^* = \psi + \delta w_x,$$

Cauchy's inequality tells us that

$$\xi^2 \leq (1 + \delta^2)(\psi^2 + w_x^2)$$

and that

$$\int_0^1 \xi(x, t)^2\,dx \leq (1 + \delta^2)E(t) \leq \varepsilon(1 + \delta^2)A^2 \qquad \text{for} \quad 0 \leq t \leq T,$$

which completes the proof.

An extension to theories with a finite speed of propagation. A number of theories have the effect of replacing the heat equation by the telegraph equation

$$\kappa\theta_{xx} = \theta_t + \tau\theta_{tt}, \tag{19}$$

in which κ is the thermal diffusivity and τ is a characteristic time. Accounts of such theories can be found in the articles of Müller [8] and Meixner [9], for example.

Our final extension of (P) is

THEOREM 5. Suppose that $\kappa, \tau > 0$, that $\theta(x, t) \in C^2$ on the rectangle $0 \leq x \leq a, 0 \leq t \leq T$, where it satisfies eq. (19), and that is satisfies the boundary conditions $\theta(0, t) = \theta(a, t) = 0$ on the interval $0 \leq t \leq T$. Then

$$\max\left\{\max_{0\leqslant x\leqslant a} |\theta + (\kappa\tau)^{1/2}\theta_x + \tau\theta_t|, \qquad \max_{0\leqslant x\leqslant a} |\theta - (\kappa\tau)^{1/2}\theta_x + \tau\theta_t|\right\}$$

is a decreasing function of t for $0 \leq t \leq T$.

To prove the theorem we introduce

$$p = \theta + (\kappa\tau)^{1/2}\theta_x + \tau\theta_t, \qquad q = \theta - (\kappa\tau)^{1/2}\theta_x + \tau\theta_t,$$

which, as we can verify easily, satisfy the equations

$$p_t = \left(\frac{\kappa}{\tau}\right)^{1/2} p_x - \frac{1}{2\tau}(p - q), \tag{20}$$

$$q_t = -\left(\frac{\kappa}{\tau}\right)^{1/2} q_x + \frac{1}{2\tau}(p - q). \tag{21}$$

We propose to show that each of the functions

$$I_n(t) = \int_0^a (p(x, t)^{2n} + q(x, t)^{2n})\, dx,$$

where n is a positive integer, decreases for $0 \leq t \leq T$. If we use Eqs. (20), (21) we find the derivative of $I_n(t)$ to be

$$\begin{aligned}\frac{dI_n}{dt} &= 2n \int_0^a (p^{2n-1}p_t + q^{2n-1}q_t)\, dx \\ &= 2n \int_0^a \left[\left(\frac{\kappa}{\tau}\right)^{1/2}(p^{2n-1}p_x - q^{2n-1}q_x) - \frac{1}{2\tau}(p - q)(p^{2n-1} - q^{2n-1})\right] dx \\ &= \int_0^a \left[\left(\frac{\kappa}{\tau}\right)^{1/2} \frac{\partial}{\partial x}(p^{2n} - q^{2n}) - \frac{n}{\tau}(p - q)(p^{2n-1} - q^{2n-1})\right] dx,\end{aligned}$$

and, on using the elementary inequality

$$(p - q)(p^{2n-1} - q^{2n-1}) \geq 0,$$

we deduce that

$$\frac{dI_n}{dt} \leq \left(\frac{\kappa}{\tau}\right)^{1/2} \int_0^a \frac{\partial}{\partial x}\left(p^{2n} - q^{2n}\right) dx. \tag{22}$$

Since the boundary conditions imply

$$p(0, t) = (\kappa\tau)^{1/2}\theta_x(0, t) = -q(0, t),$$

$$p(a, t) = (\kappa\tau)^{1/2}\theta_x(a, t) = -q(a, t),$$

the right-hand side of (22) vanishes and so it is indeed the case that $I_n(t)$ decreases for $0 \leq t \leq T$.

Now let

$$J_n = \int_0^a p^{2n}\,dx, \qquad K_n = \int_0^a q^{2n}\,dx,$$

so that

$$I_n = J_n + K_n,$$
$$\max\{J_n, K_n\} \le I_n \le 2\max\{J_n, K_n\},$$
$$\max\{J_n^{1/2n}, K_n^{1/2n}\} \le I_n^{1/2n} \le 2^{1/2n}\max\{J_n^{1/2n}, K_n^{1/2n}\}.$$

On letting n tend to infinity and appealing to Riesz's lemma again we arrive at the relation

$$I_n(t)^{1/2n} \to \max\left\{\max_{0\leqslant x\leqslant a}|p(x,t)|, \qquad \max_{0\leqslant x\leqslant a}|q(x,t)|\right\} \qquad \text{as} \quad n\to\infty$$

which suffices to complete the proof.

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