Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Extensions of Banach contraction principle to partial cone metric spaces over a non-normal solid cone

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**Abstract**
In this paper, we present some extensions of Banach contraction principle to partial cone metric spaces over a non-normal solid cone, which improve many recent fixed point results in cone metric spaces and partial cone metric spaces. An example is given to support the usability of our results.
**MSC:** 06A07; 47H10

**Keywords:** Banach contraction principle; partial cone metric space; non-normal cone; solid cone

## 1 Introduction

The Banach contraction principle is the most celebrated fixed point theorem, which has been extended in various directions. In 2007, Huang and Zhang [1] introduced cone metric spaces and extended the Banach contraction principle to cone metric spaces over a normal solid cone, being unaware that cone metric spaces already existed under the name of $K$-metric spaces and $K$-normed spaces that were introduced and used in the middle of the 20th century in [2–9]. Furthermore, Huang and Zhang defined the convergence via interior points of the cone. Such an approach allows the investigation of the case that the cone is not necessarily normal, for example, the authors in [10–18] established many fixed point results and common fixed point results in cone metric spaces over a non-normal cone. In 2012, based on the definition of cone metric spaces and partial metric spaces, which were introduced by Matthews [19], Sonmez [20, 21] defined a partial cone metric space and considered the extensions of Banach contraction principle to partial cone metric spaces.

It is worth mentioning that in most of the preceding references concerned with fixed point results of contractions in cone metric spaces and partial cone metric spaces, the contractions are always assumed to be restricted with a constant. In [9], Agarwal considered a contraction restricted with a positive linear mapping and proved the following fixed point theorem in cone metric spaces.

**Theorem 1** (See [9]) *Let $(X,d)$ be a complete cone metric space over $\mathbb{R}^n_+$ and $T: X \to X$. If there exists a linear bounded mapping $L: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ with the spectral radius $r(L) < 1$ such*

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*that*

$$d(Tx, Ty) \preceq Ld(x, y), \quad \forall x, y \in X. \tag{1}$$

*Then $T$ has a unique fixed point $x^* \in X$.*

It is clear that $\mathbb{R}^n_+$ is a normal solid cone of $\mathbb{R}^n$ endowed with the usual norm. Motivated by [10–18, 20, 21], we in this paper shall extend Theorem 1 to partial cone metric spaces over a non-normal solid cone of an abstract normed vector space.

## 2 Preliminaries

Let $E$ be a topological vector space. A cone of $E$ is a nonempty closed subset $P$ of $E$ such that

(i) $ax + by \in P$ for each $x, y \in P$ and each $a, b \geq 0$, and
(ii) $P \cap (-P) = \{\theta\}$, where $\theta$ is the zero element of $E$.

Each cone $P$ of $E$ determines a partial order $\preceq$ on $E$ by $x \preceq y \Longleftrightarrow y - x \in P$ for each $x, y \in X$.

A cone $P$ of a topological vector space $E$, is solid [22] if $\operatorname{int} P \neq \emptyset$, where $\operatorname{int} P$ is the interior of $P$. For each $x, y \in E$ with $y - x \in \operatorname{int} P$, we write $x \ll y$. A cone $P$ of a normed vector space $(E, \|\cdot\|)$, is normal [22] if there exists $N > 0$ such that $x \preceq y$ implies that $\|x\| \leq N\|y\|$ for each $x, y \in P$, and the minimal $N$ is called a normal constant of $P$.

**Lemma 1** *Let $P$ be a solid cone of a normed vector space $(E, \|\cdot\|)$, and let $\{u_n\}$ be a sequence in $E$. Then $u_n \xrightarrow{\|\cdot\|} \theta$ implies that for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_0$ such that $\epsilon \pm u_n \in \operatorname{int} P$, i.e., $u_n \ll \epsilon$ for all $n \geq n_0$.*

*Proof* For each $\epsilon \in \operatorname{int} P$, there exists some $\varepsilon > 0$ such that $\|x\| < \varepsilon$ implies that $\epsilon \pm x \in \operatorname{int} P$ for each $x \in E$. If $u_n \xrightarrow{\|\cdot\|} \theta$, then for this $\varepsilon$, there exists a positive integer $n_0$ such that $\|u_n\| < \varepsilon$ for each $n \geq n_0$, and hence $\epsilon \pm u_n \in \operatorname{int} P$ for each $n \geq n_0$, *i.e.*, $-\epsilon u_n \ll \epsilon$ for each $n \geq n_0$. The proof is complete. □

**Remark 1** The converse of Lemma 1 is true provided that $P$ is normal. In fact, for each $\varepsilon > 0$, there exists some $\epsilon \in \operatorname{int} P$ such that $\|\epsilon\| < \frac{\varepsilon}{2N+1}$, where $N$ denotes the normal constant of $P$. Note that for this $\epsilon$, there exists a positive integer $n_0$ such that $-\epsilon \ll u_n \ll \epsilon$ for each $n \geq n_0$, and so $\theta \ll u_n + \epsilon \preceq 2\epsilon$. Then $\|u_n\| \leq \|u_n + \epsilon\| + \|\epsilon\| \leq (2N + 1)\|\epsilon\| < \varepsilon$ for each $n \geq n_0$ by the normality of $P$. This forces that $u_n \xrightarrow{\|\cdot\|} \theta$.

The following example shows that the converse of Lemma 1 may not be true if $P$ is non-normal.

**Example 1** Let $E = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$ and $P = \{u \in E : u(t) \geq 0, t \in [0, 1]\}$, which is a non-normal solid cone [22]. Let $u_n(t) = \frac{\sin nt}{n}$. Clearly, $\|u_n\| = 1$, and so $u_n \overset{\|\cdot\|}{\nrightarrow} \theta$. On the other hand, let $v_n(t) \equiv \frac{1}{n}$, then $v_n \in P$, $v_n \xrightarrow{\|\cdot\|} \theta$ and $-v_n \preceq u_n \preceq v_n$. By Lemma 1, for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_0$ such that $\theta \preceq v_n \ll \epsilon$ for all $n \geq n_0$, and hence $-\epsilon \ll u_n \ll \epsilon$ for all $n \geq n_0$.

Let $X$ be a nonempty set and $P$ be a cone of a topological vector space $E$. A cone metric [1] on $X$ is a mapping $p : X \times X \to P$ such that for each $x, y, z \in X$,

(d1) $d(x,y) = \theta \Longleftrightarrow x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,y) \preceq d(x,z) + d(z,y)$.

The pair $(X,d)$ is called a cone metric space over $P$. A partial cone metric [20, 21] on $X$ is a mapping $p : X \times X \to P$ such that for each $x, y, z \in X$,
(p1) $p(x,y) = p(x,x) = p(y,y) \Longleftrightarrow x = y$;
(p2) $p(x,y) = p(y,x)$;
(p3) $p(x,x) \preceq p(x,y)$;
(p4) $p(x,y) \preceq p(x,z) + p(z,y) - p(z,z)$.

The pair $(X,p)$ is called a partial cone metric space over $P$.

Each cone metric is certainly a partial cone metric. The following example shows that there does exist some partial cone metric which is not a cone metric.

**Example 2** Let $E = C^1_{\mathbb{R}}[0,1]$ with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$, and $X = P = \{u \in E : u(t) \geq 0, t \in [0,1]\}$. Define a mapping $p : X \times X \to P$ by

$$p(x,y) = \begin{cases} x, & x = y, \\ x + y, & \text{otherwise}. \end{cases}$$

For each $x, y \in X$, $p(x,y) = p(y,x) = x$ and $p(x,x) = p(x,y) = x$ when $x = y$, and $p(x,y) = p(y,x) = x + y$ and $p(x,x) = x \preceq x + y = p(x,y)$ when $x \neq y$, *i.e.*, (p2) and (p3) are satisfied. For each $x, y \in X$, $p(x,y) = p(x,x) = p(y,y) = x$ whenever $x = y$, and $x = y$ whenever $p(x,x) = p(y,y)$, *i.e.*, (p1) is satisfied. For each $x, y, z \in X$,

$$p(x,y) = x = p(x,z) + p(y,z) - p(z,z), \quad \text{when } x = y = z,$$

$$p(x,y) = x \preceq x + y + z = p(x,z) + p(y,z) - p(z,z), \quad \text{when } x = y, y \neq z,$$

$$p(x,y) = x + y = p(x,z) + p(y,z) - p(z,z), \quad \text{when } x \neq y, y = z,$$

$$p(x,y) = x + y = p(x,z) + p(y,z) - p(z,z), \quad \text{when } x \neq y, x = z,$$

$$p(x,y) = x + y \preceq x + y + z = p(x,z) + p(y,z) - p(z,z), \quad \text{when } x \neq y, y \neq z, x \neq z,$$

*i.e.*, (p4) is satisfied for each $x, y, z \in X$. Hence $p$ is partial cone metric, but not a cone metric, since $p(x,x) \neq \theta$ for each $x \in X$ with $x \neq \theta$.

Each partial cone metric $p$ on $X$ over a solid cone generates a topology $\tau_p$ on $X$, which has a base of the family of open $p$-balls $\{B_p(x,\epsilon) : x \in X, \theta \ll \epsilon\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) \ll p(x,x) + \epsilon\}$ for each $x \in X$ and each $\epsilon \in \operatorname{int} P$.

**Definition 1** Let $(X,p)$ be a partial cone metric space over a solid cone $P$ of a topological vector space $E$.

(i) A sequence $\{x_n\}$ in $X$ converges [20] to $x \in X$ (denote by $x_n \stackrel{\tau_p}{\to} x$), if for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_0$ such that $p(x_n,x) \ll p(x,x) + \epsilon$ for each $n \geq n_0$. A sequence $\{x_n\}$ in $X$ strongly converges [21] to $x \in X$ (denote by $x_n \stackrel{s\text{-}\tau_p}{\to} x$), if $\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = p(x,x)$.

(ii) A sequence $\{x_n\}$ in $X$ is $\theta$-Cauchy, if for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_0$ such that $p(x_n,x_m) \ll \epsilon$ for all $m, n \geq n_0$. The partial cone metric space $(X,p)$ is

$\theta$-complete, if each $\theta$-Cauchy sequence $\{x_n\}$ of $X$ converges to a point $x \in X$ such that $p(x,x) = \theta$.

It follows from Lemma 1 and Remark 1 that each strongly convergent sequence $\{x_n\}$ of a partial cone metric space $X$ is convergent whenever $E$ is a normed vector space, and the converse is true provided that $P$ is a normal. The following example will show that there exists some sequence of a partial cone metric, which is convergent but not strongly convergent if $P$ is non-normal.

**Example 3** Let $(X,p)$, $E$ and $P$ be the same ones as those in Example 2, and let $u_n(t) = \frac{1+\sin nt}{n+2}$. Then $u_n \stackrel{\tau_p}{\to} \theta$, but $u_n \stackrel{s\text{-}\tau_p}{\not\to} \theta$. In fact, it is clear that $u_n \in P$, $p(u_n,\theta) = u_n$, $u_n \preceq v$ and $v_n \stackrel{\|\cdot\|}{\to} \theta$, where $v(t) \equiv \frac{2}{n+2}$. Then by Lemma 1, for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_0$ such that $\theta \preceq p(u_n,\theta) \preceq v_n \ll \epsilon$ for all $n \geq n_0$, *i.e.*, $u_n \stackrel{\tau_p}{\to} \theta$. On the other hand, $\|p(u_n,\theta) - p(\theta,\theta)\| = \|u_n\| = 1$, and hence $u_n \stackrel{s\text{-}\tau_p}{\not\to} \theta$.

**Definition 2** Let $(X,p)$ be a partial cone metric space over a solid cone $P$ of a normed vector space $(E, \|\cdot\|)$. A sequence $\{x_n\}$ in $X$ is Cauchy [20, 21], if there exists $u \in P$ with $\|u\| < \infty$ such that $\lim_{m,n\to\infty} p(x_n,x_m) = u$. The partial cone metric space $(X,p)$ is complete [20, 21], if each Cauchy sequence $\{x_n\}$ of $X$ strongly converges to a point $x \in X$ such that $p(x,x) = u$.

If $P$ is a normal solid cone of a normed vector space $(E, \|\cdot\|)$, then each complete partial cone metric space is $\theta$-complete by Lemma 1 and Remark 1. But the converse is not true, the following example shows that a partial cone metric space which is $\theta$-complete, is not necessarily complete.

**Example 4** Let $X = \{(x_1,x_2,\dots,x_k) : x_i \geq 0, x_i \in \mathbb{Q}, i = 1,2,\dots,k\}$, $E = \mathbb{R}^k$ with the norm $\|x\| = \sqrt{\sum_{i=1}^k x_i^2}$, $P = \mathbb{R}_+^k$, where $\mathbb{Q}$ denotes the set of rational numbers. Define a mapping $p : X \times X \to P$ as follows:

$$p(x,y) = (x_1 \vee y_1, x_2 \vee y_2, \dots, x_k \vee y_k), \quad \forall x,y \in X.$$

Clearly, $(X,p)$ is a partial cone metric space, $p(x,x) = x$ for each $x \in X$, $p(x,\theta) = \theta \Longleftrightarrow x = \theta$, $P$ is normal.

Let $\{y_n\}$ be a sequence in $(X,p)$, where $y_n = (y_{n1}, y_{n2}, \dots, y_{nk})$. If $\{y_n\}$ is $\theta$-Cauchy, then by Remark 1 and the normality of $P$, $\lim_{m,n\to\infty} p(y_n,y_m) = \theta$, and so for each $\varepsilon > 0$, there exists $n_0$ such that $\|p(y_n,y_m)\| < \varepsilon$ for each $m,n \geq n_0$. Thus, $y_{ni} \vee y_{mi} = p(y_{ni},y_{mi}) < \varepsilon$ for each $m,n \geq n_0$ and each $1 \leq i \leq k$. This means $\lim_{n\to\infty} y_{ni} = 0$ for each $1 \leq i \leq k$, *i.e.*, $\lim_{n\to\infty} y_n = \theta$. Therefore, $\lim_{n\to\infty} p(y_n,\theta) = \lim_{n\to\infty} y_n = \theta = p(\theta,\theta)$, *i.e.*, $y_n \stackrel{\tau_p}{\to} \theta$, and hence $(X,p)$ is $\theta$-complete since $\theta \in X$.

Let $y_{ni} = i(1 + \frac{1}{n})^n$ for each $n$ and each $1 \leq i \leq k$, and $\widetilde{e} = (e, 2e, \dots, ke)$. It is clear that $\lim_{m,n\to\infty} p(y_n,y_m) = \widetilde{e}$, and hence $\{y_n\}$ is a Cauchy sequence in $(X,p)$. If there exists $x \in X$ such that $p(x,x) = \widetilde{e}$, then $x = \widetilde{e}$, which contradicts to the fact that $\widetilde{e} \notin X$ since $e \notin \mathbb{Q}$. This means $p(x,x) \neq \widetilde{e}$ for each $x \in X$, and so there does not exist $x \in X$ such that $\lim_{m,n\to\infty} p(y_n,y_m) = p(x,x)$. Therefore, $(X,p)$ is not complete.

## 3 Extensions of Banach contraction principle

In this section, we present some extensions of Banach contraction principle in the setting of partial cone metric spaces over a non-normal solid cone of an abstract normed vector space.

**Theorem 2** *Let $(X,p)$ be a $\theta$-complete partial cone metric space over a solid cone $P$ of a normed vector space $(E,\|\cdot\|)$ and $T: X \to X$. If there exists a linear bounded mapping $L: P \to P$ with the spectral radius $r(L) < 1$ such that*

$$p(Tx,Ty) \preceq Lp(x,y), \quad \forall x,y \in X. \tag{2}$$

*Then $T$ has a unique fixed point $x^* \in X$. In addition, for each $x_0 \in X$, let*

$$x_n = Tx_{n-1} = T^n x_0, \quad \forall n, \tag{3}$$

*then there exists a positive integer $n_0$ such that $y_n \stackrel{\tau_p}{\to} x^*$, where $\{y_n\}$ is a subsequence of $\{x_n\}$ defined by $y_n = T^{nn_0}x_0$.*

*Proof* By $r(L) < 1$ and Gelfand's formula, there exists $0 < \beta < 1$ such that

$$\lim_{n\to\infty} \sqrt[n]{\|L^n\|} = r(L) \le \beta,$$

which implies that there exists a positive integer $n_0$ such that

$$\|L^n\| \le \beta^n, \quad \forall n \ge n_0. \tag{4}$$

Clearly,

$$y_n = T^{nn_0}x_0 = T^{n_0}T^{(n-1)n_0}x_0 = T^{n_0}y_{n-1}, \quad \forall n. \tag{5}$$

By (2), (5) and $L(P) \subset P$,

$$p(y_n,y_{n+1}) = p\bigl(T^{n_0}x_{(n-1)n_0}, T^{n_0}x_{nn_0}\bigr) \preceq L^{n_0}p(y_{n-1},y_n) \preceq \cdots \preceq L^{nn_0}p(y_0,y_1), \quad \forall n,$$

and so by (p4),

$$p(y_n,y_m) \preceq p(y_n,y_{n+1}) + p(y_{n+1},y_{n+2}) + \cdots + p(y_{m-1},y_m) \preceq \sum_{i=n}^{m-1} L^{in_0}p(y_0,y_1), \quad \forall m > n. \tag{6}$$

By (4),

$$\left\|\sum_{i=n}^{m-1} L^{ik}p(y_0,y_1)\right\| \le \|p(y_0,y_1)\| \sum_{i=n}^{m-1} \|L^{n_0}\|^i \le \|p(y_0,y_1)\| \sum_{i=n}^{m-1} \beta^{in_0} = \frac{\|p(y_0,y_1)\|(\beta^{nn_0} - \beta^{(m+1)n_0})}{1-\beta^{n_0}},$$

which implies that $\sum_{i=n}^{m-1} L^{ik}p(y_0, y_1) \xrightarrow{\|\cdot\|} \theta (n \to \infty)$ by $\beta < 1$. Then by (6) and Lemma 1, for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_1$ such that

$$p(y_n, y_m) \preceq \sum_{i=n}^{m-1} L^{ik} p(y_0, y_1) \ll \epsilon, \quad \forall m, n \geq n_1,$$

which implies that $\{y_n\}$ is a $\theta$-Cauchy sequence in $(X, p)$. Moreover by the $\theta$-completeness of $(X, p)$, there exists some $x^* \in X$ such that $y_n \xrightarrow{\tau_p} x^*$ and $p(x^*, x^*) = \theta$, and so there exists a positive integer $n_2 \geq n_1$ such that

$$p(y_n, x^*) \ll \frac{\epsilon}{2}, \quad \forall n \geq n_2. \tag{7}$$

Since $r(L) < 1$, then $(I - L)(P) \subset P$. Thus, by (p4), (2), (5) and (7),

$$\begin{aligned} p(T^{n_0}x^*, x^*) &\preceq p(y_{n+1}, T^{n_0}x^*) + p(y_{n+1}, x^*) \\ &\preceq L^{n_0} p(y_n, x^*) + p(y_{n+1}, x^*) \\ &\preceq p(y_n, x^*) + p(y_{n+1}, x^*) \ll \epsilon, \quad \forall n \geq n_2, \end{aligned}$$

which together with the arbitrary property of $\epsilon$ implies that $p(T^{n_0}x^*, x^*) = \theta$, and hence $T^{n_0}x^* = x^*$ by (p1) and (p3), *i.e.*, $x^*$ is a fixed point of $T^{n_0}$. Let $x \in X$ be a fixed point of $T^{n_0}$, *i.e.*, $T^{n_0}x = x$. Note that $\|L^{n_0}\| < 1$ by (4), then the inverse of $I - L^{n_0}$ exists, denote it by $(I - L^{n_0})^{-1}$. Moreover by Neumann's formula, $(I - L^{n_0})^{-1}(P) \subset P$. By (2), we have $p(x, x^*) = p(T^{n_0}x, T^{n_0}x^*) \preceq L^{n_0} p(x, x^*)$, and hence $(I - L^{n_0})p(x, x^*) \preceq \theta$. Act it with $(I - L^{n_0})^{-1}$, then $p(x, x^*) \preceq \theta$. This implies that $p(x, x^*) = \theta$, and hence $x = x^*$ by (p1) and (p3). Hence $x^*$ is the unique fixed point of $T^{n_0}$.

Note that $x^*$ is a fixed point of $T^{n_0}$, then $T^{n_0}(Tx^*) = T(T^{n_0}x^*) = Tx^*$, *i.e.*, $Tx^*$ is also a fixed point of $T^{n_0}$. By the uniqueness of fixed point of $T^{n_0}$, we have $Tx^* = x^*$, *i.e.*, $x^*$ is also a fixed point of $T$. Let $y$ be a fixed point of $T$. It is clear that $y$ is also a fixed point of $T^{n_0}$, and so $y = x^*$ by the uniqueness of fixed point of $T^{n_0}$. Hence $x^*$ is the unique fixed point of $T$. The proof is complete. □

**Remark 2** It is clear that Theorem 1 is exactly a special case of Theorem 2 with $E = \mathbb{R}^n$ and $P = \mathbb{R}^n_+$. Let $Lu = cu$ for some constant $c \in [0, 1)$, then $r(L) = c < 1$, and so Theorem 1 of [1], Theorem 6 of [20] and Theorem 7 of [21] directly follow from Theorem 2. In addition, the normality of $P$ necessarily assumed in [1, 9, 20, 21] has been removed in Theorem 2. Therefore, Theorem 2 indeed improves the corresponding results in [1, 9, 20, 21].

The following example shows the usability of Theorem 2.

**Example 5** Let $(X, p)$, $E$ and $P$ be the same ones as those in Example 2. Let $(Tx)(t) = (Lx)(t) \int_0^t x(s)\,ds$ for each $x \in X$, where $t \in [0, a]$, $a > 0$. Clearly, $\theta$ is the unique fixed point of $T$.

For each $x, y \in X$, $p(Tx, Ty) = \int_0^t x(s)\,ds = Lp(x, y)$ whenever $x = y$, and $p(Tx, Ty) = \int_0^t [x(s) + y(s)]\,ds = Lp(x, y)$ whenever $x \neq y$, *i.e.*, (2) is satisfied. It is clear that $(L^n x)(t) \leq \frac{t^n}{n!}\|x\|_\infty$ for each $t \in [0, a]$, and hence $\|L^n x\|_\infty \leq \frac{a^n}{n!}\|x\|_\infty$. Note that $(L^n x)'(t) = (L^{n-1}x)(t)$, then

$$\|L^n x\| = \|L^n x\|_\infty + \|(L^n x)'\|_\infty \leq \left(\frac{a^n}{n!} + \frac{a^n}{(n-1)!}\right)\|x\|_\infty \leq \left(\frac{a^n}{n!} + \frac{a^n}{(n-1)!}\right)\|x\|,$$

which implies that $\|L^n\| \leq \frac{a^n}{n!} + \frac{a^n}{(n-1)!}$. Therefore by Gelfand's formula, $r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} = 0$ since $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$, and hence $T$ has a unique fixed point in $X$ by Theorem 2.

However, the existence of fixed point of $T$ cannot derive from the fixed point results in [1–21, 23], since $P$ is non-normal, and $p$ is not a cone metric by Example 1 and Example 2, and there does not exist a constant $c \in [0,1)$ such that $p(Tx,Ty) \preceq cp(x,y)$.

**Theorem 3** *Let $(X,p)$ be a $\theta$-complete partial cone metric space over a solid cone $P$ of a normed vector space $(E, \|\cdot\|)$ and $T: X \to X$. If there exist four nonnegative constants $c_1$, $c_2$, $c_3$ and $c_4$ with $c_1 + c_2 + c_3 + 2c_4 < 1$ such that*

$$p(Tx,Ty) \preceq c_1 p(x,y) + c_2 p(x,Tx) + c_3 p(y,Ty) + c_4\big[p(x,Ty) + p(y,Tx)\big], \quad \forall x,y \in X. \tag{8}$$

*Then $T$ has a unique fixed point $x^* \in X$, and for each $x_0 \in X$, $x_n \stackrel{\tau_p}{\to} x^*$, where $x_n$ is defined by* (3).

*Proof* By (3), (8) and (p4),

$$\begin{aligned}
&p(x_n, x_{n+1}) \\
&\quad = p(Tx_{n-1}, Tx_n) \\
&\quad \preceq c_1 p(x_{n-1}, x_n) + c_2 p(x_{n-1}, x_n) + c_3 p(x_n, x_{n+1}) + c_4\big[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\big] \\
&\quad \preceq c_1 p(x_{n-1}, x_n) + c_2 p(x_{n-1}, x_n) + c_3 p(x_n, x_{n+1}) + c_4\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big], \quad \forall n,
\end{aligned}$$

and so

$$p(x_n, x_{n+1}) \preceq cp(x_{n-1}, x_n), \quad \forall n,$$

where $c = \frac{c_1+c_2+c_4}{1-c_3-c_4} < 1$ by $c_1 + c_2 + c_3 + 2c_4 < 1$. Moreover by (p4),

$$p(x_n, x_m) \preceq \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \preceq \sum_{i=n}^{m-1} c^i p(x_0, x_1) \preceq \frac{c^n p(x_0, x_1)}{1-c}, \quad \forall m > n. \tag{9}$$

Since $c < 1$, then $\frac{c^n p(x_0,x_1)}{1-c} \stackrel{\|\cdot\|}{\to} \theta$, and hence by Lemma 1, for each $\epsilon \in \operatorname{int} P$, there exists a positive integer $n_1$ such that $\frac{c^n p(x_0,x_1)}{1-c} \ll \epsilon$ for all $n \geq n_1$. Thus, by (9),

$$p(x_n, x_m) \preceq \frac{c^n p(x_0, x_1)}{1-c} \ll \epsilon, \quad \forall m, n \geq n_1,$$

*i.e.*, $\{x_n\}$ is a $\theta$-Cauchy sequence. Therefore, by the $\theta$-completeness of $(X,p)$, there exists $x^* \in X$ such that $x_n \stackrel{\tau_p}{\to} x^*$ and $p(x^*, x^*) = \theta$, and so there exists a positive integer $n_2 \geq n_1$ such that

$$p\big(x_n, x^*\big) \ll \frac{(1-c_2-c_4)\epsilon}{2(1+c_1+2c_4)}, \quad \forall n \geq n_2, \tag{10}$$

and

$$p(x_n, x_{n+1}) \ll \frac{(1-c_2-c_4)\epsilon}{2c_3}, \quad \forall n \geq n_2. \tag{11}$$

By (p4) and (8),

$$\begin{aligned}
p(Tx^*, x^*) &\preceq p(Tx^*, x_{n+1}) + p(x_{n+1}, x^*) \\
&\preceq c_1 p(x^*, x_n) + c_2 p(x^*, Tx^*) + c_3 p(x_n, x_{n+1}) \\
&\quad + c_4 \bigl[p(x^*, x_{n+1}) + p(x_n, Tx^*)\bigr] + p(x^*, x_{n+1}) \\
&\preceq c_1 p(x^*, x_n) + c_2 p(x^*, Tx^*) + c_3 p(x_n, x_{n+1}) \\
&\quad + c_4 \bigl[p(x^*, x_{n+1}) + p(x_n, x^*) + p(x^*, Tx^*)\bigr] + p(x^*, x_{n+1}) \\
&= (c_1 + c_4) p(x^*, x_n) + (c_2 + c_4) p(x^*, Tx^*) \\
&\quad + c_3 p(x_n, x_{n+1}) + (1 + c_4) p(x^*, x_{n+1}), \quad \forall n,
\end{aligned}$$

and so

$$p(Tx^*, x^*) \preceq \frac{(c_1 + c_4) p(x^*, x_n) + c_3 p(x_n, x_{n+1}) + (1 + c_4) p(x^*, x_{n+1})}{1 - c_2 - c_4}, \quad \forall n.$$

Then by (10) and (11),

$$p(Tx^*, x^*) \ll \epsilon, \quad \forall n \geq n_2,$$

which together with the arbitrary property of $\epsilon$ implies that $p(Tx^*, x^*) = \theta$, and so $Tx^* = x^*$ by (p1) and (p3). Let $x$ be a fixed point of $T$. Then by (8) and (p3),

$$\begin{aligned}
p(x^*, x) &= p(Tx^*, Tx) \\
&\preceq c_1 p(x^*, x) + c_2 p(x^*, Tx^*) + c_3 p(x, Tx) + c_4 \bigl[p(x^*, Tx) + p(x, Tx^*)\bigr] \\
&= (c_1 + 2c_4) p(x^*, x) + c_3 p(x, x) \preceq (c_1 + c_3 + 2c_4) p(x^*, x).
\end{aligned}$$

This forces that $p(x^*, x) = \theta$ since $c_1 + c_3 + 2c_4 < 1$, and so $x = x^*$ by (p1) and (p3). Hence $x^*$ is the unique fixed point of $T$. The proof is complete. □

**Remark 3** Theorem 3 and Theorem 4 of [1] are special cases of Theorem 3 in the setting of cone metric spaces with $c_1 = c_4 = 0$, $c_2 = c_3$ and $c_1 = c_2 = c_3 = 0$, respectively, and Theorem 7 of [20] and Theorem 8 of [21] are special cases of Theorem 3 with $c_1 = c_4 = 0$, $c_2 = c_3$. In addition, $P$ is not necessarily normal in Theorem 3. Compared with the corresponding results of [20, 21], the partial cone metric space $X$ is only assumed to be $\theta$-complete, but not complete in Theorem 2 and Theorem 3.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors have contributed in obtaining the new results presented in this article. All authors read and approved the final manuscript.

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**Acknowledgements**
The work was supported by the Natural Science Foundation of China (11161022), the Natural Science Foundation of Jiangxi Province (20114BAB211006, 20122BAB201015), the Educational Department of Jiangxi Province (GJJ12280, GJJ13297) and Program for Excellent Youth Talents of JXUFE (201201).

Received: 24 June 2013 Accepted: 9 September 2013 Published: 07 Nov 2013

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10.1186/1687-1812-2013-250
**Cite this article as:** Jiang and Li: **Extensions of Banach contraction principle to partial cone metric spaces over a non-normal solid cone.** *Fixed Point Theory and Applications* 2013, 2013:250