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Extensions of Hermite–Hadamard inequalities for harmonically convex functions via generalized fractional integrals

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Abstract

In the paper, the authors establish some new Hermite–Hadamard type inequalities for harmonically convex functions via generalized fractional integrals. Moreover, the authors prove extensions of the Hermite–Hadamard inequality for harmonically convex functions via generalized fractional integrals without using the harmonic convexity property for the functions. The results offered here are the refinements of the existing results for harmonically convex functions.

Keywords: Harmonically convex functions; Generalized fractional integrals; Hermite–Hadamard inequalities

1 Introduction

The Hermite–Hadamard inequality, which is the first basic result of convex mappings with a natural geometric interpretation and extensive use, has attracted attention with great interest in elementary mathematics. Many mathematicians have devoted their efforts to standardization, refining, imitation, and expansion into various categories of works such as convex mappings.

Inequalities found by C. Hermite and J. Hadamard for convex mappings are very important in literature (see [1]). These inequalities state that if $\mathcal{F}: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$, then

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if $\mathcal{F}$ is concave. For further studies of this area, one can consult [2–22].

For brevity, in the upcoming results, we use the subsequent notations: Mappings $\Lambda, \Lambda^*, \Psi, \Psi^*: [0,1] \to \mathbb{R}$ are defined by

$$\Lambda(x) = \int_0^x \frac{\varphi((\kappa_2-\kappa_1)\tau)}{\tau}\,d\tau < +\infty, \qquad \Lambda^*(x) = \int_0^x \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{\kappa_1\kappa_2}\tau)}{\tau}\,d\tau < \infty,$$

$$\Psi(x)=\int_0^x \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\,d\tau<+\infty, \qquad \Psi^*(x)=\int_0^x \frac{\varphi(\frac{\kappa_2-\kappa_1}{2}\tau)}{\tau}\,d\tau<\infty,$$

and

$$\mathcal{G}(x)=\frac{1}{x}, \qquad \phi(x)=\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right),$$
$$m=\inf_{\tau\in[\kappa_1,\kappa_2]}\phi''(\tau), \qquad M=\sup_{\tau\in[\kappa_1,\kappa_2]}\phi''(\tau).$$

Now we give the definition of the generalized fractional integrals (GFIs) given by Sarikaya and Ertuğral in [23].

Definition 1 The left-sided and right-sided GFIs are denoted by ${}_{\kappa_1+}I_\varphi$ and ${}_{\kappa_2-}I_\varphi$ and defined as follows:

$$ {}_{\kappa_1+}I_\varphi\mathcal{F}(x)=\int_{\kappa_1}^x \frac{\varphi(x-\tau)}{x-\tau}\mathcal{F}(\tau)\,d\tau, \quad x>\kappa_1, \tag{1.2}$$
$$ {}_{\kappa_2-}I_\varphi\mathcal{F}(x)=\int_x^{\kappa_2} \frac{\varphi(\tau-x)}{\tau-x}\mathcal{F}(\tau)\,d\tau, \quad x<\kappa_2, \tag{1.3}$$

where a function $\varphi:[0,\infty)\to[0,\infty)$ satisfies the condition $\int_0^1 \frac{\varphi(\tau)}{\tau}\,d\tau<\infty$.

Recently, the authors gave some refinements of Hermite–Hadamard inequalities for GFIs under the condition of convexity, as follows.

Theorem 1 ([23]) *For a convex function $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ on $[\kappa_1,\kappa_2]$ with $\kappa_1<\kappa_2$, the subsequent inequalities hold for GFIs*:

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\le\frac{1}{2\Lambda(1)}\left[{}_{\kappa_1+}I_\varphi\mathcal{F}(\kappa_2)+{}_{\kappa_2-}I_\varphi\mathcal{F}(\kappa_1)\right]\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1.4}$$

Theorem 2 ([24]) *For a convex function $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ on $[\kappa_1,\kappa_2]$ with $\kappa_1<\kappa_2$, the subsequent inequalities hold for GFIs*:

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\le\frac{1}{2\Psi^*(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2})+}I_\varphi\mathcal{F}(\kappa_2)+{}_{(\frac{\kappa_1+\kappa_2}{2})-}I_\varphi\mathcal{F}(\kappa_1)\right]\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{1.5}$$

Theorem 3 ([25]) *For a convex function $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ on $[\kappa_1,\kappa_2]$ with $\kappa_1<\kappa_2$, the subsequent inequalities hold for GFIs*:

$$\begin{aligned}\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)&\le\frac{1}{2\Psi^*(1)}\left[{}_{\kappa_1+}I_\varphi\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}_{\kappa_2-}I_\varphi\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\&\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.\end{aligned} \tag{1.6}$$

In [26], İşcan gave the following definition of harmonically convex functions and Hermite–Hadamard inequalities for harmonically convex functions.

Definition 2 ([26]) A mapping $\mathcal{F}: I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is called harmonically convex if

$$\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\tau\kappa_1 + (1-\tau)\kappa_2}\right) \le \tau\mathcal{F}(\kappa_2) + (1-\tau)\mathcal{F}(\kappa_1) \tag{1.7}$$

for all κ_1, κ_2 in I and τ in $[0,1]$. If inequality (1.7) holds in the reversed direction, then $\mathcal{F}$ is called a harmonically concave function.

Theorem 4 ([26]) *For a harmonically convex mapping $\mathcal{F}: I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$, the following double inequality holds*:

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \le \frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}\,dx \le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}, \tag{1.8}$$

where $\kappa_1, \kappa_2 \in I$ and $\kappa_1 < \kappa_2$.

In [27], İşcan and Wu gave the inequalities of Hermite–Hadamard type for harmonically convex functions via Riemann–Liouville fractional integrals.

Theorem 5 ([27]) *Let $\mathcal{F}: I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $\mathcal{F} \in L([\kappa_1,\kappa_2])$, where $\kappa_1, \kappa_2 \in I$ with $\kappa_1 < \kappa_2$. If $\mathcal{F}$ is a harmonically convex function on $[\kappa_1,\kappa_2]$, the following double inequality holds for the Riemann–Liouville fractional integrals*:

$$\begin{aligned}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) &\le \frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\alpha}\left\{J^{\alpha}_{\frac{1}{\kappa_1}-}(\mathcal{F}\circ\mathcal{G})\left(\frac{1}{\kappa_2}\right) + J^{\alpha}_{\frac{1}{\kappa_2}+}(\mathcal{F}\circ\mathcal{G})\left(\frac{1}{\kappa_1}\right)\right\}\\ &\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2},\end{aligned} \tag{1.9}$$

where $\alpha > 0$.

In [28], Zhao et al. gave the following Hermite–Hadamard type inequalities for harmonically convex functions by utilizing GFIs.

Theorem 6 *Let $\mathcal{F}: I \subseteq (0,+\infty) \to \mathbb{R}$ be a mapping such that $\mathcal{F} \in L([\kappa_1,\kappa_2])$. If $\mathcal{F}$ is a harmonically convex mapping on $[\kappa_1,\kappa_2]$, then the following inequalities hold for the GFIs*:

$$\begin{aligned}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) &\le \frac{1}{2\Lambda^*(1)}\left[{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{1}{\kappa_2}\right) + {}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{1}{\kappa_1}\right)\right]\\ &\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.\end{aligned} \tag{1.10}$$

In [29], F. Chen gave the following useful lemma and the lower and upper bounds of the left- and right-hand sides of inequalities (1.9) as follows.

Lemma 1 *A mapping $\mathcal{F}: [\kappa_1,\kappa_2] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is called harmonically convex if and only if $\phi(x)$ is convex on $[\kappa_1,\kappa_2]$.*

Theorem 7 *Let $\mathcal{F}: [\kappa_1,\kappa_2] \subseteq (0,+\infty) \to \mathbb{R}$ be a positive twice differentiable function with $\kappa_1 < \kappa_2$ and $\mathcal{F} \in L([\kappa_1,\kappa_2])$.If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold* *for the Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m\alpha}{2(\kappa_2-\kappa_1)^\alpha}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\left[(\kappa_2-x)^{\alpha-1}+(x-\kappa_1)^{\alpha-1}\right]dx\\
&\quad\le\frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^\alpha\left[J^\alpha_{\frac{1}{\kappa_2}+}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^\alpha_{\frac{1}{\kappa_1}-}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&\quad\le\frac{M\alpha}{2(\kappa_2-\kappa_1)^\alpha}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\left[(\kappa_2-x)^{\alpha-1}+(x-\kappa_1)^{\alpha-1}\right]dx.
\end{aligned}\tag{1.11}$$

Theorem 8 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$.If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold for the Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{-M\alpha}{2(\kappa_2-\kappa_1)^\alpha}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-x)(x-\kappa_1)\left[(\kappa_2-x)^{\alpha-1}+(x-\kappa_1)^{\alpha-1}\right]dx\\
&\quad\le\frac{\Gamma(\alpha+1)}{2}\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^\alpha\left[J^\alpha_{\frac{1}{\kappa_2}+}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^\alpha_{\frac{1}{\kappa_1}-}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\
&\qquad-\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}\\
&\quad\le\frac{-m\alpha}{2(\kappa_2-\kappa_1)^\alpha}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-x)(x-\kappa_1)\left[(\kappa_2-x)^{\alpha-1}+(x-\kappa_1)^{\alpha-1}\right]dx.
\end{aligned}\tag{1.12}$$

In [30], Budak et al. gave the following inequalities for harmonically convex mappings.

Theorem 9 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$.If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold for GFIs*:

$$\begin{aligned}
&\frac{m}{2\Lambda^*(x)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)\left[\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}+\frac{\varphi(\frac{\kappa_2-x}{\kappa_1\kappa_2})}{\kappa_2-x}\right]dx\\
&\quad\le\frac{1}{2\Lambda^*(x)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\mathcal{F}o\mathcal{G})\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_\alpha(\mathcal{F}o\mathcal{G})\left(\frac{1}{\kappa_1}\right)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&\quad\le\frac{M}{2\Lambda^*(x)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)\left[\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}+\frac{\varphi(\frac{\kappa_2-x}{\kappa_1\kappa_2})}{\kappa_2-x}\right]dx
\end{aligned}$$

and

$$\begin{aligned}
&\frac{m}{2\Lambda^*(x)}\int_{\kappa_1}^{\kappa_2}\left[\varphi\left(\frac{x-\kappa_1}{\kappa_1\kappa_2}\right)(\kappa_2-x)+(x-\kappa_1)\varphi\left(\frac{\kappa_2-x}{\kappa_1\kappa_2}\right)\right]dx\\
&\quad\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Lambda^*(x)}\left[{}_{\frac{1}{\kappa_1}-}I_\alpha(\mathcal{F}o\mathcal{G})\left(\frac{1}{\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_\alpha(\mathcal{F}o\mathcal{G})\left(\frac{1}{\kappa_1}\right)\right]\\
&\quad\le\frac{M}{2\Lambda^*(x)}\int_{\kappa_1}^{\kappa_2}\left[\varphi\left(\frac{x-\kappa_1}{\kappa_1\kappa_2}\right)(\kappa_2-x)+(x-\kappa_1)\varphi\left(\frac{\kappa_2-x}{\kappa_1\kappa_2}\right)\right]dx.
\end{aligned}$$

For recent findings and implications for integral inequalities via harmonically convex mappings and other classes of functions, see ([31–42]) and the references given therein.

2 Hermite–Hadamard type inequalities

In this portion, we deal with some new inequalities of Hermite–Hadamard type for harmonically convex mappings by applying GFIs.

Theorem 10 *Let $\mathcal{F}: I \subseteq (0,+\infty) \to \mathbb{R}$ be a function such that $\mathcal{F} \in L([\kappa_1,\kappa_2])$. If $\mathcal{F}$ is a harmonically convex function on $[\kappa_1,\kappa_2]$, then the following inequalities hold for the GFIs*:

$$\begin{aligned}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) &\le \frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1) + {}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right] \\ &\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \end{aligned} \tag{2.1}$$

Proof From harmonic convexity, we have

$$\mathcal{F}\left(\frac{2xy}{x+y}\right) \le \frac{1}{2}\left[\mathcal{F}(x)+\mathcal{F}(y)\right].$$

For $x = \frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}$ and $y = \frac{2\kappa_1\kappa_2}{(2-\tau)\kappa_1+\tau\kappa_2}$, we get

$$2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \le \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}\right) + \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(2-\tau)\kappa_1+\tau\kappa_2}\right). \tag{2.2}$$

Multiplying by $\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}$ both sides of inequality (2.2) and integrating the resultant one with respect to τ over $[0,1]$, we obtain

$$\begin{aligned} &2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\int_0^1 \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\,d\tau \\ &\quad\le \int_0^1 \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}\right)d\tau \\ &\qquad+ \int_0^1 \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(2-\tau)\kappa_1+\tau\kappa_2}\right)d\tau. \end{aligned} \tag{2.3}$$

For $\frac{1}{u} = \frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}$ and $\frac{1}{v} = \frac{2\kappa_1\kappa_2}{(2-\tau)\kappa_1+\tau\kappa_2}$, we obtain

$$\begin{aligned} &2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\Psi(1) \\ &\quad\le \int_{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}^{\frac{1}{\kappa_1}} \frac{\varphi(\frac{1}{\kappa_1}-u)}{\frac{1}{\kappa_1}-u}\mathcal{F}\left(\frac{1}{u}\right)du + \int_{\frac{1}{\kappa_2}}^{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}} \frac{\varphi(v-\frac{1}{\kappa_2})}{v-\frac{1}{\kappa_2}}\mathcal{F}\left(\frac{1}{v}\right)dv \\ &\quad= \left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1) + {}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]. \end{aligned}$$

Hence, we proved the first inequality. To prove the second inequality of (2.1), first we note that since $\mathcal{F}$ is a harmonically convex function, we have

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}\right) \le \left(\frac{2-\tau}{2}\right)\mathcal{F}(\kappa_1) + \left(\frac{\tau}{2}\right)\mathcal{F}(\kappa_2) \tag{2.4}$$

and

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_2+(2-\tau)\kappa_1}\right)\le\left(\frac{\tau}{2}\right)\mathcal{F}(\kappa_1)+\left(\frac{2-\tau}{2}\right)\mathcal{F}(\kappa_2). \tag{2.5}$$

Adding (2.4) and (2.5), we get

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}\right)+\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(2-\tau)\kappa_1+\tau\kappa_2}\right)\le\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2). \tag{2.6}$$

Multiplying by $\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}$ both sides of inequality (2.6) and integrating the resultant one with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\tau\kappa_1+(2-\tau)\kappa_2}\right)d\tau+\int_0^1\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(2-\tau)\kappa_1+\tau\kappa_2}\right)d\tau\\&\quad\le\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)\right]\int_0^1\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\,d\tau.\end{aligned}$$

By changing the variables of integration, we have the second inequality of (2.1). □

Remark 1 Under the assumptions of Theorem 10, if we put $\varphi(\tau)=\tau$, then Theorem 10 reduces to Theorem 4.

Corollary 1 *Under the assumptions of Theorem 10, if we set $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}$, then we have the following inequality for Riemann–Liouville fractional integrals*:

$$\begin{aligned}&\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\quad\le 2^{\alpha-1}\Gamma(\alpha+1)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^\alpha\left[J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\&\quad\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.\end{aligned}$$

Corollary 2 *Under the assumptions of Theorem 10, if we set $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequality for the k-Riemann–Liouville fractional integrals*:

$$\begin{aligned}&\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\quad\le 2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}}\left[J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+,k}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-,k}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\&\quad\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.\end{aligned}$$

Theorem 11 *Let $\mathcal{F}:I\subseteq(0,+\infty)\to\mathbb{R}$ be a function such that $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If $\mathcal{F}$ is a harmonically convex function on $[\kappa_1,\kappa_2]$, then the following inequalities hold for the GFIs*:

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\le\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_1}-}I_\varphi(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_2}+}I_\varphi(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]$$

$$\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{2.7}$$

Proof Since $\mathcal{F}$ is a harmonically convex function on $[\kappa_1,\kappa_2]$, we have

$$\mathcal{F}\left(\frac{2xy}{x+y}\right) \leq \frac{1}{2}\big[\mathcal{F}(x)+\mathcal{F}(y)\big].$$

For $x=\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}$ and $y=\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}$, we get

$$2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \leq \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}\right)+\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}\right). \tag{2.8}$$

Multiplying by $\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}$ both sides of inequality (2.8) and integrating the resultant one with respect to τ over $[0,1]$, we obtain

$$\begin{aligned}2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\Psi(1) \leq{}& \int_0^1 \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}\right)d\tau \\ &+\int_0^1 \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}\right).\end{aligned}$$

By setting $\frac{1}{u}=\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}$ and $\frac{1}{v}=\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}$, we have

$$\begin{aligned}&2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\Psi(1) \\ &\quad\leq \int_{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}^{\frac{1}{\kappa_1}} \frac{\varphi(u-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})}{u-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}\mathcal{F}\left(\frac{1}{u}\right)du+\int_{\frac{1}{\kappa_2}}^{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}} \frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-v)}{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-v}\mathcal{F}\left(\frac{1}{v}\right)dv \\ &\quad= \left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right].\end{aligned}$$

Hence we have the first inequality in (2.7).

To prove the second inequality in (2.7), first we note that $\mathcal{F}$ is a harmonically convex function, we get

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}\right) \leq \left(\frac{1+\tau}{2}\right)\mathcal{F}(\kappa_1)+\left(\frac{1-\tau}{2}\right)\mathcal{F}(\kappa_2) \tag{2.9}$$

and

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}\right) \leq \left(\frac{1-\tau}{2}\right)\mathcal{F}(\kappa_1)+\left(\frac{1+\tau}{2}\right)\mathcal{F}(\kappa_2). \tag{2.10}$$

Adding (2.9) and (2.10), we have

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}\right)+\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}\right) \leq \mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2). \tag{2.11}$$

Multiplying by $\frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}$ both sides of inequality (2.11) and integrating the resultant one with respect to τ over $[0,1]$, we obtain

$$\int_0^1 \frac{\varphi(\frac{(\kappa_2-\kappa_1)}{2\kappa_1\kappa_2}\tau)}{\tau}\left[\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1-\tau)\kappa_1+(1+\tau)\kappa_2}\right)+\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{(1+\tau)\kappa_1+(1-\tau)\kappa_2}\right)\right] \\ \leq \Psi(1)\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)\right].$$

By changing the variable of integration, we have the second inequality in (2.7). □

Remark 2 Under the assumptions of Theorem 11, if we put $\varphi(\tau)=\tau$, then Theorem 10 reduces to Theorem 4.

Corollary 3 *Under the assumptions of Theorem 11, if we set $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}$, then we have the following inequalities for the Riemann–Liouville fractional integrals*:

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\ \leq 2^{\alpha-1}\Gamma(\alpha+1)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^\alpha\left[J^\alpha_{(1/\kappa_1)-}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+J^\alpha_{(1/\kappa_2)+}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right] \\ \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.$$

Corollary 4 *Under the assumptions of Theorem 11, if we put $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequalities for the k-Riemann–Liouville fractional integrals*:

$$\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\ \leq 2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}} \\ \times\left[J^\alpha_{(1/\kappa_1)-,k}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+J^\alpha_{(1/\kappa_2)+,k}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right] \\ \leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.$$

3 Extension of Hermite–Hadamard type inequalities

In this section, we give the following inequalities which give the above and below bounds for the left- and right-hand sides of inequalities (2.1) and (2.7). We prove inequalities (2.1) and (2.7) under the condition $\phi'(\kappa_1+\kappa_2-x)\geq\phi'(x)$ instead of the harmonic convexity of $\mathcal{F}$.

Theorem 12 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold for GFIs*:

$$\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx$$

$$\leq \frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_1) + {}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right] - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)$$
$$\leq \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx. \tag{3.1}$$

Proof By using the change of variables, we have

$$\begin{aligned}
&\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_1) + {}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\
&= \frac{1}{2\Psi(1)}\left[\int_{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}^{\frac{1}{\kappa_1}}\frac{\varphi(\frac{1}{\kappa_1}-x)}{\frac{1}{\kappa_1}-x}\mathcal{F}\left(\frac{1}{x}\right)dx + \int_{\frac{1}{\kappa_2}}^{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}\frac{\varphi(x-\frac{1}{\kappa_2})}{x-\frac{1}{\kappa_2}}\mathcal{F}\left(\frac{1}{x}\right)dx\right]\\
&= \frac{1}{2\Psi(1)}\left[\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\frac{\varphi(\frac{\kappa_2-x}{\kappa_1\kappa_2})}{\kappa_2-x}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx + \int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx\right]\\
&= \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx.
\end{aligned}\tag{3.2}$$

By equality (3.2), we get

$$\begin{aligned}
&\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_1) + {}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right] - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&= \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&= \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx.
\end{aligned}\tag{3.3}$$

Using the fact that

$$\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)=\phi(x)-\phi\left(\frac{\kappa_1+\kappa_2}{2}\right)=-\int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\tau)\,d\tau$$

and

$$\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)=\phi(\kappa_1+\kappa_2-x)-\phi\left(\frac{\kappa_1+\kappa_2}{2}\right)=\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1+\kappa_2-x}\phi'(\tau)\,d\tau,$$

we have

$$\begin{aligned}
&\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&= \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1+\kappa_2-x}\phi'(\tau)\,d\tau-\int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\tau)\,d\tau\\
&= \int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\kappa_1+\kappa_2-u)\,du-\int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\tau)\,d\tau\\
&= \int_x^{\frac{\kappa_1+\kappa_2}{2}}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau.
\end{aligned}\tag{3.4}$$

We also have

$$\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)=\int_{\tau}^{\kappa_1+\kappa_2-\tau}\phi''(u)\,du. \tag{3.5}$$

By using equality (3.5) and the assumption $m<\phi''(u)<M$, $u\in[\kappa_1,\kappa_2]$, we obtain

$$m\int_{\tau}^{\kappa_1+\kappa_2-\tau}du\le\int_{\tau}^{\kappa_1+\kappa_2-\tau}\phi''(u)\,du\le M\int_{\tau}^{\kappa_1+\kappa_2-\tau}du$$

i.e.

$$m(\kappa_1+\kappa_2-2\tau)\le\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\le M(\kappa_1+\kappa_2-2\tau). \tag{3.6}$$

Integrating inequality (3.6) with respect to τ on $[x,\frac{\kappa_1+\kappa_2}{2}]$, we get

$$m\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\le\int_x^{\frac{\kappa_1+\kappa_2}{2}}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau\le M\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2.$$

By equality (3.4), we have

$$\begin{aligned}m\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2&\le\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\le M\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2.\end{aligned} \tag{3.7}$$

Multiplying inequality (3.7) by $\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{2\Psi(1)(x-\kappa_1)}$ and integrating the resultant inequality with respect to x on $[\kappa_1,\frac{\kappa_1+\kappa_2}{2}]$, we establish

$$\begin{aligned}&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&\quad\le\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&\quad\le\frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx.\end{aligned}$$

That is, we get

$$\begin{aligned}&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&\quad\le\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\quad\le\frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx,\end{aligned}$$

which gives inequality (3.1). □

Remark 3 Under the assumptions of Theorem 12, if we put $\varphi(\tau)=\tau$, then we have the following inequalities:

$$\frac{m(\kappa_2-\kappa_1)^2}{24}\le\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}\,dx-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\le\frac{M(\kappa_2-\kappa_1)^2}{24}. \tag{3.8}$$

Corollary 5 *Under the assumptions of Theorem 12, if we set $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, then we have the following inequalities for the Riemann–Liouville fractional integrals:*

$$\begin{aligned}\frac{m(\kappa_2-\kappa_1)^2}{4(\alpha+1)(\alpha+2)}&\le 2^{\alpha-1}\Gamma(\alpha+1)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\alpha}\\&\quad\times\left[J^{\alpha}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^{\alpha}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\le\frac{M(\kappa_2-\kappa_1)^2}{4(\alpha+1)(\alpha+2)}.\end{aligned}$$

Corollary 6 *Under the assumptions of Theorem 12, if we put $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequalities for the k-Riemann–Liouville fractional integrals:*

$$\begin{aligned}&\frac{m(\kappa_2-\kappa_1)^2}{4(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\\&\quad\le 2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}}\\&\quad\times\left[J^{\alpha}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+,k}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^{\alpha}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-,k}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\quad\le\frac{M(\kappa_2-\kappa_1)^2}{4(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}.\end{aligned}$$

Theorem 13 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold for the GFIs:*

$$\begin{aligned}&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-x)\varphi\left(\frac{x-\kappa_1}{\kappa_1\kappa_2}\right)dx\\&\quad\le\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\&\quad\le\frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-x)\varphi\left(\frac{x-\kappa_1}{\kappa_1\kappa_2}\right)dx.\end{aligned}\tag{3.9}$$

Proof By using the change of variables, we have

$$\begin{aligned}&\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\&\quad=\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\end{aligned}$$

$$\times\left[\int_{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}^{\frac{1}{\kappa_1}}\frac{\varphi(\frac{1}{\kappa_1}-x)}{\frac{1}{\kappa_1}-x}\mathcal{F}\left(\frac{1}{x}\right)dx+\int_{\frac{1}{\kappa_2}}^{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}\frac{\varphi(x-\frac{1}{\kappa_2})}{x-\frac{1}{\kappa_2}}\mathcal{F}\left(\frac{1}{x}\right)dx\right]$$
$$=\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}$$
$$-\frac{1}{2\Psi(1)}\left[\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\frac{\varphi(\frac{\kappa_2-x}{\kappa_1\kappa_2})}{\kappa_2-x}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx+\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\left(\frac{\kappa_1\kappa_2}{x}\right)dx\right]$$
$$=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)\right.$$
$$\left.-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx. \tag{3.10}$$

By using the equalities

$$\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}(\kappa_1)=\phi(x)-\phi(\kappa_1)=\int_{\kappa_1}^{x}\phi'(\tau)\,d\tau$$

and

$$\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)=\phi(\kappa_2)-\phi(\kappa_1+\kappa_2-x)=\int_{\kappa_1+\kappa_2-x}^{\kappa_2}\phi'(\tau)\,d\tau,$$

we have

$$\phi(\kappa_1)+\phi(\kappa_2)-\phi(x)-\phi(\kappa_1+\kappa_2-x)$$
$$=\int_{\kappa_1+\kappa_2-x}^{\kappa_2}\phi'(\tau)\,d\tau-\int_{\kappa_1}^{x}\phi'(\tau)\,d\tau=\int_{\kappa_1}^{x}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau. \tag{3.11}$$

Integrating (3.6) with respect to τ over $[\kappa_1,x]$, we get

$$m\int_{\kappa_1}^{x}(\kappa_1+\kappa_2-2\tau)\,d\tau\le\int_{\kappa_1}^{x}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau$$
$$\le M\int_{\kappa_1}^{x}(\kappa_1+\kappa_2-2\tau)\,d\tau,$$

which implies that

$$m(x-\kappa_1)(\kappa_2-x)\le\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)$$
$$\le M(x-\kappa_1)(\kappa_2-x). \tag{3.12}$$

Multiplying inequality (3.12) by $\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{2\Psi(1)(x-\kappa_1)}$ and integrating the resultant inequality with respect to x on $[\kappa_1,\frac{\kappa_1+\kappa_2}{2}]$, we establish

$$\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{(x-\kappa_1)}\,dx$$
$$\le\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{(x-\kappa_1)}\,dx$$

$$\leq \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{(x-\kappa_1)}\,dx.$$

That can be written as

$$\begin{aligned}
&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{(x-\kappa_1)}\,dx\\
&\quad\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\Big[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_\varphi(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\Big]\\
&\quad\leq \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{(x-\kappa_1)}\,dx,
\end{aligned}$$

which gives inequalities (3.9). □

Remark 4 Under the assumptions of Theorem 13, if we put $\varphi(\tau)=\tau$, then we have the following inequalities:

$$\frac{m(\kappa_2-\kappa_1)^2}{12}\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\frac{\mathcal{F}(x)}{x^2}\,dx\leq \frac{M(\kappa_2-\kappa_1)^2}{24}. \tag{3.13}$$

Corollary 7 *Under the assumptions of Theorem* 13, *if we set* $\varphi(\tau)=\frac{\tau^\alpha}{\Gamma(\alpha)}$, *then we have the following inequalities for the Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m(\kappa_2-\kappa_1)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}\\
&\quad\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-2^{\alpha-1}\Gamma(\alpha+1)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^\alpha\\
&\qquad\times\Big[J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\Big]\\
&\quad\leq \frac{M(\kappa_2-\kappa_1)^2\alpha(\alpha+3)}{8(\alpha+1)(\alpha+2)}.
\end{aligned}$$

Corollary 8 *Under the assumptions of Theorem* 13, *if we put* $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, *then we have the following inequalities for the k-Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m(\kappa_2-\kappa_1)^2\frac{\alpha}{k}(\frac{\alpha}{k}+3)}{8(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\\
&\quad\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}}\\
&\qquad\times\Big[J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+,k}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+J^\alpha_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-,k}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\Big]\\
&\quad\leq \frac{M(\kappa_2-\kappa_1)^2\frac{\alpha}{k}(\frac{\alpha}{k}+3)}{8(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}.
\end{aligned}$$

Now by using Theorems 12 and 13, we prove inequality (2.1) under the condition $\phi'(\kappa_1+\kappa_2-x)\geq\phi'(x)$ instead of the harmonic convexity of $\mathcal{F}$.

Theorem 14 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If $\phi'(\kappa_1+\kappa_2-x)\geq\phi'(x),\forall x\in[\kappa_1,\frac{\kappa_1+\kappa_2}{2}]$, then we have the following inequalities for GFIs*:

$$\begin{aligned}\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)&\leq\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\&\leq\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}.\end{aligned}\tag{3.14}$$

Proof From (3.3) and (3.4), one has

$$\begin{aligned}&\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\int_{x}^{\frac{\kappa_1+\kappa_2}{2}}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau\right)\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&\geq0,\end{aligned}$$

which gives the first inequality in (3.14). On the other hand, by equalities (3.10) and (3.11), we have

$$\begin{aligned}&\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\left[{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_1)+{}_{(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})(1/\kappa_2)\right]\\&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\int_{\kappa_1}^{x}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau\right)\frac{\varphi(\frac{x-\kappa_1}{\kappa_1\kappa_2})}{x-\kappa_1}\,dx\\&\geq0.\end{aligned}$$

This gives the second inequality in (3.14) and completes the proof. □

Theorem 15 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold for the GFIs*:

$$\begin{aligned}&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)\varphi\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2}\right)dx\\&\leq\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\&\leq\frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)\varphi\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2}\right)dx.\end{aligned}\tag{3.15}$$

Proof By using the change of variables, we have

$$\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{2\Psi(1)}\left[\int_{\frac{1}{\kappa_2}}^{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}} \frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-x)}{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-x}\mathcal{F}\left(\frac{1}{x}\right)dx + \int_{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}^{\frac{1}{\kappa_1}} \frac{\varphi(x-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})}{x-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}\mathcal{F}\left(\frac{1}{x}\right)dx\right] \\
&= \frac{1}{2\Psi(1)}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx\right. \\
&\quad \left.+ \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2} \frac{\varphi(\frac{x}{\kappa_1\kappa_2}-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})}{x-\frac{\kappa_1+\kappa_2}{2}}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx\right] \\
&= \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx.
\end{aligned} \tag{3.16}
$$

By equality (3.16), we get

$$
\begin{aligned}
&\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right) + {}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right] - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&= \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&= \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right. \\
&\quad \left. - 2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx.
\end{aligned} \tag{3.17}
$$

Using the fact that

$$
\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right) - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) = \phi(x) - \phi\left(\frac{\kappa_1+\kappa_2}{2}\right) = -\int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\tau)\,d\tau
$$

and

$$
\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right) - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) = \phi(\kappa_1+\kappa_2-x) - \phi\left(\frac{\kappa_1+\kappa_2}{2}\right) = \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1+\kappa_2-x}\phi'(\tau)\,d\tau,
$$

we have

$$
\begin{aligned}
&\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right) + \mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right) - 2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&= \int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1+\kappa_2-x}\phi'(\tau)\,d\tau - \int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\tau)\,d\tau \\
&= \int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\kappa_1+\kappa_2-u)\,du - \int_x^{\frac{\kappa_1+\kappa_2}{2}}\phi'(\tau)\,d\tau \\
&= \int_x^{\frac{\kappa_1+\kappa_2}{2}}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau.
\end{aligned} \tag{3.18}
$$

We also have

$$
\phi'(\kappa_1+\kappa_2-\tau) - \phi'(\tau) = \int_\tau^{\kappa_1+\kappa_2-\tau}\phi''(u)\,du. \tag{3.19}
$$

By using equality (3.19) and the assumption $m < \phi''(u) < M$, $u \in [\kappa_1, \kappa_2]$, we obtain

$$m\int_{\tau}^{\kappa_1+\kappa_2-\tau} du \le \int_{\tau}^{\kappa_1+\kappa_2-\tau} \phi''(u)\,du \le M\int_{\tau}^{\kappa_1+\kappa_2-\tau} du$$

i.e.

$$m(\kappa_1+\kappa_2-2\tau) \le \phi'(\kappa_1+\kappa_2-\tau) - \phi'(\tau) \le M(\kappa_1+\kappa_2-2\tau). \tag{3.20}$$

Integrating inequality (3.20) with respect to τ on $[x, \frac{\kappa_1+\kappa_2}{2}]$, we get

$$m\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2 \le \int_x^{\frac{\kappa_1+\kappa_2}{2}} \left[\phi'(\kappa_1+\kappa_2-\tau) - \phi'(\tau)\right]d\tau \le M\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2.$$

By equality (3.18), we have

$$\begin{aligned} m\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2 &\le \mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right) + \mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right) - 2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\ &\le M\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2. \end{aligned} \tag{3.21}$$

Multiplying inequality (3.21) by $\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{2\Psi(1)(\frac{\kappa_1+\kappa_2}{2}-x)}$ and integrating the resultant inequality with respect to x on $[\kappa_1, \frac{\kappa_1+\kappa_2}{2}]$, we establish

$$\begin{aligned} &\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2 \frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx \\ &\le \frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left[\mathcal{F}(x) + \mathcal{F}(\kappa_1+\kappa_2-x) - 2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx \\ &\le \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2}-x\right)^2 \frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx. \end{aligned}$$

That is, we get

$$\begin{aligned} &\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2}-x\right)\varphi\left(\frac{x}{\kappa_1\kappa_2}-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)dx \\ &\le \frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right) + {}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right] - \mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\ &\le \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}} \left(\frac{\kappa_1+\kappa_2}{2}-x\right)\varphi\left(\frac{x}{\kappa_1\kappa_2}-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)dx, \end{aligned}$$

which gives inequality (3.15). □

Remark 5 Under the assumptions of Theorem 15, if we put $\varphi(\tau) = \tau$, then inequality (3.15) reduces to inequality (3.8).

Corollary 9 *Under the assumptions of Theorem 15, if we set $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$, then we have the following inequalities for the Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m\alpha(\kappa_2-\kappa_1)^2}{8(\alpha+2)} \\
&\leq 2^{\alpha-1}\Gamma(\alpha+1)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^\alpha \\
&\quad\times\left[J^\alpha_{(1/\kappa_1)-}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+J^\alpha_{(1/\kappa_2)+}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&\leq \frac{M\alpha(\kappa_2-\kappa_1)^2}{8(\alpha+2)}.
\end{aligned}$$

Corollary 10 *Under the assumptions of Theorem 15, if we put $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequalities for the k-Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m\frac{\alpha}{k}(\kappa_2-\kappa_1)^2}{8(\frac{\alpha}{k}+2)} \\
&\leq 2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)\left(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\right)^{\frac{\alpha}{k}} \\
&\quad\times\left[J^\alpha_{(1/\kappa_1)-,k}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+J^\alpha_{(1/\kappa_2)+,k}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right) \\
&\leq \frac{M\frac{\alpha}{k}(\kappa_2-\kappa_1)^2}{8(\frac{\alpha}{k}+2)}.
\end{aligned}$$

Theorem 16 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If ϕ'' is bounded in $[\kappa_1,\kappa_2]$, then the following inequalities hold for the GFIs*:

$$\begin{aligned}
&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-x)(x-\kappa_1)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx \\
&\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_\varphi(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_\varphi(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right] \\
&\leq \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\kappa_2-x)(x-\kappa_1)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx. \qquad (3.22)
\end{aligned}$$

Proof By using the change of variables, we have

$$\begin{aligned}
&\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_\varphi(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_\varphi(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right] \\
&=\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2} \\
&\quad-\frac{1}{2\Psi(1)}\left[\int_{\frac{1}{\kappa_2}}^{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-x)}{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-x}\mathcal{F}\left(\frac{1}{x}\right)dx+\int_{\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}^{\frac{1}{\kappa_1}}\frac{\varphi(x-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})}{x-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}}\mathcal{F}\left(\frac{1}{x}\right)dx\right] \\
&=\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}
\end{aligned}$$

$$-\frac{1}{2\Psi(1)}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx\right.$$
$$\left.+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\frac{\varphi(\frac{x}{\kappa_1\kappa_2}-\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2})}{x-\frac{\kappa_1+\kappa_2}{2}}\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)dx\right]$$
$$=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)\right.$$
$$\left.-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}dx. \tag{3.23}$$

By using the equalities

$$\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}(\kappa_1)=\phi(x)-\phi(\kappa_1)=\int_{\kappa_1}^{x}\phi'(\tau)\,d\tau$$

and

$$\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)=\phi(\kappa_2)-\phi(\kappa_1+\kappa_2-x)=\int_{\kappa_1+\kappa_2-x}^{\kappa_2}\phi'(\tau)\,d\tau,$$

we have

$$\phi(\kappa_1)+\phi(\kappa_2)-\phi(x)-\phi(\kappa_1+\kappa_2-x)$$
$$=\int_{\kappa_1+\kappa_2-x}^{\kappa_2}\phi'(\tau)\,d\tau-\int_{\kappa_1}^{x}\phi'(\tau)\,d\tau=\int_{\kappa_1}^{x}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau. \tag{3.24}$$

Integrating (3.6) with respect to τ over $[\kappa_1,x]$, we get

$$m\int_{\kappa_1}^{x}(\kappa_1+\kappa_2-2\tau)\,d\tau\le\int_{\kappa_1}^{x}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau\le M\int_{\kappa_1}^{x}(\kappa_1+\kappa_2-2\tau)\,d\tau,$$

which implies that

$$m(x-\kappa_1)(\kappa_2-x)\le\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)$$
$$\le M(x-\kappa_1)(\kappa_2-x). \tag{3.25}$$

Multiplying inequality (3.25) by $\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{2\Psi(1)(\frac{\kappa_1+\kappa_2}{2}-x)}$ and integrating the resultant inequality with respect to x on $[\kappa_1,\frac{\kappa_1+\kappa_2}{2}]$, we establish

$$\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}dx$$
$$\le\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}dx$$
$$\le\frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}dx.$$

That can be written as

$$\begin{aligned}
&\frac{m}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx \\
&\quad\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\biggl[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)+{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)\biggr] \\
&\quad\le \frac{M}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(x-\kappa_1)(\kappa_2-x)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx,
\end{aligned}$$

which gives inequalities (3.22). □

Remark 6 Under the assumptions of Theorem 16, if we put $\varphi(\tau)=\tau$, then inequality (3.22) reduces to inequality (3.13).

Corollary 11 *Under the assumptions of Theorem 16, if we set $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, then we have the following inequalities for the Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m(\kappa_2-\kappa_1)^2}{4(\alpha+2)} \\
&\quad\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-2^{\alpha-1}\Gamma(\alpha+1)\biggl(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\biggr)^{\alpha} \\
&\quad\quad\times\biggl[J^{\alpha}_{(1/\kappa_1)-}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)+J^{\alpha}_{(1/\kappa_2)+}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)\biggr] \\
&\quad\le \frac{M(\kappa_2-\kappa_1)^2}{4(\alpha+2)}.
\end{aligned}$$

Corollary 12 *Under the assumptions of Theorem 16, if we put $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we have the following inequalities for the k-Riemann–Liouville fractional integrals*:

$$\begin{aligned}
&\frac{m(\kappa_2-\kappa_1)^2}{4(\frac{\alpha}{k}+2)} \\
&\quad\le \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)\biggl(\frac{\kappa_1\kappa_2}{\kappa_2-\kappa_1}\biggr)^{\frac{\alpha}{k}} \\
&\quad\quad\times\biggl[J^{\alpha}_{(1/\kappa_1)-,k}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)+J^{\alpha}_{(1/\kappa_2)+,k}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)\biggr] \\
&\quad\le \frac{M(\kappa_2-\kappa_1)^2}{4(\frac{\alpha}{k}+2)}.
\end{aligned}$$

Now, by using Theorems 15 and 16, we prove inequality (2.7) under the condition $\phi'(\kappa_1+\kappa_2-x)\ge\phi'(x)$ instead of the harmonic convexity of $\mathcal{F}$.

Theorem 17 *Let $\mathcal{F}:[\kappa_1,\kappa_2]\subseteq(0,+\infty)\to\mathbb{R}$ be a positive twice differentiable function with $\kappa_1<\kappa_2$ and $\mathcal{F}\in L([\kappa_1,\kappa_2])$. If $\phi'(\kappa_1+\kappa_2-x)\ge\phi'(x),\forall x\in[\kappa_1,\frac{\kappa_1+\kappa_2}{2}]$, then we have the following inequalities for GFIs*:

$$\mathcal{F}\biggl(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\biggr)\le\frac{1}{2\Psi(1)}\biggl[{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)+{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\biggl(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\biggr)\biggr]$$

$$\leq \frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{3.26}$$

Proof From (3.17) and (3.18), we get

$$\begin{aligned}
&\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]-\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\\
&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)+\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)-2\mathcal{F}\left(\frac{2\kappa_1\kappa_2}{\kappa_1+\kappa_2}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx\\
&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\int_{x}^{\frac{\kappa_1+\kappa_2}{2}}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau\right)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx\\
&\geq 0,
\end{aligned}$$

which proves the first inequality in (3.26). On the other hand, by equalities (3.23) and (3.24), we have

$$\begin{aligned}
&\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}-\frac{1}{2\Psi(1)}\left[{}_{\frac{1}{\kappa_2}+}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)+{}_{\frac{1}{\kappa_1}-}I_{\varphi}(\mathcal{F}\circ\mathcal{G})\left(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}\right)\right]\\
&=\frac{1}{2\Psi(1)}\\
&\quad\times\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{x}\right)-\mathcal{F}\left(\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2-x}\right)\right]\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx\\
&=\frac{1}{2\Psi(1)}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\int_{\kappa_1}^{x}\left[\phi'(\kappa_1+\kappa_2-\tau)-\phi'(\tau)\right]d\tau\right)\frac{\varphi(\frac{\kappa_1+\kappa_2}{2\kappa_1\kappa_2}-\frac{x}{\kappa_1\kappa_2})}{\frac{\kappa_1+\kappa_2}{2}-x}\,dx\\
&\geq 0.
\end{aligned}$$

This proves the second inequality in (3.26) and completes the proof. □

4 Conclusion

In this work, the authors established Hermite–Hadamard type inequalities for harmonically convex functions by using generalized fractional integrals. Furthermore, the authors proved some extensions of newly proved inequalities without using the condition of harmonic convexity for the functions. It is an interesting and new problem, and the upcoming researchers can offer similar inequalities for harmonically convex functions on the co-ordinates via generalized fractional integrals in their future research.

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