EXTENSIONS OF JENTZSCH'S THEOREM

BY
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1. **Introduction.** In this paper, the projective metric of Hilbert(1) is applied to prove various extensions of Jentzsch's theorem on integral equations(2) with positive kernels. In particular, it is shown that Jentzsch's theorem reduces to the Picard fixpoint theorem(3), relative to this projective metric.

A natural setting for generalizing Jentzsch's theorem seems to be provided by the theory of vector lattices(4). A bounded linear transformation P of a vector lattice L into itself will be called *uniformly positive* if, for some fixed $e>0$ in L and finite real number K, independent of f, we have

$$\lambda e \leqq fP \leqq K\lambda e \qquad \text{for any } f > 0 \text{ and some } \lambda = \lambda(f) > 0. \tag{1}$$

Theorem 3 below shows that Jentzsch's theorem applies, in generalized form, to any such operator P.

The method is also applied to various other cases: in §5, to a class of integro-functional equations to which the usual proof would not be applicable; in §8, to a class of semigroups including various multiplicative processes(5).

2. **Projective metrics on line.** For convenient reference, we derive some basic formulas regarding the effect of projective transformations on projective metrics. In homogeneous coordinates, the first positive quadrant joins (0, 1) with (1, 0) by "points" (f_1, f_2). This is mapped onto the hyperbolic line $-\infty<u<+\infty$ by the correspondence $\mathrm{Ln}(f_2/f_1)=u$. We define

$$\theta(f, g) = \left| \mathrm{Ln}\, v - \mathrm{Ln}\, u \right| = \left| \mathrm{Ln}\, (f_2 g_1/f_1 g_2) \right| . \tag{2}$$

Since f_2g_1/f_1g_2 is the cross-ratio $R(f_2/f_1, g_2/g_1; 0, \infty)$, $\theta(f, g)$ is invariant under all projective transformations mapping the interval $0<f_2/f_1<+\infty$ onto itself.

We next consider a general projective transformation

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(1) Math. Ann. vol. 57 (1903) pp. 137–150. For a modern exposition, see H. Busemann and P. J. Kelly, *Projective geometries and projective metrics*, New York, 1953, §§28, 29, 50; or H. Busemann, *The geometry of geodesics*, New York, 1955, §18. I am indebted to Professors Busemann, Coxeter and Menger for helpful references.

(2) J. Reine Angew. Math. vol. 141 (1912) pp. 235–244, or W. Schmeidler, *Integralgleichungen*, p. 298.

(3) É. Picard, *Traité d'analyse*, 2d ed. vol. 1 p. 170. The present approach was announced in Abstract 62-2-190 of Bull. Amer. Math. Soc., where the phrase "hyperbolic metric" was used, because of the relation to Hilbert's hyperbolic geometries.

(4) In the sense of G. Birkhoff, *Lattice theory*, rev. ed., Chap. XV. Interpretations of the usual Banach spaces as vector lattices are explained there.

(5) In the sense of C. J. Everett and S. Ulam, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 403–407.

(3) $$P: \; y = xP = (ax+b)/(cx+d), \qquad x = f_2/f_1,$$

which maps $0<x<\infty$ onto a proper subinterval of $0<y<\infty$. Without loss of generality, since $y\to 1/y$ is an isometry for $\theta(y, y')$, we can assume that order is preserved. That is, we can assume $x>x'$ implies $xP\geqq x'P$. Since $OP=b/d$, clearly b and d have the same sign; since $xP=0$ has no positive solution, b and a have the same sign; since $xP=\infty$ has no positive solution, d and c have the same sign. Hence we can assume a, b, c, d *positive* in (3). Furthermore, since

(4) $$y' = dy/dx = (ad-bc)/(cx+d)^2,$$

we can assume $ad>bc$.

We now consider the ratio of hyperbolic distance differentials $d\theta(y)/d\theta(x) = xdy/ydx$. By (3) and (4), this is $d\theta(y)/d\theta(x)=(ad-bc)x/(ax+b)(cx+d)$. By the differential calculus, the *maximum* of this (i.e., the minimum of its reciprocal) occurs when $ac=bdx^{-2}$, or $x=(bd/ac)^{1/2}$. Substituting above and simplifying, we see that P *contracts* all hyperbolic distances, by a factor whose supremum is

(5) $$N(P) = \frac{\nu-1}{\nu+2(\nu)^{1/2}+1}, \qquad \text{where } \nu = (ad/bc) > 1.$$

We call $N(P)$ the *projective norm* of P. Because of its definition, as the supremum of distance ratios

(6) $$N(P) = \sup\,[\theta(fP, gP)/\theta(f, g)],$$

we have immediately

(6′) $$N(PP') \leqq N(P)N(P').$$

Also, if $\lambda=\operatorname{Ln}(ad/bc)$ is the length of the segment onto which P maps the first quadrant, then by (5), we have

(7) $$N(P) = \frac{\nu^{1/4}-\nu^{1/4}}{\nu^{1/4}+\nu^{1/4}} = \frac{2\sinh(\lambda/4)}{2\cosh(\lambda/4)} = \tanh\frac{\lambda}{4}\cdot$$

The intervention of hyperbolic functions is most appropriate!

3. **Convex cones.** Now let C be any *bounded closed convex cone* of a real vector space L, of finite *or* infinite dimensions. It is convenient to make a central projection of C onto its (convex) intersection $C\cap H$ with a hyperplane H, cutting each ray of C in exactly one point; we can then discuss C and $C\cap H$ interchangeably, as subspaces of projective space.

Since C is a bounded closed convex set, *every line intersects C in a closed segment.* Hence, if $f\neq g$ in H, the intersection of the line $l(f, g)$ with C can be mapped onto the line $0\leqq x\leqq\infty$ of §2 so that $fA<gA$ by a *unique* affine transformation A. We define

$$\theta(f, g; C) = \theta(fA, gA). \tag{8}$$

If f or g is a boundary point, $\theta(f, g; C) = \infty$. We call $\theta(f, g; C)$ the *projective metric* associated with C.

The following result is well-known(6).

LEMMA 1. *For any $a \in C$, the set A of rays $f \in C$ satisfying $\theta(a, f; c) < +\infty$ is a metric space relative to the distance $\theta(f, g; C)$.*

It is well known(6) that, if $C \cap H$ is an ellipsoid, then $\theta(f, g; C)$ makes C into a hyperbolic geometry. It seems not to have been observed, however, that the following example leads to the Perron-Frobenius(7) theory of positive matrices.

EXAMPLE 1. Let C be the cone R of "positive" $f \neq 0$ satisfying $f_1 \geqq 0, \cdots, f_n \geqq 0$ in real n-space, and let H be the hyperplane $\sum f_i = 1$. In this case, the disconnected components of $C \cap H$ are the interiors of its cells, where $C \cap H$ is regarded as a *simplex*.

The theorem of Jentzsch can be deduced very simply, as we shall see below, from the following special case.

EXAMPLE 2. Let L be the space of continuous functions, in the usual Banach lattice norm $\|f\| = \sup |f(x)|$, and let C be the cone L^+ of non-negative functions. Then the functions which are identically *positive* form a connected component under $\theta(f, g; L^+)$.

Similarly, generalizations of Jentzsch's theorem can be deduced by considering other special cases, such as the following.

EXAMPLE 3. Let B be the Banach space of bounded measurable functions on the unit interval, with the norm $\|f\| = \sup |f(x)|$. For any positive constant M, let C be the cone of functions satisfying

$$0 < \sup f(x) \leqq M \inf f(x).$$

We omit proving that the cones in question are closed in the relevant Banach spaces, if the origin is included.

4. **Fixpoint theorem.** Let P be any bounded linear transformation of a Banach space B, which maps a closed convex cone C of B into itself. The C-norm $N(P; C)$ of P is defined as

$$N(P; C) = \sup \left[\theta(fP, gP; C)/\theta(f, g; C)\right], \tag{9}$$

for pairs $f, g \in C$ with finite $\theta(f, g; C)$.

LEMMA 1. *If the transform CP of C under P has finite diameter Δ under $\theta(f, g; C)$, then*

$$N(P, C) = \tanh(\Delta/4) < 1. \tag{9a}$$

(6) See the refs. of Footnote 1.

(7) G. Frobenius, Sitzungsberichte der Berlin Akad. Wiss. (1908) pp. 471–476 and (1909) pp. 514–518 and references given there.

Proof. If $\theta(f, g; C) < +\infty$, then f and g lie on a segment $s(a, b)$ of C. The image segment $s(aP, bP) \leqq CP$; hence $\theta(aP, bP; C) \leqq \Delta$. By (7), we infer

$$\theta(fP, gP; C)/\theta(f, g; C) \leqq \tanh(\Delta/4).$$

Hence, by (9), $N(P, C) \leqq \tanh(\Delta/4)$. To show that equality holds, we take a sequence of inverse images f_n, g_n of suitable nearby pairs of points on segments $s(c_n, d_n)$ of lengths $\Delta - 2^{-n}$ or more, and use (7) again.

If CP has infinite diameter, then similar considerations show that $N(P, C) = 1$. (The fact that $N(P, C) \leqq 1$ is immediate from (7).)

THEOREM 1 (PROJECTIVE CONTRACTION THEOREM). *Let $N(P^r; C) < 1$ for some r, and let C be complete relative to $\theta(f, g; C)$. Then, for any $f \in C$, the sequence of fP^n converges geometrically to a unique fixpoint (characteristic ray) $c \in C$.*

Proof. If $N(P^r; C) < 1$, then CP^r has a finite hyperbolic diameter, by what we have just seen. Hence $\theta(fP^r, fP^{r+1}; C) < +\infty$. More generally, if $q > 0$ is the integral part of (n/r), then (writing $\theta(f, g; C)$ as $\theta(f, g)$)

$$\theta(fP^n, fP^{n+1}) \leqq N(P^r; C)^{q-1}\theta(fP^r, fP^{r+1}).$$

Hence, as in the proof of Picard's Fixpoint Theorem[3], $\{fP^n\}$ is a Cauchy sequence. By the assumption of completeness, the Cauchy sequence converges to a limit $c \in C$. Since P is *bounded*, $cP = c$, and

$$\|fP^n - c\| < K\rho^n,$$

where $\rho = N(P^r; C)^{1/r}$, and $K < +\infty$. The uniqueness of c is immediate, since $cP = c$ and $c^*P = c^*$ imply

$$\theta(c, c^*) = \theta(cP^r, c^*P^r) \leqq N(P^r, C)\theta(c, c^*),$$

all relative to C. Since $N(P^r, C) < 1$, this implies $\theta(c, c^*) = 0$.

COROLLARY 1. *If some CP^r has finite projective diameter relative to C, then the conclusion of Theorem* 1 *holds.*

5. **Applications.** To apply Theorem 1, one must verify that the cone C involved is *complete* in the projective metric $\theta(f, g; C)$. This is obvious in the case of Example 1, from the known[6] facts about finite-dimensional projective metrics. The cases of Examples 2–3 are also easily covered. We now give some applications of Theorem 1 based on these special examples.

If P is the linear transformation corresponding to a matrix $\|p_{ij}\|$, with positive entries, then RP is a compact subset interior to R. The rays L touching R, in Example 1 of §3, are also a compact set, and the lengths $\theta(LP)$ of their transforms vary continuously with L; hence RP has finite projective diameter. We conclude, by Theorem 1, that P admits a *positive* eigenvector $c \in R$, with $cP = \gamma c$. Obviously, it is sufficient that P be non-negative, and some power P^r positive.

Again, as in Jentzsch's theorem, let P be the operator defined by $[fP](x) = \int_0^1 p(x, y)f(y)dy$, $p(x, y) > 0$, with

$$0 < I = \inf p(x, y) \leqq \sup p(x, y) = KI = S. \tag{10}$$

Choose L as in Example 2 of §3. Then, if $e(x) \equiv 1$, and $f(x) \geqq 0$ with $\int f(x)dx = \phi > 0$, clearly $(I\phi)e \leqq fP \leqq (S\phi)e$. Hence $\theta(e, fP; L^+) \leqq \mathrm{Ln}\, K$, and Theorem 1 applies to show that $fP^n \to c$, where $cP = \gamma c$, $c \in R$.

Note that the preceding proof does not assume Fredholm's theory of integral equations. It will be generalized in Theorem 3 below.

Projective metrics can be applied flexibly(8) to a variety of positive transformations. The following application is fairly typical.

THEOREM 2. *Let $p(x, y)$ satisfy* (10); *let $T_1, \cdots, T_n$ be any one-one Borel transformations of the unit interval onto itself, and let $a_1, \cdots, a_n$ be positive constants with sum A. Then the integro-functional equation*

$$[fP](x) = \int_0^1 p(x, y)f(y)dy + \sum a_i f(xT_i) \tag{11}$$

admits a unique positive characteristic function, such that

$$\int_0^1 p(x, y)c(y)dy + \sum a_i c(xT_i) = \gamma c(x). \tag{12}$$

Proof. Choose any $M > K$, and let B and C be defined for this M as in Example 3 of §4. Then, if $f(x) \in C$ and $g(x) = [fP](x)$, we have

$$\sup g(x) \leqq KI \int f(y)dy + MA \inf f(x),$$

$$\inf g(x) \geqq I \int f(y)dy + A \inf f(x).$$

In view of the inequality $\int f(y)dy \geqq \inf f(x)$, we infer

$$\frac{\sup g(x)}{\inf g(x)} \leqq \frac{KI + MA}{I + A} < M, \qquad \text{if } K < M.$$

Hence, the projective contraction theorem applies, and so the conclusion of Theorem 2 follows.

6. **Banach lattices.** In a different direction, one can generalize Jentzsch's

(8) In this respect, the technique of projective metrics is analogous to the Leray-Schauder technique applied by E. Rothe, Amer. J. Math. vol. 66 (1944) pp. 245–254, and by M. G. Krein and M. A. Rutman, Uspehi Matematičeskih Nauk. vol. 3 (1948) pp. 3–95, to prove other generalizations of Jentzsch's theorem. It differs from the Leray-Schauder technique in being *constructive* and in not assuming complete continuity of P.

theorem to Banach lattices. In making this generalization, the following lemma will prove convenient.

LEMMA 2. *In any vector lattice L, let L^+ denote the cone of positive elements. If f and g are in the same connected component of L^+, then they are* STRONGLY COMPARABLE *in the sense that*

$$\lambda f \leqq g \leqq R\lambda f \quad \text{and} \quad \mu g \leqq f \leqq R\mu f, \qquad R < +\infty. \tag{13}$$

Actually, the smallest such $R = \exp[\theta(f, g; L^+)]$.

Proof. The plane through f and g intersects L^+ in a domain affine equivalent to a quadrant of the (x, y)-plane, with $g_2/g_1 \geqq f_2/f_1$. In this quadrant, we easily calculate $(f_1, f_2) \leqq (f_1, g_2 f_1/g_1) \leqq R(f_1, f_2)$, etc. To complete the proof, consider the (x, y)-plane as a projective line.

COROLLARY 1. *On the unit sphere of any Banach lattice*[4] *L, we have*

$$\|f - g\| \leqq e^\theta - 1, \qquad \text{where } \theta = \theta(f, g; L^+). \tag{14}$$

Proof. Suppose $\|f\| = \|g\| = 1$ in (13). Then by the monotonicity of $\|f\|$ as a function of $|f|$, $\lambda \leqq 1 \leqq R\lambda$. Consequently

$$\|f - g\| = \|f \cup g - f \cap g\| \leqq \|R\lambda f - \lambda f\| = (R - 1)\lambda\|f\|.$$

Since $R = e^\theta$ and $\lambda \leqq 1$, the proof is complete.

COROLLARY 2. *In the metric $\theta(f, g; L^+)$, any θ-connected component of the unit sphere of any Banach lattice is a complete metric space.*

THEOREM 3. *Any uniformly positive bounded linear transformation P of a Banach lattice L into itself admits a unique positive unit vector c such that*

$$cP = \gamma c, \qquad \gamma > 0. \tag{15}$$

For any $f > 0$, $\|(fP^n/\|fP^n\|) - c\| < M\rho^n$, for some finite M and positive $\rho < 1$.

Proof. Choose C as the set L^+ of positive elements. By Theorem 1 and Corollary 2 of Lemma 2, it suffices to show that CP has finite projective diameter. Since $\theta(fP, gP; C) = \theta(fP/\lambda(f), gP/\lambda(g); C)$ we can assume that $\lambda(f) = \lambda(g) = 1$ in (1). Hence, if K is defined by (1), the segment $(Kf - g, (K-1)f, (K-1)g, Kg - f)$ is in C. But, by the projective invariance of cross-ratios,

$$R(Kf - g, (K - 1)f, (K - 1)g, Kg - f) = R(-1, 0, K - 1, K),$$

since the two quadruples are perspective. Hence[1] $\theta(fP, gP; C) \leqq \operatorname{Ln} K$, and the projective diameter of CP is at most $\operatorname{Ln} K$. This completes the proof.

7. **Complementary invariant subspace.** Let P be again any uniformly positive linear operator on a Banach lattice L, and let c be the associated positive characteristic vector, with positive characteristic value γ. For any positive $f > 0$, we can define λ_n and μ_n as the largest and smallest real numbers, respectively, such that

(16a) $$\lambda_n\gamma^n c \leqq fP^n \leqq \mu_n\gamma^n c.$$

Clearly, $0<\lambda_n \leqq \mu_n$, for all $n \geqq 1$ (see also (16d) below). Applying P to (16a), we get $\lambda_n\gamma^{n+1}c \leqq fP^{n+1} \leqq \mu\gamma^{n+1}c$, whence

(16b) $$\lambda_n \leqq \lambda_{n+1} \leqq \mu_{n+1} = \mu_n, \text{ or } \lambda_n \uparrow \quad \text{and} \quad \mu_n \downarrow .$$

Now consider $r_n = fP^n - \lambda_n\gamma^n c$ and $s_n = \mu_n\gamma^n c - fP^n$. Clearly, $0 \leqq r_n$, $0 \leqq s_n$, and $r_n + s_n = (\mu_n - \lambda_n)\gamma^n c$. By (1) and Lemma 2 of §6, $\alpha_n > 0$ and $\beta_n > 0$ exist (the case $r_n = s_n = 0$ is trivial), such that

(16c) $$\alpha_n c \leqq r_n P \leqq e^{\Delta}\alpha_n c \quad \text{and} \quad \beta_n c \leqq s_n P \leqq e^{\Delta}\beta_n c,$$

whence

(16d) $$(\alpha_n + \beta_n)c \leqq (r_n + s_n)P = \gamma^{n+1}(\mu_n - \lambda_n)c \leqq e^{\Delta}(\alpha_n + \beta_n)c.$$

On the other hand, (16c) implies

$$(\lambda_n\gamma^{n+1} + \alpha_n)c \leqq (\lambda_n\gamma^n c + r_n P) = fP^{n+1} = (\mu_n\gamma^n c - s_n P) \leqq (\mu_n\gamma^{n+1} + \beta_n)c,$$

whence $\lambda_{n+1} \geqq \lambda_n + \gamma^{-n-1}\alpha_n$, $\mu_{n+1} \leqq \mu_n - \gamma^{-n-1}\beta_n$. Subtracting these inequalities, and using (16d), we get

$$\mu_{n+1} - \lambda_{n+1} \leqq (\mu_n - \lambda_n) - \gamma^{-n-1}(\alpha_n + \beta_n) \leqq (1 - e^{-\Delta})(\mu_n - \lambda_n).$$

Induction on n gives $(\mu_{n+1} - \lambda_{n+1}) \leqq (1 - e^{-\Delta})^n(\mu_1 - \lambda_1)$, so that *$\lambda_n \uparrow M$ and $\mu_n \downarrow M$ for some finite $M = M(f)$.* Now, referring back to (16a), we conclude

LEMMA 3. *For each $f > 0$, there exist positive constants $M = M(f)$, $K = K(f)$, and $\rho = (1 - e^{-\Delta})\gamma < \gamma$ independent of f, such that*

(17) $$\left| fP^n - M\gamma^n c \right| \leqq K\rho^n c, \qquad 0 \leqq \rho < \gamma.$$

But now, $f = f^+ + f^-$ for any $f \in L$, where $f^+ = f \cup 0 \geqq 0$ and $f^- = f \cap 0 \leqq 0$. Writing $fP^n = f^+P^n + f^-P^n$, we deduce the following

COROLLARY. *The inequality* (17) *holds for each $f \in L$, and $M(f)$ is a* POSITIVE LINEAR FUNCTIONAL.

THEOREM 4. *Any uniformly positive linear operator P, acting on a Banach lattice L, decomposes L into an invariant axis with positive basis-element (eigenvector) c and associated positive eigenvalue γ, and a complementary invariant subspace S on which the spectral norm*[9] *of P is at most* $(1 - e^{-\Delta})\gamma < \gamma$.

Proof. Let S be the subspace on which $M(f) = 0$. Then, by (17) with $M = 0$, $SP \leqq S$. The last conclusion also follows from Lemma 3.

8. **Multiplicative processes.** We now consider one-parameter semigroups $\{P_t\}$ of non-negative linear operators, like those involved in multiplicative processes[5]. For simplicity, we shall consider only one-parameter semigroups

[9] L. Loomis, *Abstract harmonic analysis*, p. 75. The conclusion holds in any vector lattice which is complete in $\theta(f, g; L^+)$.

on Banach lattices, though the method can easily be adapted to other cases.

Accordingly, let P_t $(t>0)$ map a Banach lattice L linearly into itself, so that

$$f > 0 \text{ implies } fP_t > 0. \tag{18}$$

We assume the (Chapman-Kolmogorov) semigroup condition $(fP_t)P_\tau = fP_{t+\tau}$. The special case

$$f(\mathbf{x};\, t+\tau) = \int p(\mathbf{x};\, \mathbf{y};\, \tau) f(\mathbf{y};\, t)\, dR(\mathbf{y}), \tag{19}$$

with $p(\mathbf{x}, \mathbf{y};\, \tau) > 0$ for all $\tau > 0$, is typical for many applications (e.g., to multigroup diffusion).

THEOREM 5. *If, for some $t=T$, P_T is uniformly positive, then there exists a positive eigenvector $c>0$ and a unique "asymptotic growth coefficient" δ, such that*

$$\|fP_t - e^{\delta t} m(f)c\| \leqq K^* e^{\sigma t}, \qquad 0 \leqq \sigma < \delta, \tag{20}$$

for every f, a suitable "effective initial size" $m(f)$, and $t \geqq T$.

Proof. By Theorem 4, the discrete semigroup of $P_T^n = P_{nT}$ has the desired property; $m(f)$ is given by (17), with $e^{\delta T} = \gamma$. Furthermore, if C is the "cone" of non-negative $f = f^+$ in L, then CP_T has finite projective diameter Δ. Hence, for any $t > nT$, we have

$$\Delta[CP_t] = \Delta[(CP_{t-nT})P_{nT}] \leqq \Delta[CP_{nT}] \to 0 \tag{21}$$

where $\Delta[S] = \sup\, \hat{f}_{,g\in S}\, \theta(f, g;\, L^+)$, denotes the projective diameter of a cone S. It follows that the c for P_T (in the sense of Theorem 3) is the (unique) c for P_t, which is also "uniformly positive," and with the same δ.

Finally, we can write $f = c + r$, where r is in the complementary invariant subspace of §7. Applying (17), with $M = m(f)$, we get

$$|\, fP_T^n - m(f)\gamma^n c\,| \leqq K\rho^n c, \qquad 0 \leqq \rho < \gamma.$$

Hence, for any t with $(n+1)T \leqq t < (n+2)T$, we have

$$\|fP_t - e^{\delta t} m(f)c\| = \|(fP_T^n - m(f)\gamma^n c)P_{(t-nT)}\| \leqq K\rho^n \|c\| \cdot \|P_{t-nT}\|.$$

The uniform boundedness of $\|P_{t-nT}\|$ follows, however, from (16d). There follows

$$\|fP_t - e^{\delta t} m(f)c\| \leqq K^* e^{\sigma t}, \qquad 0 \leqq \sigma < \delta, \tag{22}$$

where $K^* = (K/\rho^2)\|c\| \sup_{T \leqq t < 2T} \|P_t\|$, and $e^{\sigma T} = \rho$. This completes the proof.

Theorem 5 should be compared with the main result of Everett and Ulam(5). Our assumption of "uniform positivity" corresponds to *uniform* *mixing at a finite stage*. For various physical applications, it would be desirable to weaken this hypothesis.

REMARK. It is perhaps worth noting that all the preceding results apply to complete vector lattices(10). The essential step is the following extension of Corollary 2 of Lemma 2, §6.

LEMMA 4. *Let L be any complete vector lattice. Relative to $\theta(f, g; L^+)$, any connected component of L^+ is a complete metric space.*

Proof. From any convergent subsequence $\{g_k\}$, we can extract a hyperconvergent subsequence $\{f_n\} = \{g_{k(n)}\}$, such that $\theta(f_n, f_{n+1}; L^+) < 2^{-n}$. There follows, as in (13),

$$|f_{n+1} - f_n| \leqq ((R_n)^{1/2} - 1)|f_n| < 2^{-n}f_n, \qquad f_n = |f_n| \in L^+.$$

By the triangle inequality and induction, $|f_{n+h} - f_n| \leqq f_m/2^{n-1}$ for any $m \leqq n$; hence all f_n satisfy $0 < f_n < 4f_1$. The vector lattice being complete, $f_\infty = Vf_n$ therefore exists, and $|f_\infty - f_n| \leqq |f_n|/2^{n-2} \leqq f_1/2^{n-4}$, whence $f_n \to f_\infty$ in the sense of relative uniform convergence *and* in the intrinsic topology.

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(10) In the sense of G. Birkhoff, *Lattice theory*, Chap. XV, §3.