Extensions of Korpelevich's Extragradient Method for the Variational Inequality Problem in Euclidean Space

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Abstract

We present two extensions of Korpelevich's extragradient method for solving the Variational Inequality Problem (VIP) in Euclidean space. In the first extension we replace the second orthogonal projection onto the feasible set of the VIP in Korpelevich's extragradient method with a specific subgradient projection. The second extension allows projections onto the members of an infinite sequence of subsets which epi-converges to the feasible set of the VIP. We show that in both extensions the convergence of the method is preserved and present directions for further research.

Keywords: Epi-convergence, extragradient method, Lipschitz mapping, subgradient, variational inequality.

1 Introduction

In this paper we are concerned with the Variational Inequality Problem (VIP) of finding a point x^* such that

$$x^* \in S \text{ and } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in S, \tag{1.1}$$

where $f : R^n \to R^n$ is a given mapping, S is a nonempty, closed and convex subset of R^n and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. This problem, denoted by VIP(S, f), is a fundamental problem in Optimization Theory. Many algorithms for solving the VIP are projection algorithms that employ projections onto the feasible set S of the VIP, or onto some related set, in order to iteratively reach a solution.

Korpelevich [10] proposed an algorithm for solving the VIP, known as the Extragradient Method; see also Facchinei and Pang [3, Chapter 12]. In each iteration of her algorithm, in order to get the next iterate x^{k+1}, two orthogonal projections onto S are calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$y^k = P_S(x^k - \tau f(x^k)), \tag{1.2}$$

$$x^{k+1} = P_S(x^k - \tau f(y^k)), \tag{1.3}$$

where τ is some positive number and P_S denotes the Euclidean nearest point projection onto S. Although convergence was proved in [10] under the assumptions of Lipschitz continuity and pseudo-monotonicity, there is still the need to calculate two projections onto the closed convex set S.

We present two extensions of Korpelevich's extragradient method. In our first algorithmic extension we note that projection methods are particularly useful if the set S is simple enough so that projections onto it are easily executed. But if S is a general closed convex set, a minimum Euclidean distance problem has to be solved (twice in Korpelevich's extragradient method) in order to obtain the next iterate. This might seriously affect the efficiency of the method. Therefore, we replace the (second) projection (1.3) onto S by a projection onto a specific constructible half-space which is actually one of the subgradient half-spaces, as will be explained. We call this (Algorithm 2.1) the **subgradient extragradient** algorithm.

In our second algorithmic extension we develop a projection method for solving VIP(S, f), with projections related to approximations of the set S. This extension allows projections onto the members of an infinite sequence of subsets $\{S_k\}_{k=0}^{\infty}$ of S which epi-converges to the feasible set S of the VIP. We call this extension (Algorithm 4.3) the **perturbed extragradient** algorithm. The proof methods of both extensions are quite similar and we try to present both in a self-contained manner without repetitious texts. Our work is admittedly a theoretical development although its potential numerical advantages are obvious.

The paper is organized as follows. In Sections 2 and 4 the two algorithmic extensions are presented. They are analyzed in Sections 3 and 5, respectively. In Section 6 we present a hybrid of the two extensions (Algorithm 6.1) and a **two-subgradient extragradient** algorithm (Algorithm 6.2) about which we are able to prove only boundedness. Finally, we present a conjecture.

1.1 Relation with previous work

The literature on the VIP is vast and Korpelevich's extragradient method has received great attention by many authors who improved it in various ways; see, e.g., [7, 8, 14] and references therein, to name but a few. In general, projection algorithms that use metric projections onto the set S require that f be Lipschitz continuous, meaning that there exists an $L \geq 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|, \tag{1.4}$$

and strongly monotone, meaning that there exists an $\alpha \geq 0$ such that

$$\langle f(x) - f(y), x - y\rangle \geq \alpha\|x - y\|^2. \tag{1.5}$$

By adding the extra projection onto S, Korpelevich was able to replace the strong monotonicity assumption on f by a weaker assumption called pseudo-monotonicity, meaning that

$$\langle f(y), x - y\rangle \geq 0 \;\Rightarrow\; \langle f(x), x - y\rangle \geq 0. \tag{1.6}$$

The next development, proposed by Iusem and Svaiter [7], consists of removing the Lipschitz continuity assumption. This is important not only because f might fail to be Lipschitz continuous, but also because the constant L might be difficult to estimate, and even when L is known, $1/L$ (and consequently the step-size τ) might be very small, so that progress toward a solution becomes exceedingly slow. Solodov and Svaiter [14] presented an algorithm, which is an improvement of [7], so that their next iterate x^{k+1}

is closer to the solution set of VIP(S, f) than the next iterate computed by the method of [7]. They were able to drop the Lipschitz continuity by using an Armijo search in each iteration in order to construct some hyperplane to project onto. Though in [14] the assumptions required for convergence were weakened, there is still the need to compute the metric projection onto S at least once in each iteration and another projection onto an intersection of a hyperplane with S. In some other developments, Iiduka, Takahashi and Toyoda [5] introduced an iterative method for solving the VIP(S, f) in Hilbert space, but again they have to calculate the projection onto S twice. The main difference between their method and Korpelevich's method is that the second step (1.3) of Korpelevich's method is replaced by

$$x^{k+1} = P_S(\alpha_k x^k + (1 - \alpha_k)y^k), \tag{1.7}$$

for some sequence $\{\alpha_k\}_{k=0}^{\infty} \subseteq [-1, 1]$. Noor [11, 12] suggested and analyzed an extension of the extragradient method which still employs two orthogonal projections onto S, but (1.3) is replaced by

$$x^{k+1} = P_S(y^k - \tau f(y^k)). \tag{1.8}$$

So, Noor's and all other extensions of Korpelevich's method mentioned above, still require two projections onto S or that one projection is replaced by a projection onto a set which is the intersection of S with some hyperplane found through a line search.

2 The subgradient extragradient algorithmic extension

Our first algorithmic extension is a modification of the extragradient method, which we call the **subgradient extragradient** algorithm. The name derives from the replacement of the second projection onto S in (1.3) with a specific subgradient projection. Let the set S be given by

$$S = \{x \in R^n \mid c(x) \leq 0\}, \tag{2.1}$$

where $c : R^n \to R$ is a convex function. It is known that every closed convex set can be represented in this way, i.e., take $c(x) = \text{dist}(x, S)$, where dist is the distance function; see, e.g., [4, Subsection 1.3(c)]. We denote the subdifferential set of c at a point x by

$$\partial c(x) := \{\xi \in R^n \mid c(y) \geq c(x) + \langle \xi, y - x \rangle \text{ for all } y \in R^n\}. \tag{2.2}$$

For $z \in R^n$, take any $\xi \in \partial c(z)$ and define

$$T(z) := \{w \in R^n \mid c(z) + \langle \xi, w - z \rangle \leq 0\}. \tag{2.3}$$

This is a half-space the bounding hyperplane of which separates the set S from the point z. In the next algorithm we replace the second orthogonal projection onto S in (1.3) by a specific selection of a subgradient half-space.

Algorithm 2.1 ***The subgradient extragradient algorithm***

Step 0: *Select an arbitrary starting point* $x^0 \in R^n$ *and* $\tau > 0$, *and set* $k = 0$.

Step 1: *Given the current iterate* x^k, *compute*

$$y^k = P_S(x^k - \tau f(x^k)), \tag{2.4}$$

construct the half-space T_k *the bounding hyperplane of which supports* S *at* y^k,

$$T_k := \{w \in R^n \mid \langle (x^k - \tau f(x^k)) - y^k, w - y^k \rangle \leq 0\} \tag{2.5}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}(x^k - \tau f(y^k)). \tag{2.6}$$

Step 2: *If* $x^k = y^k$, *then stop. Otherwise, set* $k \leftarrow (k+1)$ *and return to* ***Step 1.***

Remark 2.2 *Observe that if* c *is not differentiable at* y^k, *then* $(x^k - \tau f(x^k)) - y^k \in \partial c(y^k)$; *otherwise* $\{(x^k - \tau f(x^k)) - y^k\} = \partial c(y^k) = \{\nabla c(y^k)\}$.

Figure 1 illustrates the iterative step of this algorithm.

We need to assume the following conditions in order to prove convergence of our algorithm.

Condition 2.3 *The solution set of (1.1), denoted by* $SOL(S, f)$, *is nonempty.*

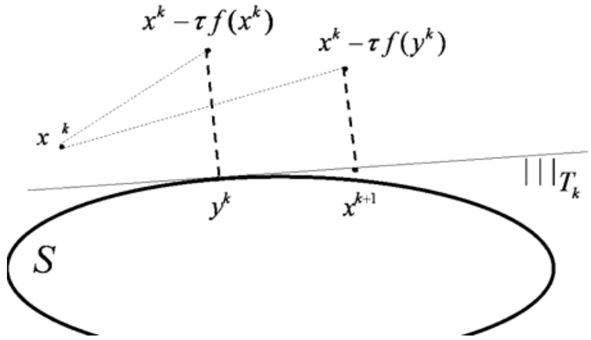

Figure 1: In the iterative step of Algorithm 2.1, x^{k+1} is a subgradient projection of the point $x^k - \tau f(y^k)$ onto the set S.

Condition 2.4 *The mapping f is pseudo-monotone on S with respect to $SOL(S, f)$.*

Substituting $y = x^*$ in (1.6), we get

$$\langle f(x), x - x^* \rangle \geq 0 \text{ for all } x \in S \text{ and for all } x^* \in SOL(S, f). \tag{2.7}$$

Condition 2.5 *The mapping f is Lipschitz continuous on R^n with constant $L > 0$.*

3 Convergence of the subgradient extragradient algorithm

In a recent paper [2] we have studied further extensions of Korpelevich's method including weak convergence of the subgradient extragradient Algorithm 2.1. The proof there is similar to the one given here only until a certain point. We give here a full proof for convenience, since steps in it are needed in later sections. First we show that the stopping criterion in **Step 2** of Algorithm 2.1 is valid.

Lemma 3.1 *If $x^k = y^k$ in Algorithm 2.1, then $x^k \in SOL(S, f)$.*

Proof. If $x^k = y^k$, then $x^k = P_S(x^k - \tau f(x^k))$, so $x^k \in S$. By the variational characterization of the projection with respect to S, we have

$$\left\langle w - x^k, (x^k - \tau f(x^k)) - x^k \right\rangle \leq 0 \text{ for all } w \in S, \tag{3.1}$$

which implies that

$$\tau \left\langle w - x^k, f(x^k) \right\rangle \geq 0 \text{ for all } w \in S. \tag{3.2}$$

Since $\tau > 0$, (3.2) implies that $x^k \in \mathrm{SOL}(S, f)$. ■

The next lemma is central to our proof of the convergence theorem.

Lemma 3.2 *Let $\{x^k\}_{k=0}^{\infty}$ and $\{y^k\}_{k=0}^{\infty}$ be the two sequences generated by Algorithm 2.1 and let $x^* \in SOL(S, f)$. Then, under Conditions 2.3–2.5, we have*

$$\left\| x^{k+1} - x^* \right\|^2 \leq \left\| x^k - x^* \right\|^2 - (1 - \tau^2 L^2) \left\| y^k - x^k \right\|^2 \text{ for all } k \geq 0. \tag{3.3}$$

Proof. Since $x^* \in \mathrm{SOL}(S, f)$, $y^k \in S$ and f is pseudo-monotone with respect to $\mathrm{SOL}(S, f)$,

$$\left\langle f(y^k), y^k - x^* \right\rangle \geq 0 \text{ for all } k \geq 0. \tag{3.4}$$

So,

$$\left\langle f(y^k), x^{k+1} - x^* \right\rangle \geq \left\langle f(y^k), x^{k+1} - y^k \right\rangle. \tag{3.5}$$

By the definition of T_k (2.5), we have

$$\left\langle x^{k+1} - y^k, \left(x^k - \tau f(x^k)\right) - y^k \right\rangle \leq 0 \text{ for all } k \geq 0. \tag{3.6}$$

Thus,

$$\begin{aligned} \left\langle x^{k+1} - y^k, (x^k - \tau f(y^k)) - y^k \right\rangle &= \left\langle x^{k+1} - y^k, x^k - \tau f(x^k) - y^k \right\rangle \\ &\quad + \tau \left\langle x^{k+1} - y^k, f(x^k) - f(y^k) \right\rangle \\ &= \tau \left\langle x^{k+1} - y^k, f(x^k) - f(y^k) \right\rangle. \end{aligned} \tag{3.7}$$

Denoting $z^k = x^k - \tau f(y^k)$, we obtain

$$\begin{aligned} \left\| x^{k+1} - x^* \right\|^2 &= \left\| P_{T_k}(z^k) - x^* \right\|^2 \\ &= \left\langle P_{T_k}(z^k) - z^k + z^k - x^*, P_{T_k}(z^k) - z^k + z^k - x^* \right\rangle \\ &= \left\| z^k - x^* \right\|^2 + \left\| z^k - P_{T_k}(z^k) \right\|^2 + 2 \left\langle P_{T_k}(z^k) - z^k, z^k - x^* \right\rangle. \end{aligned} \tag{3.8}$$

Since

$$2\left\|z^k - P_{T_k}(z^k)\right\|^2 + 2\left\langle P_{T_k}(z^k) - z^k, z^k - x^*\right\rangle$$
$$= 2\left\langle z^k - P_{T_k}(z^k), x^* - P_{T_k}(z^k)\right\rangle \le 0 \text{ for all } k \ge 0, \tag{3.9}$$

we get

$$\left\|z^k - P_{T_k}(z^k)\right\|^2 + 2\left\langle P_{T_k}(z^k) - z^k, z^k - x^*\right\rangle \le -\left\|z^k - P_{T_k}(z^k)\right\|^2 \text{ for all } k \ge 0. \tag{3.10}$$

Hence,

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 &\le \left\|z^k - x^*\right\|^2 - \left\|z^k - P_{T_k}(z^k)\right\|^2 \\ &= \left\|(x^k - \tau f(y^k)) - x^*\right\|^2 - \left\|(x^k - \tau f(y^k)) - x^{k+1}\right\|^2 \\ &= \left\|x^k - x^*\right\|^2 - \left\|x^k - x^{k+1}\right\|^2 + 2\tau\left\langle x^* - x^{k+1}, f(y^k)\right\rangle \\ &\le \left\|x^k - x^*\right\|^2 - \left\|x^k - x^{k+1}\right\|^2 + 2\tau\left\langle y^k - x^{k+1}, f(y^k)\right\rangle, \end{aligned} \tag{3.11}$$

where the last inequality follows from (3.5). So,

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 &\le \left\|x^k - x^*\right\|^2 - \left\|x^k - x^{k+1}\right\|^2 + 2\tau\left\langle y^k - x^{k+1}, f(y^k)\right\rangle \\ &= \left\|x^k - x^*\right\|^2 - \left(\left\langle x^k - y^k + y^k - x^{k+1}, x^k - y^k + y^k - x^{k+1}\right\rangle\right) \\ &+ 2\tau\left\langle y^k - x^{k+1}, f(y^k)\right\rangle \\ &= \left\|x^k - x^*\right\|^2 - \left\|x^k - y^k\right\|^2 - \left\|y^k - x^{k+1}\right\|^2 \\ &+ 2\left\langle x^{k+1} - y^k, x^k - \tau f(y^k) - y^k\right\rangle, \end{aligned} \tag{3.12}$$

and by (3.7)

$$\begin{aligned}\left\|x^{k+1} - x^*\right\|^2 \le \left\|x^k - x^*\right\|^2 - \left\|x^k - y^k\right\|^2 - \left\|y^k - x^{k+1}\right\|^2 \\ + 2\tau\left\langle x^{k+1} - y^k, f(x^k) - f(y^k)\right\rangle. \end{aligned} \tag{3.13}$$

Using the Cauchy–Schwarz inequality and Condition 2.5, we obtain

$$2\tau\left\langle x^{k+1} - y^k, f(x^k) - f(y^k)\right\rangle \le 2\tau L\left\|x^{k+1} - y^k\right\|\left\|x^k - y^k\right\|. \tag{3.14}$$

In addition,

$$\begin{aligned}0 &\le \left(\tau L\left\|x^k - y^k\right\| - \left\|y^k - x^{k+1}\right\|\right)^2 \\ &= \tau^2L^2\left\|x^k - y^k\right\|^2 - 2\tau L\left\|x^{k+1} - y^k\right\|\left\|x^k - y^k\right\| + \left\|y^k - x^{k+1}\right\|^2, \end{aligned} \tag{3.15}$$

so,

$$2\tau L \left\| x^{k+1} - y^k \right\| \left\| x^k - y^k \right\| \leq \tau^2 L^2 \left\| x^k - y^k \right\|^2 + \left\| y^k - x^{k+1} \right\|^2. \tag{3.16}$$

Combining the above inequalities and using Condition 2.5, we see that

$$\begin{aligned} \left\| x^{k+1} - x^* \right\|^2 &\leq \left\| x^k - x^* \right\|^2 - \left\| x^k - y^k \right\|^2 - \left\| y^k - x^{k+1} \right\|^2 \\ &+ 2\tau L \left\| x^{k+1} - y^k \right\| \left\| x^k - y^k \right\| \\ &\leq \left\| x^k - x^* \right\|^2 - \left\| x^k - y^k \right\|^2 - \left\| y^k - x^{k+1} \right\|^2 \\ &+ \tau^2 L^2 \left\| x^k - y^k \right\|^2 + \left\| y^k - x^{k+1} \right\|^2 \\ &= \left\| x^k - x^* \right\|^2 - \left\| x^k - y^k \right\|^2 + \tau^2 L^2 \left\| x^k - y^k \right\|^2. \end{aligned} \tag{3.17}$$

Finally, we get

$$\left\| x^{k+1} - x^* \right\|^2 \leq \left\| x^k - x^* \right\|^2 - (1 - \tau^2 L^2) \left\| y^k - x^k \right\|^2, \tag{3.18}$$

which completes the proof. ∎

Theorem 3.3 *Assume that Conditions 2.3–2.5 hold and let* $0 < \tau < 1/L$. *Then any sequence* $\{x^k\}_{k=0}^{\infty}$ *generated by Algorithm 2.1 converges to a solution of (1.1).*

Proof. Let $x^* \in \mathrm{SOL}(S, f)$ and define $\rho := 1 - \tau^2 L^2$. Since $0 < \tau < 1/L$ we have $\rho \in (0, 1)$. By Lemma 3.2, the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded. Therefore, it has at least one accumulation point $\bar{x}$. From Lemma 3.2 it follows that

$$\rho \left\| y^k - x^k \right\|^2 \leq \left\| x^k - x^* \right\|^2 - \left\| x^{k+1} - x^* \right\|^2. \tag{3.19}$$

Summing up, we get for all integer $K \geq 0$,

$$\rho \sum_{k=0}^{K} \left\| y^k - x^k \right\|^2 \leq \left\| x^0 - x^* \right\|^2. \tag{3.20}$$

Since the sequence $\left\{ \sum_{k=0}^{K} \left\| y^k - x^k \right\|^2 \right\}_{K \geq 0}$ is monotonically increasing and bounded,

$$\rho \sum_{k=0}^{\infty} \left\| y^k - x^k \right\|^2 \leq \left\| x^0 - x^* \right\|^2, \tag{3.21}$$

which implies that

$$\lim_{k\to\infty} \left\| y^k - x^k \right\| = 0. \tag{3.22}$$

So, if $\bar{x}$ is the limit point of some subsequence $\{x^{k_j}\}_{j=0}^{\infty}$ of $\{x^k\}_{k=0}^{\infty}$, then

$$\lim_{j\to\infty} y^{k_j} = \bar{x}. \tag{3.23}$$

By the continuity of f and P_S, we have

$$\bar{x} = \lim_{j\to\infty} y^{k_j} = \lim_{j\to\infty} P_S(x^{k_j} - \tau f(x^{k_j})) = P_S(\bar{x} - \tau f(\bar{x})). \tag{3.24}$$

As in the proof of Lemma 3.1, it follows that $\bar{x} \in \mathrm{SOL}(S, f)$. We now apply Lemma 3.2 with $x^* = \bar{x}$ to deduce that the sequence $\{\left\| x^k - \bar{x} \right\|\}_{k=0}^{\infty}$ is monotonically decreasing and bounded, hence convergent. Since

$$\lim_{k\to\infty} \left\| x^k - \bar{x} \right\| = \lim_{j\to\infty} \left\| x^{k_j} - \bar{x} \right\| = 0, \tag{3.25}$$

the whole sequence $\{x^k\}_{k=0}^{\infty}$ converges to $\bar{x}$. ∎

Remark 3.4 *In the convergence theorem we assume that f is Lipschitz continuous on R^n with constant $L > 0$ (Condition 2.5). If we assume that f is Lipschitz continuous only on S with constant $L > 0$, we can use a Lipschitzian extension of f to R^n in order to evaluate the mapping at x^k. This extension exists by Kirszbraun's theorem [9], which states that there exists a Lipschitz continuous map on R^n, $\tilde{f} : R^n \to R^n$, that extends f and has the same Lipschitz constant L as f. Alternatively, we can take $\widetilde{f} = fP_S$ which is Lipschitz continuous on R^n with constant $L > 0$.*

4 The perturbed extragradient algorithmic extension

Our next algorithmic extension is a modification of the extragradient method, which we call the **perturbed extragradient** algorithm. Following [13], we denote by $\mathrm{NCCS}(R^n)$ the family of all nonempty, closed and convex subsets of R^n.

Definition 4.1 *[1, Proposition 3.21] Let* S *and* $\{S_k\}_{k=0}^{\infty}$ *be a set and a sequence of sets in* $NCCS(R^n)$*, respectively. The sequence* $\{S_k\}_{k=0}^{\infty}$ *is said to* `epi-converge to the set` S *(denoted by* $S_k \overset{epi}{\to} S$*) if the following two conditions hold:*

(i) for every $x \in S$*, there exists a sequence* $\{x^k\}_{k=0}^{\infty}$ *such that* $x^k \in S_k$ *for all* $k \geq 0$*, and* $\lim_{k\to\infty} x^k = x$*.*

(ii) If $x^{k_j} \in S_{k_j}$ *for all* $j \geq 0$*, and* $\lim_{j\to\infty} x^{k_j} = x$*, then* $x \in S$*.*

The next proposition is [13, Proposition 7], but its Banach space variant already appears in [6, Proposition 7].

Proposition 4.2 *Let* S *and* $\{S_k\}_{k=0}^{\infty}$ *be a set and a sequence of sets in* $NCCS(R^n)$*, respectively. If* $S_k \overset{epi}{\to} S$ *and* $\lim_{k\to\infty} x^k = x$*, then*

$$\lim_{k\to\infty} P_{S_k}(x^k) = P_S(x). \tag{4.1}$$

We now formulate the perturbed extragradient algorithm.

Algorithm 4.3 ***The perturbed extragradient algorithm***

Step 0*: Let* $\{S_k\}_{k=0}^{\infty}$ *be a sequence of sets in* $NCCS(R^n)$ *such that* $S_k \overset{epi}{\to} S$*. Select a starting point* $x^1 \in S_0$ *and* $\tau > 0$*, and set* $k = 1$*.*

Step 1*: Given the current iterate* $x^k \in S_{k-1}$*, compute*

$$y^k = P_{S_k}(x^k - \tau f(x^k)) \tag{4.2}$$

and

$$x^{k+1} = P_{S_k}(x^k - \tau f(y^k)). \tag{4.3}$$

Step 2*: Set* $k \leftarrow (k+1)$ *and return to* ***Step 1.***

Figure 2 illustrates the iterative step of this algorithm.

We will need the following additional assumption for the convergence theorem.

Condition 4.4 f *is* `Lipschitz continuous` *on* S *with constant* $L > 0$*.*

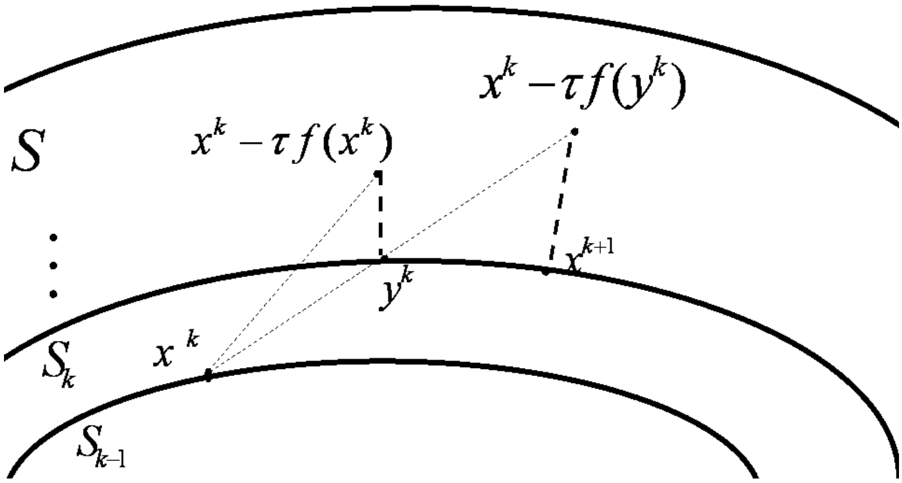

Figure 2: In the iterative step of Algorithm 4.3, x^{k+1} is obtained by performing the projections of the original Korpelevich method with respect to the set S_k.

5 Convergence of the perturbed extragradient algorithm

First we observe that Lemma 3.2 holds for Algorithm 4.3 under Conditions 2.3–2.5. The following lemma uses, instead of Condition 2.5, Condition 4.4, which requires Lipschitz continuity on S and not on the whole space R^n. This entails the main difference between the proofs of Lemmata 3.2 and 5.1, which is that (3.6) becomes an inequality and this propagates down the rest of the proof. We give, however, the next proof in full for the convenience of the readers.

Lemma 5.1 *Assume that* $S_k \subseteq S_{k+1} \subseteq S$ *for all* $k \geq 0$*, that* $S_k \overset{epi}{\rightarrow} S$*, that Conditions 2.3–2.4 and Condition 4.4 hold. Let* $\{x^k\}_{k=0}^{\infty}$ *and* $\{y^k\}_{k=0}^{\infty}$ *be two* *sequences generated by Algorithm 4.3. Let $x^* \in SOL(S,f)$. Then*

$$\left\|x^{k+1}-x^*\right\|^2 \leq \left\|x^k-x^*\right\|^2-(1-\tau^2L^2)\left\|y^k-x^k\right\|^2 \text{ for all } k\geq 0. \quad (5.1)$$

Proof. Since $x^* \in \mathrm{SOL}(S,f)$, $y^k \in S_k \subseteq S$ and f is pesudo-monotone with respect to $\mathrm{SOL}(S,f)$,

$$\left\langle f(y^k), y^k-x^*\right\rangle \geq 0 \text{ for all } k\geq 0. \quad (5.2)$$

So,

$$\left\langle f(y^k), x^{k+1}-x^*\right\rangle \geq \left\langle f(y^k), x^{k+1}-y^k\right\rangle. \quad (5.3)$$

By the variational characterization of the projection with respect to S_k, we have

$$\left\langle x^{k+1}-y^k, \left(x^k-\tau f(x^k)\right)-y^k\right\rangle \leq 0. \quad (5.4)$$

Thus,

$$\begin{aligned}\left\langle x^{k+1}-y^k, (x^k-\tau f(y^k))-y^k\right\rangle &= \left\langle x^{k+1}-y^k, x^k-\tau f(x^k)-y^k\right\rangle \\ &\quad + \tau\left\langle x^{k+1}-y^k, f(x^k)-f(y^k)\right\rangle \\ &\leq \tau\left\langle x^{k+1}-y^k, f(x^k)-f(y^k)\right\rangle. \quad (5.5)\end{aligned}$$

Denoting $z^k = x^k-\tau f(y^k)$, we obtain exactly equations (3.8)–(3.12) with P_{T_k} replaced by P_{S_k}. By (5.5) we obtain (3.13) for the present lemma too. Using the Cauchy–Schwarz inequality and Condition 4.4, we can repeat the remainder of the proof as in the proof of Lemma 3.2 and obtain finally

$$\left\|x^{k+1}-x^*\right\|^2 \leq \left\|x^k-x^*\right\|^2-(1-\tau^2L^2)\left\|y^k-x^k\right\|^2, \quad (5.6)$$

which completes the proof. ■

Next, we present our convergence theorem for the perturbed extragradient algorithm.

Theorem 5.2 *Assume that $S_k \subseteq S_{k+1} \subseteq S$ for all $k\geq 0$, that $S_k \overset{epi}{\to} S$, and that Conditions 2.3–2.4 and Condition 4.4 hold. Let $0<\tau<1/L$. Then any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 4.3, converges to a solution of (1.1).*

Proof. Let $x^* \in \mathrm{SOL}(S,f)$ and define $\rho := 1 - \tau^2 L^2$. Using Lemma 5.1 instead of Lemma 3.2, we can obtain, by similar arguments, that here also

$$\lim_{k\to\infty} \left\| y^k - x^k \right\| = 0.$$

So, if $\bar{x}$ is the limit point of some subsequence $\{x^{k_j}\}_{j=0}^{\infty}$ of $\{x^k\}_{k=0}^{\infty}$, then

$$\lim_{j\to\infty} y^{k_j} = \bar{x}. \tag{5.7}$$

Using the continuity of f and P_{S_k}, and Proposition 4.2, we have

$$\bar{x} = \lim_{j\to\infty} y^{k_j} = \lim_{j\to\infty} P_{S_{k_j}}(x^{k_j} - \tau f(x^{k_j})) = P_S(\bar{x} - \tau f(\bar{x})). \tag{5.8}$$

As in the proof of Lemma 3.1, it follows that $\bar{x} \in \mathrm{SOL}(S,f)$. We now apply Lemma 5.1 with $x^* = \bar{x}$ to deduce that the sequence $\{\left\| x^k - \bar{x} \right\|\}_{k=0}^{\infty}$ is monotonically decreasing and bounded, hence convergent. Since

$$\lim_{k\to\infty} \left\| x^k - \bar{x} \right\| = \lim_{j\to\infty} \left\| x^{k_j} - \bar{x} \right\| = 0, \tag{5.9}$$

the whole sequence $\{x^k\}_{k=0}^{\infty}$ converges to $\bar{x}$. ■

6 Further algorithmic possibilities

As a matter of fact, Algorithm 4.3 can be naturally modified by combining the two algorithmic extensions studied above into a **hybrid perturbed subgradient extragradient** algorithm, namely, to allow the second projection in Algorithm 4.3 to be replaced by a specific subgradient projection with respect to S_k.

Algorithm 6.1 ***The hybrid perturbed subgradient extragradient algorithm***

Step 0: *Select an arbitrary starting point* $x^1 \in S_0$ *and* $\tau > 0$*, and set* $k = 1$.

Step 1: *Given the current iterate* x^k*, compute*

$$y^k = P_{S_k}(x^k - \tau f(x^k)) \tag{6.1}$$

construct the half-space T_k *the bounding hyperplane of which supports* S_k *at* y^k,

$$T_k := \{w \in R^n \mid \left\langle \left(x^k - \tau f(x^k)\right) - y^k, w - y^k \right\rangle \leq 0\} \tag{6.2}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}(x^k - \tau f(y^k)). \tag{6.3}$$

Step 2: *Set* $k \leftarrow (k+1)$ *and return to* ***Step 1.***

Figure 3 illustrates the iterative step of this algorithm.

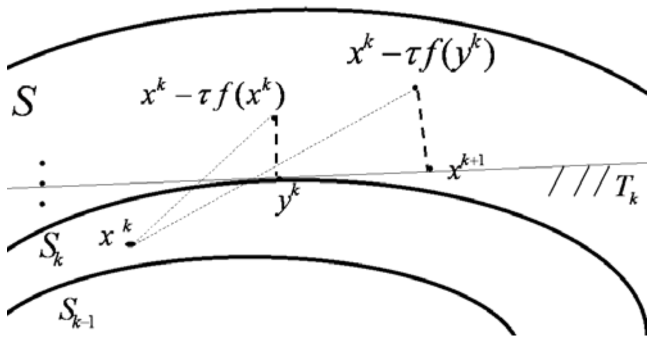

Figure 3: In the iterative step of Algorithm 6.1, x^{k+1} is obtained by performing one subgradient projection and one projection onto the set S_k in each iterative step.

We proved the convergence of this algorithm by using similar arguments to those we employed in the previous proofs. Therefore we omit the proof of its convergence.

Another possibility is the following one. In Algorithm 2.1 we replaced the second projection onto S with a specific subgradient projection. It is natural to ask whether it is possible to replace the first projection onto S as well and, furthermore, if this could be done for any choice of a subgradient half-space. To accomplish this, one might consider the following algorithm.

Algorithm 6.2 ***The two-subgradient extragradient algorithm***

Step 0: *Select an arbitrary starting point* $x^0 \in R^n$ *and set* $k = 0$.

***Step 1**: Given the current iterate x^k, choose $\xi^k \in \partial c(x^k)$, consider $T_k := T\left(x^k\right)$ as in (2.3), and then compute*

$$y^k = P_{T_k}(x^k - \tau f(x^k)) \tag{6.4}$$

and

$$x^{k+1} = P_{T_k}(x^k - \tau f(y^k)). \tag{6.5}$$

Step 2**: If $x^k = y^k$, then stop. Otherwise, set $k \leftarrow (k+1)$ and return to **Step 1.

Figure 4 illustrates the iterative step of this algorithm.

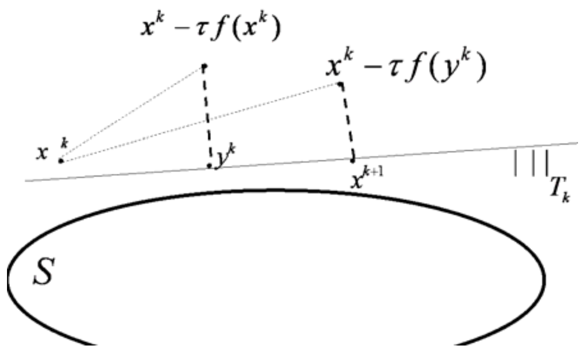

Figure 4: In the iterative step of Algorithm 6.2, x^{k+1} is obtained by performing two subgradient projections in each iterative step.

We now observe that under Conditions 2.3–2.5, where Condition 2.4 is on R^n, that Lemma 3.1 and 3.2 still hold, that is, the generated sequence $\{x^k\}_{k=0}^{\infty}$ is bounded and $\lim_{k\to\infty} \left\|y^k - x^k\right\| = 0$. It is still an open question whether these sequences converge to $x^* \in \mathrm{SOL}(S, f)$. First we show that **Step 2** of Algorithm 6.2 is valid.

Lemma 6.3 *If $x^k = y^k$ for some k in Algorithm 6.2, then $x^k \in SOL(S, f)$.*

Proof. If $x^k = y^k$, then $x^k = P_{T_k}(x^k - \tau f(x^k))$, so $x^k \in T_k$. Therefore by the definition of T_k (see (2.3)), we have $c(x^k) + \langle \xi^k, x^k - x^k \rangle \leq 0$, so

$c(x^k) \leq 0$ and by the representation of the set S, $x^k \in S$. By the variational characterization of the projection with respect to T_k, we have

$$\left\langle w - y^k, (x^k - \tau f(x^k)) - y^k \right\rangle \leq 0 \text{ for all } w \in T_k \tag{6.6}$$

and

$$\tau \left\langle w - x^k, f(x^k) \right\rangle \geq 0 \text{ for all } w \in T_k. \tag{6.7}$$

Now we claim that $S \subseteq T_k$. Let $x \in S$, and consider $\xi^k \in \partial c(x^k)$. By the definition of the subdifferential set of c at a point x^k (see (2.2)), we get for all $y \in R^n$, $c(y) \geq c(x^k) + \langle \xi^k, y - x^k \rangle$, so, in particular, for $x \in S \subseteq R^n$,

$$c(x) \geq c(x^k) + \langle \xi^k, x - x^k \rangle. \tag{6.8}$$

By the representation of the set S (see (2.1)), we obtain

$$0 \geq c(x) \geq c(x^k) + \langle \xi^k, x - x^k \rangle \tag{6.9}$$

which means that $x \in T_k$ and so $S \subseteq T_k$, as claimed. Since $S \subseteq T_k$, we have by (6.7),

$$\tau \left\langle w - x^k, f(x^k) \right\rangle \geq 0 \text{ for all } w \in S. \tag{6.10}$$

Since $\tau > 0$ and $x^k \in S$, we finally get $x^k \in SOL(S, f)$. ■

The proof of the next lemma is similar to that of Lemma 5.1 above.

Lemma 6.4 *Let $\{x^k\}_{k=0}^\infty$ and $\{y^k\}_{k=0}^\infty$ be two sequences generated by Algorithm 6.2. Let $x^* \in SOL(S, f)$. Then under Conditions 2.3, 2.4 and 4.4, we have for every $k \geq 0$,*

$$\left\| x^{k+1} - x^* \right\|^2 \leq \left\| x^k - x^* \right\|^2 - (1 - \tau^2 L^2) \left\| y^k - x^k \right\|^2. \tag{6.11}$$

It is not difficult to show, by following the arguments given in Theorem 3.3, that under the conditions of this lemma and if $0 < \tau < 1/L$, then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 6.2 is bounded and

$$\lim_{k \to \infty} \left\| y^k - x^k \right\| = 0. \tag{6.12}$$

The second algorithmic extension of Korpelevich's extragradient method proposed here can be further studied along the lines of the following conjecture.

Conjecture 6.5 *The set inclusion condition that* $S_k \subseteq S_{k+1} \subseteq S$ *for all* $k \geq 0$, *which appears in our analysis of the perturbed extragradient algorithm, could probably be removed by employing techniques similar to those of [15], i.e., using the definition of the* γ**-distance between** S_1 **and** S_2, *where* $\gamma \geq 0$, S_1, $S_2 \in NCCS(R^n)$,

$$d_\gamma(S_1, S_2) := \sup\{\|P_{S_1}(x) - P_{S_2}(x)\| \mid \|x\| \leq \gamma\} \tag{6.13}$$

and the equivalence between the conditions (a) $\lim_{k\to\infty} d_\gamma(S_k, S) = 0$ *for all* $\gamma \geq 0$ *and (b)* $S_k \overset{epi}{\to} S$. *See [13, 15].*

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