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Extensions of Measures with Values in a Topological Group with Applications to Vector Measures

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The extension problem for countably additive scalar measures has its roots in integration theory. To apply the Lebesgue construction it was necessary to extend scalar set functions, usually defined explicitly only on a ring, to the σ-algebra of measureable sets. The existence of such an extension for countable additive set functions is insured by the Caratheodory outer measure construction. It was not until much later that the integral was defined directly from a countable additive set function on a ring (see Bogdanowicz [4] and [5]). As a result of the Bogdanowicz construction, the original Caratheodory extension theorem became a corollary to rather than an essential part of the definition of the integral.

The extension problem for vector measures has had a more difficult development. The key to the solution was discovered in the condition of strong boundedness introduced by Rickart [16]. It is somewhat curious that Rickart introduced strong boundedness in the context of decomposition of set functions rather than in relation to extension theory. Brooks [7] and Oberle [14] established the equivalence of the condition of strong boundedness with the existence of a Bartle-Dunford-Schwartz control measure (see [2]) which could be extended to the generated σ-ring by the classical Caratheodory construction. The original vector measure if countably additive is then extendable to the σ-ring via uniform continuity. The result generated an extensive study of the theory of topological rings of sets, see Drewnowski [8], Labuda [13], Oberle [14], and Bogdan and Oberle [6]. The most inclusive statement of the extension theorem for vector measures has been given by Kluvanek [12]. Kluvanek has shown that for a weakly countably additive vector measure defined on a ring, the condition of strong boundedness is equivalent to the existence of a countably additive extension to the generated σ-ring.

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The Kluvanek theorem is comprehensive in that it establishes numerous equivalent conditions to strong boundedness—thereby pulling together the numerous specialized extension theorems that are scattered throughout the literature. The reader need only consult the bibliography of the Kluvanek paper to gain an appreciation of the scope of the interest in the vector measure extension problem.

Even though the Kluvanek extension is quite powerful, it fails to include as a special case the extension of the Lebesgue measure from the ring generated by half open intervals to the delta ring of Lebesgue summable sets. A special case of such an extension theorem for vector measures was accomplished by Gould [11] via a construction analogous to the Caratheodory outer measure procedure. Gould's extension was accomplished only for vector measure taking values in a certain class of range spaces (which turned out to be those Banach spaces which do not contain a copy of the space of the null convergent sequences of scalars). Since the domain of the Gould extension is a delta ring with convergence conditions analogous to the Lebesgue summable sets, the Gould result extends the classical Lebesgue extension. Although the Gould extension is accomplished via an assumed property of the range space, the condition equivalent to the existence of the extension is easily recognized as a variant of the Rickart strong boundedness condition.

The next step in the study of the extension problem is found in the theory of group valued measures. Sion [17] and Fox and Rogers [10] have used the strong boundedness condition to establish variations of the Kluvanek theorem for group valued measures. The Sion construction is a variation of the Caratheodory construction. However, when the range space is a Banach space, the conditions assumed by Sion restrict the class of extendable measures to those whose extension has finite semivariation on each measurable set. Although a large class of vector measures, in particular the Lebesgue measure on the real line do not satisfy the Sion criteria, the basic construction may be modified to yield a general extension theorem.

The purpose of this paper is to show that certain countably additive group valued measures may be extended from a ring to a delta ring possessing monotone convergence properties similar to the delta ring of Lebesgue summable sets. As in the scalar case, the domain of the extension is generally larger than the generated delta ring and generally smaller than the generated σ-ring. As expected, the condition required to accomplish the extension is a variant of the condition of strong boundedness. It will be seen that not only does the extension theorem to be established generalize the classical Lebesgue extension but also includes the Kluvanek and Gould extension theorems as well as the Sion extension for group valued measures. Because the development is quite lengthy and technical in nature, specific application to the theory of vector measures will be limited to illustrations. In particular, the analogues of the various equivalent formulations of strong boundedness given by Kluvanek (some of which no longer remain true) will not be discussed. The reader is referred to [1], [2], [6], [7], [8], [12], [13], [15], and [18] to aid any indepth study of the extension problem for vector measures.

Let V be a ring of subsets of an abstract space X and let V_σ denote the class of all countable unions of sets from the ring V. Let E be a commutative, complete topological group and denote by $a(V, E)$, respectively $ca(V, E)$, the class of finitely additive, respectively countably additive, functions on the ring V into the group E. For any family of subsets W of the space X, we say that a sequence $A_n \in V$, $n=1, 2, 3, \cdots$, is W-dominated if there exists a set $B \in W$ such that $A_n \subset B$, for all $n=1, 2, 3, \cdots$. A function $\mu \in a(V, E)$ is said to be Rickart on the ring V relative to the family W if $\lim_n \mu(A_n)=\theta$ for each disjoint, W-dominated sequence $A_n \in V$, $n \in N$.

Section 1 of this paper contains some equivalent formulations of the Rickart condition and results analogous to those established by Rickart, [16], who introduced a similar class of Banach space valued, finitely additive functions. This section also contains modified statements of the Sion, [17], extension theorem with an outline of the extension procedure and its relation to other recent extension theorems appearing in the literature.

Section 2 contains a discussion of topological rings of sets generated by group valued charges. The development given parallels the constructions of Bogdan (published in [6]) who developed the theory of topological rings of sets for families subadditive, real valued functions in connection with the Vitali-Hahn-Saks theorem.

In section 3, the extension problem is discussed in the context of topological rings of sets as developed in section 2. It is shown that a group valued measure on a ring admits an extension to a measure on a delta ring which is sequentially complete in the uniform structure generated by the extension.

Throughout this paper, let V denote a ring of subsets of an abstract space X and let $W \subset V_\sigma$ (the family of sets expressible as a countable union of sets from the ring V) be directed upward by set inclusion. Let E be a commutative, complete topological group and let $\mathscr{G}$ denote a base of closed, symmetric neighborhoods of the origin in the group E. For any set function $\mu: D(\mu) \to E$, with domain $D(\mu)$ consisting of subsets of the space X, and any closed neighborhood b of the origin in E, we set

$$N(\mu, b) = \{A \in P(X): \mu(B) \in b \quad \text{for all} \quad B \in D(\mu) \text{ with } B \subset A\}$$

and

$$\hat{N}(\mu, b) = \{(A, B): A, B \in D(\mu) \quad \text{and} \quad A \div B \in N(\mu, b)\}$$

where $\div$ denotes the symmetric difference operation on the σ-algebra $P(X)$ of all subsets of the space X. The natural numbers will be denoted by N and the (non-negative) reals will be denoted $(R^+)R$.

§ 1. Properties of Rickart measures.

In this section, we develop results for the space of finitely additive, E-valued functions analogous to those given by Rickart [16] and Bogdan and Oberle [6] for Banach space valued functions.

LEMMA 1.1. *Let $\mu \in a(V, E)$ be Rickart on the ring V relative to the class W. Then for each W-dominated, disjoint sequence $A_n \in V$, $n \in N$, and each neighborhood $g \in \mathscr{G}$, there exists an index $n(g) \in N$ such that $A_n \in N(\mu, g)$ for all $n \geqq n(g)$.*

PROOF. Assume the contrary. Then there exists a subsequence $k_n \in N$, for $n = 1, 2, \cdots$ and a sequence $B_n \in V$, $B_n \subset A_{k_n}$ for $n = 1, 2, \cdots$ such that $\mu(B_n) \notin g$ for all $n = 1, 2, 3, \cdots$. Since the sequence $B_n \in V$, $n = 1, 2, 3, \cdots$ is disjoint and W-dominated, and the charge $\mu \in a(V, E)$ is Rickart, we have a contradiction.

LEMMA 1.2. *Let $\mu \in a(V, E)$ be Rickart on the ring V relative to the class W. Then for each disjoint W-dominated sequence $A_n \in V$, $n \in N$, and each neighborhood $g \in \mathscr{G}$, there exists a finite set $\Delta(g) \subset N$ such that for every finite set $\Delta \subset N$ with $\Delta \cap \Delta(g) = \phi$, we have $\mu(B) \in g$ uniformly with respect to the sets $B \in V$, $B \subset \bigcup_{k \in \Delta} A_k$.*

PROOF. If we assume the contrary, then for each finite set $\Delta \subset N$, there exists a finite set $\Delta' \subset N$ and a set $B \in V$, with $B \subseteqq \bigcup_{k \in \Delta'} A_k$, $\Delta \cap \Delta' = \phi$ and $\mu(B) \notin g$. Starting with any finite set $\Delta_0 \subset N$, it is possible to choose a sequence $\Delta_n \subset N$, $n = 0, 1, 2, 3, \cdots$ of finite sets and a W-dominated sequence $B_n \in V$, $n = 1, 2, 3, \cdots$ with the properties:

$$\left.\begin{array}{l} 1.\quad (\Delta_0 \cup \cdots \cup \Delta_{n-1}) \cap \Delta_n = \phi \\ 2.\quad B_n \subset \bigcup_{k \in \Delta_n} A_k \quad \text{and} \quad \mu(B_n) \notin g \end{array}\right\} \quad \text{for} \quad n = 1, 2, 3, \cdots.$$

Condition 1 insures that the sequence $B_n \in V$, $n \in N$ is disjoint so that condition 2 is a contradiction to the Rickart condition for the charge $\mu \in a(V, E)$.

The following characterization of the Rickart condition has been established by Bogdan and Oberle [6], for Banach space valued charges.

PROPOSITION 1.1. *Let V be a ring of subsets of a space X and let E be a commutative complete topological group. For any charge $\mu \in a(V, E)$ and any class W of subsets of the space X, the following are equivalent:*

1. *The charge $\mu \in a(V, E)$ is Rickart on the ring V relative to the class W.*
2. *For each disjoint, W-dominated sequence $A_n \in V$, $n \in N$ the series $\sum_n \mu(A_n)$ converges unconditionally in the group E, (that is, the net $\{\sum_{k \in \Delta} \mu(A_k): \Delta \in F(N)\}$ converges in the group E).*
3. *For each monotone, W-dominated sequence*

$$A_n \in V,\ n \in N,\ \lim_{n,m} \mu(A_n \div A_m) = \theta\ ,$$

where $\div$ denotes the symmetric difference operation in the ring V and θ denotes the null element of the group E.

PROOF. Assume condition 1 and let $A_n \in V$, $n=1, 2, 3, \cdots$ be a W-dominated, disjoint sequence. Then for each neighborhood $g \in \mathscr{G}$, Lemma 1.2 insures that there exists a finite set $\Delta(g) \subseteqq N$ with the property that each finite set $\Delta \subset N$, with $\Delta \cap \Delta(g) = \phi$ yields $\mu(\bigcup_{k \in \Delta} A_k) \in g$. The finite additivity of the charge $\mu \in a(V, E)$ and the above observation insure that the net $\{\sum_{k \in \Delta} \mu(A_k): \Delta \subset N, \Delta\text{-finite}\}$ converges in the group E. Thus, the series $\sum_n \mu(A_n)$ converges unconditionally in the group E.

Assume that condition 2 holds and let $A_n \in V$, $n=1, 2, 3, \cdots$ be a W-dominated monotone sequence and set $B_n = A_{n+1} \div A_n$, $n=1, 2, 3, \cdots$. The sequence $B_n \in V$, $n \in N$ is disjoint and W-dominated. For any neighborhood $g \in \mathscr{G}$, condition 2 insures that there exists a finite set $\Delta(g) \subset N$ such that

$$\mu\Big(\bigcup_{k \in \Delta \setminus \Delta'} B_k\Big) \in g$$

for each pair of finite sets $\Delta, \Delta' \subset N$ with $\Delta(g) \subset \Delta' \subset \Delta$. Let $n(g) \in N$ be chosen so that $n(g)\geqq\max(t;\ t\in\Delta(g))$. Then for indices m, nN with m, $n\geqq n(g)$ and $m\geqq n$, we have

$$\mu\Big(\bigcup_{k=n}^{m-1}(A_{k+1}\div A_k)\Big)\in g\ .$$

Since the sequence A_n, $n\in N$ is monotone, $\bigcup_{k=n}^{m-1} A_{k+1}\div A_k=A_m\div A_n$ so that $\mu(A_m\div A_n)\in g$ for indices m, $n\in N$ with m, $n\geqq n(g)$.

Assume that condition 3 holds and let $A_n\in V$, $n\in N$ be a disjoint, W-dominated sequence. Let $g\in\mathscr{G}$ and set $B_n=\bigcup_{k=1}^{n} A_k$, for $n=1, 2, 3, \cdots$. The sequence $B_n\in V$, $n\in N$ is W-dominated and monotone increasing. Applying condition 3, there exists an index $n(g)\in N$ such that m, $n\in N$, m, $n\geqq n(g)$ yields $\mu(B_m\div B_n)\in g$. In particular, for any index $n\in N$, $n>n(g)$, $\mu(A_n)=\mu(B_n\div B_{n-1})\in g$. Thus, the charge $\mu\in a(V, E)$ is Rickart on the ring V relative to the class W. The next two lemmas have important applications in section 3.

LEMMA 1.3. *Let $\mu\in ca(V, E)$ be Rickart on the ring V relative to the class W and let $A_n\in V$, $n\in N$ be disjoint and W-dominated. Then for each neighborhood $g\in\mathscr{G}$, there exists an index $n(g)\in N$ such that $n\geqq n(g)$ yields*

$$\mu(B)\in g$$

uniformly with respect to the sets $B\in V$, $B\subset A\backslash\bigcup_{k=1}^{n} A_k$, where $A=\bigcup_n A_n$.

PROOF. Let $g\in\mathscr{G}$ be arbitrary and choose a sequence $g_n\in\mathscr{G}$, $n\in N$ such that

$$\sum_{k=1}^{n} g_k\subset g\ ,$$

for all $n\in N$. Using the property of W-boundedness given in Lemma 1.2, there exists an increasing sequence k_n, $n=1, 2, 3, \cdots$ of non-negative integers such that

$$\mu(B)\in g_n$$

for each set $B\in V$, $B\subset\bigcup_{t=k_n+1}^{k_{n+1}} A_t$.

We have

$$A\Big\backslash\bigcup_{t=1}^{k_n} A_t=\bigcup_{t=k_n+1}^{\infty} A_t=\bigcup_{t=n}^{\infty}\ \bigcup_{s=k_t+1}^{k_{t+1}} A_s\ .$$

For any set $B\in V$ with $B\subset A\backslash\bigcup_{t=1}^{k_n} A_t$, we have

$$B=\bigcup_{t=n}^{\infty}\bigcup_{s=k_t+1}^{k_{t+1}} B\cap A_s\,.$$

Since the charge $\mu\in ca(V, E)$ is countably additive, we have

$$\mu(B)=\sum_{t=n}^{\infty}\mu\Big(\bigcup_{s=k_t+1}^{k_{t+1}} B\cap A_s\Big)=\lim_m\sum_{t=n}^{m}\mu\Big(\bigcup_{s=k_t+1}^{k_{t+1}} B\cap A_s\Big)\,.$$

However, for all $m>n$

$$\sum_{t=n}^{m}\mu\Big(\bigcup_{s=k_t+1}^{k_{t+1}} B\cap A_s\Big)\in\sum_{t=n}^{m} g_t\subset g\,.$$

Since the neighborhood $g\in\mathscr{G}$ is closed, we have $\mu(B)\in g$, the desired result.

LEMMA 1.4. *Let $\mu\in ca(V, E)$ be W-bounded on the ring V and let $A_n\in V_\sigma$, $n=1, 2, 3, \cdots$ be W-dominated and increasing. Then for each neighborhood $g\in\mathscr{G}$, there exists an index $n(g)$ such that $n\geqq n(g)$ yields*

$$\mu(B)\in g$$

uniformly with respect to the sets $B\in V$ with $B\subset A\backslash A_n$ where $A=\bigcup_n A_n$.

PROOF. We assume that for each index $n=1, 2, 3, \cdots A_n=\bigcup_m A_{n,m}$ where $A_{n,m}\in V$, $A_{n,m}\subset A_{n,m+1}$ for $m\in N$. For each index $n\in N$, set $B_n= A_{1,n}\cup A_{2,n}\cup\cdots\cup A_{n,n}$. Then the sequence $B_n\in V$, $n\in N$ is W-dominated such that $\bigcup_n B_n=\bigcup_n A_n=A$, and for $n\in N$, $B_n\subset\bigcup_{k=1}^{n} A_k=A_n$, and $A\backslash A_n\subset A\backslash B_n$. Using Lemma 1.3, there exists an index $n(g)\in N$ such that $n\geqq n(g)$ yields $\mu(B)\in g$ for all sets $B\in V$, $B\subset A\backslash B_n$. For indices $n\geqq n(g)$ this yields $\mu(B)\in g$ for all sets $B\in V$, $B\subset A/A_n$.

Denote by $P(X, W)$ the delta ring of all W-dominated subsets of the space X. Using the same definition of measurability as Sion [17], and minor modifications to his arguments, the following extension theorem may be established.

THEOREM 1.1. *Let V be a ring of subsets of the space X and let $W\subset V_\sigma$ be directed upward by set inclusion and assume $V\subset P(X, W)$. Let $\mu\in ca(V, E)$ be Rickart on the ring V relative to the class W. Then there exists a function $\mu^*: P(X, W)\to E$ with the following properties*

1. *The measure μ is extended by the function μ^*.*
2. *For each increasing W-dominated sequence $A_n\in P(X, W)$, $n\in N$ $\mu^*(\bigcup_n A_n)=\lim_n\mu^*(A_n)$.*
3. *The class $\sum(\mu^*, W)$ of μ^*-measurable, W-dominated sets is a*

delta ring containing the class $V_\sigma(W)$ of W-dominated, V_σ-sets and the function μ^* is finitely additive on the class $\sum(\mu^*, W)$.

4. *If a set $A \in P(X, W)$ is μ^*-measurable, then for each neighborhood $g \in \mathscr{G}$, there exits a set $B \in \mathscr{C}(A, W)$ with $B \backslash A \in N(\mu^*, g)$. If the family $\mathscr{G}$ is countable, then a set $A \in P(X, W)$ is measurable if and only if $A = B \cup C$ with $B \in V_{\sigma\delta}$, $B, C \in P(X, W)$, $B \cap C = \varnothing$ and $\mu^*(C) = \theta$.*

The function $\mu^*: P(X, W) \to E$ is referred to as the outer measure generated by the Rickart measure $\mu \in ca(V, E)$. A discussion of the uniformity on the class $P(X, W)$ generated by the outer measure μ^* is given by Sion [17]. Although the notion of μ^*-measurability is useful for studying additive (and, hence, countably additive) extensions of the W-bounded measure μ, the more intrinsic result is the fact that generated outer measure μ^* is continuous under W-dominated convergence in the space $P(X, W)$.

The extension via outer measure is accomplished by introducing, for each set $A \in P(X, W)$, the class

$$\mathscr{C}(A, W) = \{B \in V_\sigma \cap P(X, W) : A \subset B\} .$$

The condition $W \subset V_\sigma$ insures that the class $\mathscr{C}(A, W)$ is non-empty for each set $A \in P(X, W)$. For each set $A \in P(X, W)$, the class $\mathscr{C}(A, W)$ is directed downward by set inclusion. The measure $\mu \in ca(V, E)$ is first extended to a finitely additive function $\mu_\sigma : V_\sigma \cap P(X, W) \to E$ which is continuous with respect to W-dominated increasing convergence. Noting that for each set $A \in P(X, W)$, the net $\{\mu_\sigma(B) : B \in \mathscr{C}(A, W)\}$ is Cauchy in the group E, the outer measure is defined by the relation

$$\mu^*(A) = \lim(\mu_\sigma(B) : B \in \mathscr{C}(A, W)) .$$

The function $\mu^*: P(X, W) \to E$ then satisfies the condition of the theorem.

Sion [17] solved the extension problem for measures on a ring with values in a commutative, complete topological group for the case $W = V_\sigma$ and $X \in V_\sigma$. For a ring V of subsets of the space X and Banach space E, Bogdan and Oberle [6] studied the extension problem for a vector measure $\mu \in cab(V, E)$ for each of the conditions

1. $W = V$

and

2. $W = \{A \in P(X) : p(A, \mu) < \infty\}$

where

$$p(A, \mu)=\sup(|\mu(B)|, B \in V, B \subset A)$$

and

$$cab(V, E)=\{\mu \in ca(V, E): p(A, \mu)<\infty \quad \text{for each set} \quad A \in V\} .$$

The same extension theorem has been obtained by Gould [11], for a class of Banach spaces which include the weakly complete spaces. Fox [9] solved the extension problem for general Banach spaces and measures on an algebra of sets.

All of the above extension theorems are obtained by using the assumed properties to enlarge the domain to the desired extension ring. The extension problem has also been approached via the existence of a "control measure". Brooks [7] has shown that a vector charge $\mu \in a(V, E)$ is Rickart on the ring V relative to the family $W=\{X\}$ (such charges are called strongly bounded) if and only if the charge μ admits a control measure $v \in ab^{+}(V, R)$. Moreover, if in addition the charge μ is a measure (that is, countably additive) then the control measure may be chosen to be countably additive. In this case, the control measure admits an extension to the σ-ring generated by the ring V via the classical construction and the vector measure is then extended via uniform continuity. Uhl [18] has shown that for strongly bounded vector measure on an algebra of sets, the existence of the extension measure on the generated σ-algebra is equivalent to the existence of a finitely additive control measure which in turn is equivalent to the range of vector measures being contained in a weakly compact subset of the range space.

§2. Topological rings of sets generated by group valued charges.

Bogdan and Oberle [6] made a study of the topology on an abstract ring of sets generated by families of non-negative, subadditive, increasing set functions which vanish at the empty set (called contents). The theorems relating to completeness proved to be especially useful in establishing extensions of the classical Vitali-Hahn-Saks theorem. In this section, an analogous completeness theorem is established for the topology on an abstract ring generated by a group valued charge.

Let $\mu \in a(V, E)$ be an arbitrary charge. Then the family of sets $\{\hat{N}(\mu, g), g \in \mathscr{G}\}$ is a base for a uniformity on the ring V and the associated topology is referred to as the μ-topology on the ring V. The pair (V, μ), where V is given the μ-topology for $\mu \in a(V, E)$ will be called a topological ring of sets. Convergence in the μ-topology of a sequence $A_n \in V$, $n \in N$ to a set $A \in V$ will be denoted $A=\mu-\lim_n A_n$. If the topology on the group E is generated by an invariant metric ρ, then for any charge $\mu \in a(V, E)$, for which the ρ-semivariation

$$p_\rho(A, \mu) = \sup [\rho(\mu(B), \theta), B \in V, B \subset A]$$

is finite on the ring V, the μ-topology is equivalent to the usual $p_\rho(\cdot, \mu)$-semi-metric topology. The family of all $\mu \in a(V, E)$ for which the p-semivariation $p_\rho(\cdot, \mu)$ is finite on the ring V is denoted $ab(V, E)$.

For any sequence A_n, $n = 0, 1, 2, \cdots$ of subsets of the space X, the symbol $A_n \to A_0$ is to be understood as pointwise convergence of the associated characteristic functions. For any class W of subsets of the space X, a topological ring (V, μ) is said to be a W-dominated convergence ring of sets if for each W-dominated sequence $A_n \in V$, $n \in N$ for which $A_n \to A$, we have $A \in V$ and $\lim_n \mu(B) = \theta$ uniformly with respect to sets $B \in V$, $B \subset A \div A_n$.

PROPOSITION 2.1. *Let W be any class of subsets of the space X. A topological ring (V, μ) is a W-dominated convergence ring of sets if and only if the ring V is closed under W-dominated, countable unions and the charge μ is countably additive on the ring V.*

Let $\mathscr{B}$ be any family of closed symmetric neighborhoods of the identity in the group E. A topological ring (V, μ) is said to be $(W, \mathscr{B})$-upper complete if for each increasing sequence $A_n \in V$, $n \in N$ for which $A_n \in N(\mu, b)$, for all $n \in N$ and some set $b \in \mathscr{B}$, there exists a set $B \in W$ such that $A_n \subset B$ for all $n \in N$.

The family $\mathscr{B}$ is said to be additive if for each pair, $b_1, b_2 \in \mathscr{B}$ and numbers $n, m \in N$, there exists a set $b \in \mathscr{B}$ such that $nb_1 + mb_2 \subset b$. In a general abelian group for $n \in N$, and $b \subset E$, we set $nb = b + b + \cdots + b$(n-times).

The family of all non-negative multiples of the unit sphere in a Banach space is an example of an additive family. In the model of interest, the family $\mathscr{B}$ is intended to consist of bounded neighborhoods of the identity in a locally convex topological vector space. In this case, a topological ring (V, μ) is $(W, \mathscr{B})$-upper complete if the only increasing sequences in the ring V which map uniformly into some bounded neighborhood of the origin are the W-dominated sequences. In this situation, each charge $\mu \in cab(V, E)$ is $(W, \mathscr{B})$-upper complete for $W = \{A \in V_\sigma : p(A, \mu) < \infty\}$.

Any real valued, countably additive function μ on a ring is $(W, \mathscr{B})$-upper complete for the family W consisting of all sets $A \in V_\sigma$ for which $\sup (|\mu(B)| : B \in V, B \subset A)$ is finite and $\mathscr{B}$ consisting of all bounded neighborhoods of zero. The finite part of any abstract measure (X, Σ, μ) is a $(W, \mathscr{B})$-absolute convergence ring of sets for $W=\Sigma$ and $\mathscr{B}$ consisting of the bounded neighborhoods of zero.

A topological ring (V, μ) is said to be a $\mathscr{B}$-monotonely complete ring of sets if for each monotone sequence $A_n \in V$, $n \in N$ for which $A_n \in N(\mu, b)$ for all $n \in N$ and some set $b \in \mathscr{B}$ wet have $A=\bigcup_n A_n \in V$ if the sequence is increasing or $A=\bigcap_n A_n \in V$ if the sequence is decreasing and $\mu\text{-}\lim_n A \div A_n=\varnothing$.

PROPOSITION 2.2. *Let V be a ring of subsets closed under W-dominated, countable unions and let (V, μ) be a $(W, \mathscr{B})$-upper complete topological ring for a measure $\mu \in ca(V, E)$ which is Rickart on the ring V relative to the family W. Then the topological ring (V, μ) is a $\mathscr{B}$-monotonely complete ring of sets. Conversely, if the topological ring (V, μ) is a $\mathscr{B}$-monotonely complete ring of sets and $\mu \in ca(V, E)$ then the ring V is a delta ring and the topological ring (V, μ) is $(V, \mathscr{B})$-upper complete.*

PROOF. The proof proceeds as the proof of Lemma 1.3.

REMARK 2.1. A contrapositive argument, using the countable additivity of the charge $\mu \in ca(V, E)$ insures that we have

$$A=\bigcup_n A_n \in N(\mu, b)$$

for each increasing, W-dominated sequence $A_n \in V$, $n \in N$ for which $A_n \in N(\mu, b)$, for all $n \in N$ and some set $b \in \mathscr{B}$. The converse asserted in Proposition 2.2 is clear.

For a charge $\mu \in ca(V, E)$, we define the class of μ-null sets $\Theta(\mu)$ to be all sets $A \in P(X)$ for which $\mu(B)=\theta$ for each set $B \in V$ with $B \subset A$ and for every neighborhood $g \in \mathscr{G}$, there exists a set $B \in V_\sigma$ such that $A \subset B$ and $B \in N(\mu, g)$.

REMARK 2.2. Notice that if $A \in V$ and $B \in V \cap \Theta(\mu)$, then $\mu(A \div B)=\mu(A)$. This observation is a simple consequence of the fact that $\mu(B)=\theta$ for all $B \in V \cap \Theta(\mu)$.

Let E be a complete metrizable group with invariant metric ρ and for $\mu \in ab(V, E)$, let $p_\rho(\cdot, \mu)$ denote the ρ-semivariation of the charge μ.

A sequence $\langle A_n \in V, n \in N\rangle$ is said to be (μ, ρ)-absolutely summable if $\sum_n p_\rho(A_n, \mu)<\infty$. The topological ring (V, μ) is said to be a (W, ρ)-absolute convergence ring if every (μ, ρ)-absolutely summable sequence is W-dominated.

THEOREM 2.1. *Let E be a complete metrizable group with invariant metric ρ and let V be a ring of subsets of an abstract space X. If a topological ring (V, μ) with $\mu \in cab(V, E)$ is a V-dominated and (V, ρ)-absolute convergence ring, then the ring V is sequentially complete in the $p_\rho(\cdot, \mu)$ topology.*

Since the ρ-semivariation $p_\rho(\cdot, \mu)$ is a content, Theorem 2.1 is a special case of Theorem 1.1.1 of reference [6].

Let $\mathscr{B} \subset E$ be an additive family of closed neighborhoods of the origin in the group E. Let $W(\mu, \mathscr{B})$ be the family of sets $A \in V_\sigma$ for which there exists a set $b \in \mathscr{B}$, with $\mu(B) \in b$ for each set $B \in V$ with $B \subset A$. If E is a Banach space and $\mathscr{B}$ is the family of all positive multiples of the closed unit sphere, then for any charge $\mu \in ab(V, E)$, the family $W(\mu, \mathscr{B})$ is the family of V_σ-sets of finite semivariation.

THEOREM 2.2. *Let V be a ring of subsets of an abstract space X and let $\mathscr{B}$ be an additive family of closed neighborhoods of the origin in the group E. Then each charge $\mu \in ca(V, E)$ is $(W(\mu, \mathscr{B}), \mathscr{B})$-upper complete on the ring V.*

PROOF. Let $A_n \in V$, $n \in N$ be increasing with $A_n \in N(\mu, b)$ for all $n = 1, 2, 3, \cdots$ and some $b \in \mathscr{B}$. We must show that there exists a set $A \in W(\mu, \mathscr{B})$ such that $A_n \subset A$ for all $n = 1, 2, 3, \cdots$. Set $A = \bigcup_n A_n$ and consider any set $B \in V$ with $B \subset A$. Then $B = \lim_n B \cap A_n$ and from countable additivity $\mu(B) = \lim_n \mu(B \cap A_n) \in b$. Consequently, $A = \bigcup_n A_n \in W(\mu, \mathscr{B})$ and the theorem is established.

§ 3. Extensions of group valued measures via topological rings of sets.

In this section a general extension theorem for certain group valued measures will be established. The extension is accomplished by first extending to the family V_δ and a subfamily of V_σ and then using these classes to define the domain of the completion. The domain of the extension consists of a subfamily of measurable sets and when the base $\mathscr{G}$ is countable, this extension is characterized as the smallest extension delta ring closed under W-dominated increasing sequential convergence.

Let $P(X)$ denote the σ-algebra of all subsets of the space X and let $P(X, W)$ denote the delta ring of all members of $P(X)$ dominated by some set from the class W. Proposition 1.1, the Rickart condition, and the countable additivity insure that each measure $\mu \in ca(V, E)$, Rickart on the ring V relative to the class W, admits an additive extension $\mu_\sigma: P(X, W) \cap V_\sigma \to E$ characterized by the relation: For each set $A \in P(X, W) \cap V_\sigma$ $\mu_\sigma(A) = \lim_n \mu(A_n)$ where $A_n \in V$, $n \in N$ is any W-dominated sequence increasing to the set A (see Fox, [9], or Sion, [17]).

REMARK 3.1. 1. For any set $A \in V_\sigma \cap P(X, W)$, $B \in V$, $B \subset A$, we have $\mu_\sigma(A) = \mu(B) + \mu_\sigma(A \backslash B)$.

2. For any neighborhood $g \in \mathscr{G}$ and any set $A \in V_\sigma \cap P(X, W)$, the conditions $A \in N(\mu_\sigma, g)$ and $A \in N(\mu, g)$ are equivalent.

PROPOSITION 3.1. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W and let $A_n \in V_\sigma$, $n \in N$ be an increasing, W-dominated sequence with $A = \bigcup_n A_n$.*

1. *The sequence $\mu_\sigma(A_n)$, $n \in N$ is Cauchy in the group E and $\mu_\sigma(A) = \lim_n \mu_\sigma(A_n)$.*

2. *For each neighborhood $g \in \mathscr{G}$, there exists an index $n(g) \in N$ and a set $B \in V_\sigma$ such that $B \in N(\mu_\sigma, g)$ and $A \backslash A_n \subset B$ for all indices $n \geqq n(g)$.*

3. *If $A'_n \in V_\sigma$, $n \in N$ is another increasing sequence with $A = \bigcup_n A'_n$, then*

$$\lim_n \mu_\sigma(A_n) = \lim_n \mu_\sigma(A'_n) .$$

PROOF. 1. Let $A_n \in V_\sigma \cap P(X, W)$ be increasing and W-dominated with $A = \bigcup_n A_n$. Assume that for each index $n \in N$, $A_n = \bigcup_m A_{n,m}$ with $A_{n,m} \subset A_{n,m+1}$ and $A_{n,m} \in V$ for $m = 1, 2, 3, \cdots$. For each $n \in N$, set $B_n = A_{1,n} \cup \cdots \cup A_{n,n}$ and note that the sequence $B_n \in V$, $n \in N$ is W-dominated, increasing with $B_n \subset A_n \subset A$ and $A = \bigcup_n B_n$. From Remark 3.1(1), for each index $t \in N$, $\mu_\sigma(A_t) = \mu(B_t) + \mu_\sigma(A_t \backslash B_t)$. Consequently, for any pair of indices $m, n \in N$, we have

$$\mu_\sigma(A_m) - \mu_\sigma(A_n) = \mu_\sigma(A_m \backslash B_m) + \mu(B_m \div B_n) - \mu_\sigma(A_n \backslash B_n) .$$

Since the sequence $B_n \in V$, $n \in N$ is increasing, Lemma 1,3 insures that for each neighborhood $g \in \mathscr{G}$, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $A \backslash B_n \in N(\mu, g)$. Therefore, for indices $m, n \geqq n(g)$, $\mu(B_m \div B_n) \in g$ and from Remark 3.1, $\mu_\sigma(A_t \backslash B_t) \in g$ for $t = m, n$ so that

$$\mu_\sigma(A_m) - \mu_\sigma(A_n) \in 3g .$$

Thus, the sequence $\mu_\sigma(A_n)$, $n \in N$ is Cauchy in the group E.

To see that $\mu_\sigma(A) = \lim_n \mu_\sigma(A_n)$, we note that from the construction,

$\mu_\sigma(A)=\lim_n \mu_\sigma(B_n)$ and for each index $t \in N$ $\mu_\sigma(A_t)-\mu_\sigma(B_t)=\mu_\sigma(A_t \backslash B_t)$. Then for any neighborhood $g \in \mathscr{G}$ and index $n \geqq m(g)$ (chosen as above) we have $\mu_\sigma(A_n)-\mu_\sigma(B_n) \in g$ so that $\lim_n \mu_\sigma(A_n)=\lim_n \mu(B_n)$ and therefore $\mu_\sigma(A)= \lim_n \mu_\sigma(A_n)$.

2. Let $g \in \mathscr{G}$ be arbitrary and choose the sequence $B_n \in V$, $n \in N$ as in the proof of 2(1) above. That is, the sequence $B_n \in V$, $n \in N$ is increasing to the set A and for each index $n \in N$, $B_n \subset A_n \subset A$. From Lemma 1.4 and Remark 3.1, there exists an index $n(g)$ suc thhat $A \backslash B_n \in N(\mu_\sigma, g)$ for all indices $n \geqq n(g)$. We set $B=A \backslash B_{n(g)} \in V_\sigma$ and note $B \in N(\mu_\sigma, g)$ and $A \backslash A_n \subset B$ for all indices $n \geqq n(g)$.

3. Finally, let $A'_n \in V_\sigma \cap P(X, W)$, $n \in N$ be an increasing sequence for which $A=\bigcup_n A'_n$. Choose the sequences B_n, $B'_n \in V$, $n \in N$ corresponding to the sequences A_n, $A'_n \in V_\sigma \cap P(X, W)$ as in (1) above. From (2) above, we have $\lim_n \mu_\sigma(A_n)=\lim_n \mu(B_n)$ and $\lim_n \mu_\sigma(A'_n)=\lim_n \mu(B'_n)$. But the construction of the extension insures that $\mu_\sigma(A)=\lim_n \mu(B_n)=\lim_n \mu(B'_n)$. Consequently, $\mu_\sigma(A)=\lim_n \mu_\sigma(A_n)=\lim_n \mu_\sigma(A'_n)$.

For each measure $\mu \in ca(V, E)$ which is Rickart on the ring V relative to the class W and each decreasing sequence $A_n \in V$, $n \in N$ with $A=\bigcap_n A_n$, the sequence $\mu(A_n)$, $n \in N$ is Cauchy in the group E. Moreover, if $A'_n \in V$, $n \in N$ is another decreasing sequence with $A=\bigcap_n A'_n$, we get from the countable additivity

$$\lim_n \mu(A_n)=\lim_n \mu(A'_n) .$$

Consequently, the limit $\mu_\delta\colon A \to \mu_\delta(A)$ for sets $A \in V_\delta$ represents a group valued finitely additive function on the class V_δ for which $\mu_\sigma(A)=\mu_\delta(A)$ for all sets $A \in V_\sigma \cap V_\delta$.

LEMMA 3.1. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W.*

1. *For each set $A \in V_\delta$ with $A=\bigcap_n A_n$, $A_n \in V$, $A_{n+1} \subset A_n$ for $n \in N$ and each neighborhood $g \in \mathscr{G}$, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $A_n \backslash A \in N(\mu_\sigma, g)$.*

2. *For any set $A \in V$ with $A=B \cup C$, $B \in V_\delta$, $C \in V_\sigma$ and $B \cap C=\varnothing$, $\mu(A)=\mu_\delta(B)+\mu_\sigma(C)$.*

3. *For sets $A \in V_\sigma \cap P(X, W)$, $B \in V_\delta$ and $C \in V$ with $B \subset C \subset A$. $\mu_\sigma(A \backslash B)=\mu_\sigma(A)-\mu_\delta(B)$.*

PROOF. 1. Let $g \in \mathscr{G}$ be an arbitrary neighborhood and let $A_n \in V$, $n \in N$ be a sequence decreasing to the set $A \in V_\delta$. From the Rickart condition there exists and index $n(g)$ such that n, $m \geqq n(g)$ yields $\mu(B) \in g$

uniformly with respect to sets $B \in V$ with $B \subset A_n \div A_m$. Moreover, for each index $n \in N$,

$$A_n \backslash A = \lim_k A_n \backslash A_{n+k} .$$

From countable additivity, for any set $B \in V$, $B \subseteqq A_n \backslash A$, $\mu(B) = \lim_k \mu(B \cap (A_n \backslash A_{n+k}))$ so that $\mu(B) \in g$ for indices $n \geqq n(g)$. Recalling Remark 3.1, for indices $n \geqq n(g)$, $A_n \backslash A \in N(\mu_\sigma, g)$.

2. Let $A \in V$ have the representation $A = B \cup C$ with $B \in V_\delta$, $C \in V_\sigma$ and $B \cap C = \emptyset$. Assume that the set $B \in V_\delta$ has the representation $B = \bigcap_n B_n$ where $B_n \in V$, $B_n \subseteqq A$, $n \in N$ is decreasing. The sequence $A \backslash B_n \in V$, $n \in N$ increases to the set $A \backslash B = C \in V_\sigma$. From Proposition 1.1, $\mu_\sigma(C) = \mu_\sigma(A \backslash B) = \lim_n \mu_\sigma(A \backslash B_n) = \mu(A) - \lim_n \mu(B_n)$ so that

$$\mu_\sigma(C) = \mu(A) - \mu_\delta(B) .$$

3. Consider sets $A \in V_\sigma \cap P(X, W)$, $B \in V_\delta$ and $C \in V$ with $B \subset C \subset A$. Assume that $A = \bigcup_n A_n$ for an increasing sequence A_n with $C \subset A_n$ for $n \in N$. The sequence $A_n \backslash B \in V_\sigma$, $n \in N$ increases to the set $A \backslash B$. Also, from part (2) above, for each index $n \in N$, $\mu(A_n) = \mu_\delta(B) + \mu_\sigma(A_n \backslash B)$. Using Proposition 3.1, we have

$$\mu_\sigma(A) = \mu_\delta(B) + \mu_\sigma(A \backslash B) .$$

REMARK 3.2. The requirement that the sets A and B be separated by a set from the ring V will be removed when it has been shown that for each increasing, W-dominated sequence $A_n \in V_\delta$, $n \in N$, the sequence $\mu_\delta(A_n)$, $n \in N$ is Cauchy in the group E.

PROPOSITION 3.2. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W. For each decreasing sequence $A_n \in V_\delta$, $n \in N$ with $A = \bigcap_n A_n$, we have*

1. $\mu_\delta(A) = \lim_n \mu_\delta(A_n)$.
2. *For each neighborhood $g \in \mathscr{G}$, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $A_n \backslash A \in N(\mu_\sigma, g)$.*

PROOF. For each index $n \in N$, assume $A_n = \bigcap_m A_{n,m}$ with $A_{n,m+1} \subset A_{n,m}$ for indices $m \in N$. For each index $n \in N$, define $B_n = A_{1,n} \cap \cdots \cap A_{n,n}$ and note that the sequence $B_n \in V$, $n \in N$ decreases to the set A and moreover $A_n \subset B_n$ for $n \in N$. Now, for each neighborhood $g \in \mathscr{G}$, Lemma 3.1(1) insures that there exists an index $n(g)$ for which $n \geqq (g)$ yields $B_n \backslash A \in N(\mu_\sigma, g')$ where $g' \in \mathscr{G}$ and $\operatorname{cls} 2g' \subset g$. In particular, for $n \geqq n(g)$, $B_n \backslash A_n \in N(\mu_\sigma, g)$ and from Lemma 3.1(2) $\mu(B_n) = \mu_\delta(A_n) + \mu_\sigma(B_n \backslash A_n)$. Consequently, for indices $n \geqq n(g)$, we get $\mu(B_n) - \mu_\delta(A_n) = \mu_\sigma(B_n \backslash A_n) \in g$. Thus $\mu_\delta(A) = \lim_n \mu(B_n) = \lim_n \mu_\delta(A_n)$.

REMARK 3.3. Consider a set $A \in V_\sigma \cap P(X, W)$ and a set $B \in V_\delta$ with $B \subset A$. Assume $B = \bigcap_n B_n$ where the sequence $B_n \in V$, $n \in N$ is decreasing. The sequence $B_n \backslash A \in V_\delta$, $n \in N$ decreases to the empty set so that Proposition 3.2(1) yields $\lim_n \mu_\delta(B_n \backslash A) = \mu_\delta(\varnothing) = \theta$. Consequently, for each neighborhood $g \in \mathscr{G}$, there exists an index $n(g)$ for which $n \geqq n(g)$ yields $\mu_\delta(B_n \backslash A) \in g$. Also, for each index $m \in N$, we have $\mu(B_m) = \mu_\sigma(B_m \cap A) + \mu_\delta(B_m \backslash A)$ and since $\mu_\delta(B) = \lim_m \mu(B_m)$, there exists an index $m(g)$ such that $m \geqq m(g)$ yields $\mu(B_m) - \mu_\delta(B) \in g$. Consequently, for indices $k \geqq \max(n(g), m(g))$, we have $\mu_\sigma(B_k \cap A) - \mu_\delta(B) \in 2g$; that is, $\mu_\delta(B) = \lim_k \mu_\sigma(B_k \cap A)$. This observation strengthens Proposition 3.2(2) which is the key to the proof of the Theorem 3.1.

PROPOSITION 3.3. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W. Then for each decreasing sequence $A_n \in V_\delta$, $n \in N$ with $A = \bigcap_n A_n$ and each neighborhood $g \in \mathscr{G}$, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $A_n \backslash A \in N(\mu_\delta, g)$.*

PROOF. Let $g \in \mathscr{G}$ be arbitrary and define the sequence $B_n \in V$, $n \in N$ just as in Proposition 3.2. That is, $A_n \subset B_n$ and $B_{n+1} \subset B_n$ for indices $n \in N$ and $A = \bigcap_n A_n = \bigcap_n B_n$. From Proposition 3.2(2), there exists an index $n(g)$ such that $n \geqq n(g)$ yields $B_n \backslash A \in N(\mu_\sigma, g)$. Consider any set $C \in V_\delta$ with $C \subset A_n \backslash A$ and indices $n \geqq n(g)$. Then $C \subset B_n \backslash A$ and from Remark 3.3, if the sequence $C_m \in V$, $m \in N$ decreases to the set C, then $\mu_\delta(C) = \lim_m \mu_\sigma(C_m \cap (B_n \backslash A))$. Consequently, for indices $n \geqq n(g)$ we get $\mu_\delta(C) \in g$ or $A_n \backslash A \in N(\mu_\delta, g)$.

LEMMA 3.2. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W.*

1. *For each finite family $A_k \in V_\sigma \cap P(X, W)$, $g_k \in \mathscr{G}$, $k = 1, \cdots, n$, with $A_k \in N(\mu_\sigma, g_k)$, $k = 1, \cdots, n$, we have $\bigcup_{k=1}^n A_k \in N(\mu_\sigma, \operatorname{cls}(g_1 + \cdots + g_k))$.*

2. *For neighborhoods $g_1, g_2 \in \mathscr{G}$ and sets $A_1 \in V_\sigma \cap P(X, W)$, $A_2 \in V_\delta$ with $A_1 \in N(\mu_\sigma, g_1)$, $A_2 \in N(\mu_\delta, g_2)$, we have $A_1 \cup A_2 \in N(\mu_\sigma, \operatorname{cls}(g_1 + g_2))$.*

PROOF. 1. Consider a finite family $A_k \in V_\sigma \cap P(X, W)$, $g_k \in \mathscr{G}$, $k = 1, \cdots, n$ with $A_k \in N(\mu_\sigma, g_k)$ for $k = 1, \cdots, n$ and assume $A_k = \bigcup_m A_{k,m}$ with $A_{k,m} \in V$, $A_{k,m} \subset A_{k,m+1}$ for $m = 1, 2, 3, \cdots$ and $k = 1, 2, \cdots, n$. From the countable additivity, for any set $A \in V$, $A \subset A_1 \cup \cdots \cup A_n$, we have $\mu(A) = \lim_m \mu(A \cap \bigcup_{k=1}^n A_{k,m})$. Moreover, for each index $m \in N$, we have

$$A \cap \bigcup_{k=1}^{n} A_{k,m} = \bigcup_{k=1}^{n} A \cap \left(A_{k,m} \backslash \bigcup_{j=1}^{k-1} A_{j,m}\right)$$

so that

$$\mu\left(A \cap \bigcup_{k=1}^{n} A_{k,m}\right) = \sum_{k=1}^{n} \mu\left(A \cap \left(A_{k,m} \backslash \bigcup_{k=1}^{k-1} A_{j,m}\right)\right) .$$

The relations $A_k \in N(\mu_\sigma, g_k)$ for $k=1, \cdots, n$ yields

$$\mu\left(A \cap \bigcup_{k=1}^{n} A_{k,m}\right) \in g_1 + \cdots + g_n \quad \text{for each} \quad m \in N .$$

Consequently, $\mu(A) = \lim_m \mu(A \cap \bigcup_{k=1}^{n} A_{k,m}) \in \text{cls}(g_1 + \cdots + g_n)$. The observation in Remark 3.1(2), insures that $A_1 \cup \cdots \cup A_n \in N(\mu_\sigma, \text{cls}(g_1 + \cdots + g_n))$.

2. Consider sets $A \in V_\sigma \cap P(X, W)$, $B \in V_\delta$ with $A \in N(\mu_\delta, g_1)$ and $B \in N(\mu_\delta, g_2)$ for $g_1, g_2 \in \mathscr{G}$ and $C \in V$ with $C \subset A \cup B$. From the representation $C = ((A \backslash B) \cap C) \cup (B \cap C)$ where $(A \backslash B) \cap C \in V_\sigma$ and $B \cap C \in V_\delta$, we conclude from Lemma 3.1(2)

$$\mu(C) = \mu_\sigma((A \backslash B) \cap C) + \mu_\delta(C \cap B) \in g_1 + g_2 .$$

Consequently,

$$A \cup B \in N(\mu_\sigma, \text{cls}(g_1 + g_2)) .$$

For each set $A \in P(X, W)$ introduce the following families

$$\mathscr{C}(A, V, W) = \{B \in V_\sigma \cap P(X, W) \colon A \subset B\}$$
$$\mathscr{I}(A, V, V_\delta) = \{C \in V_\delta \colon C \subset A\}$$
$$\mathscr{E}(A, V, V_\delta, W) = \{B \backslash C \colon B \in \mathscr{C}(A, V, W), C \in \mathscr{I}(A, V, V_\delta)\} .$$

The families $\mathscr{C}(A, V, W)$ and $\mathscr{E}(A, V, V_\delta, W)$ are directed downward by set inclusion and the family $\mathscr{I}(A, V, V_\delta)$ is directed upward by set inclusion. Moreover, for each measure $\mu \in ca(V, E)$ which is Rickart on the ring V relative to the class W, the net $\{\mu_\sigma(B), B \in \mathscr{C}(A, V, W)\}$ is Cauchy in the group E (see Fox, [9], Gould, [11], and Sion, [17]). The completeness of the group E insures that the limit $\mu^*(A) = \lim (\mu_\sigma(B) \colon \mathscr{C}(A, V, W))$ exists. The mapping $\mu^* \colon A \to \mu^*(A)$ for $A \in P(X, W)$ is called the outer measure generated by Rickart measure μ. A modification of the arguments given by Sion, [17], shows that the restriction of the outer measure to the family of measurable sets is a countably additive extension of the Rickart measure μ to a delta ring. The development given here will be to generate the extension of the Rickart measure μ directly from the extensions μ_σ and μ_δ without reference to the μ^* measurable sets from the δ-ring $P(X, W)$. It is clear that the outer measure μ^* agrees with the function μ_σ on the class $V_\sigma \cap P(X, W)$. The equality of the outer measure μ^* and μ_δ on the class V_δ is contained in the next proposition.

PROPOSITION 3.4. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W. For each set $A \in V_\delta$, $\mu^*(A) = \mu_\delta(A)$.*

PROOF. Let $g \in \mathscr{G}$ be an arbitrary neighborhood and using the definition of the outer measure μ^*, choose a set $B \in \mathscr{C}(A, V, W)$ such that $B' \in \mathscr{C}(A, V, W)$ and $B' \subset B$ yields $\mu^*(A) - \mu_\sigma(B') \in g$. Assume that the set A has a representation $A = \bigcap_n A_n$ with $A_n \in V$ and $A_{n+1} \subset A_n$ for indices $n \in N$. Then the sequence $A_n \cap B \in \mathscr{C}(A, V, W)$, $n \in N$ satisfies the relation

$$\mu^*(A) - \mu_\sigma(A_n \cap B) \in g$$

for all $n \in N$.

Moreover, for each $n \in N$, $A_n = (A_n \cap B) \cup (A_n \backslash B)$, $A_n \cap B \in V_\sigma$, $A_n \backslash B \in V_\delta$ and the sequence $A_n \backslash B \in V_\delta$, $n \in N$ decreases to the empty set $\emptyset$. From Proposition 3.3, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $A_n \backslash B \in N(\mu_\delta, g)$. Applying Lemma 3.1(2) for each index $n \in N$, $\mu_\sigma(A_n \backslash (A_n \backslash B)) = \mu(A_n) - \mu_\delta(A_n \backslash B)$. Consequently, for indices $n \geqq n(g)$ we have

$$\mu^*(A) - \mu(A_n) = \mu^*(A) - \mu_\sigma(A_n \cap B) + \mu_\sigma(A_n \cap B) - \mu(A_n)$$

so that

$$\mu^*(A) - \mu(A_n) = (\mu^*(A) - \mu_\sigma(A_n \cap B)) - \mu_\delta(A_n \backslash B)$$

or

$$\mu^*(A) - \mu(A_n) \in g + g = 2g .$$

Since the neighborhood $g \in \mathscr{G}$ is arbitrary, we conclude

$$\mu^*(A) = \lim_n \mu(A_n) = \mu_\delta(A) .$$

REMARK 3.4. Let $A \in V_\delta$ and $g \in \mathscr{G}$ be arbitrary. From Lemma 3.1(1), there exists a set $B \in V$ with $A \subset B$ such that $B \backslash A \in N(\mu_\sigma, g)$. From Proposition 3.4, there exists a set $C \in \mathscr{C}(A, V, W)$ with $\mu_\sigma(C') - \mu_\delta(A) \in g$ for all sets $C' \in \mathscr{C}(A, V, W)$ with $C' \subset C$. Then for every set $D \in \mathscr{C}(A, V, W)$ with $D \subset B \cap C$, we have simultaneously

$$\mu_\sigma(D) - \mu_\delta(A) \in g$$

and

$$\mu_\sigma(D\backslash A) \in g \ .$$

THEOREM 3.1. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W. Let $A_n \in V_\delta$, $n \in N$ be an increasing, W-dominated sequence with $A = \bigcup_n A_n$. Then the sequence $\mu_\delta(A_n)$, $n \in N$ is Cauchy (and hence convergent) in the group E and if $A'_n \in V_\delta$, $n \in N$ is any other increasing sequence with $A = \bigcup_n A'_n$, then $\lim_n \mu_\delta(A_n) = \lim_n \mu_\delta(A'_n)$.*

PROOF. Let $g \in \mathscr{G}$ be arbitrary and choose sequences $g_n, g'_n \in \mathscr{G}$, $n \in N$ such that $\operatorname{cls} 3g'_n \subset g_n$, and $\sum_{k=1}^n g_k \subset g$ for all $n = 1, 2, 3, \cdots$ (see Sion, [17], Lemma 2.4). From Remark 3.4, for each index $n \in N$ we may choose a set $B_n \in \mathscr{C}(A_n, V, W)$ with $B_n \backslash A_n \in N(\mu_\sigma, g'_n)$ and $\mu_\sigma(B) - \mu_\delta(A_n) \in g'_n$ for all sets $B \in \mathscr{C}(A_n, V, W)$ with $B \subset B_n$. By restricting the sequence $B_n \in V_\sigma$, $n \in N$ to the set in W dominating the sequence A_n, $n \in N$, we may assume that the sequence $B_n \in V_\sigma$, $n \in N$ is W-dominated. Assume that for each index $m \in N$, we have a representation $B_m = \bigcup_{n=1}^\infty B_{m,n}$ with $B_{m,n} \subset V$, $B_{m,n} \subset B_{m,n+1}$ for indices $m \in N$. For indices $m \geqq 2$, $n \in N$, we set $K_{m,n} = \bigcup_{r=1}^m B_{r,n}$ and note

$$K_{m,n} \backslash B_{m,n} = \bigcup_{i=1}^{m-1} G_{i,n}^m$$

and for indices $i \in N$ with $1 \leqq i \leqq m-1$

$$B_{i,n} \backslash H_{i,n}^m = G_{i,n}^m$$

where

$$G_{i,n}^m = B_{i,n} \backslash \bigcup_{r=i+1}^m B_{r,n}$$

and

$$B_{i,n}^m = B_{i,n} \cap \bigcup_{r=i+1}^m B_{r,n} \ .$$

For indices $i \in N$ with $1 \leqq i \leqq m-1$, we set

$$H_i^m = \bigcup_{n=1}^\infty H_{i,n}^m$$

and note that $H_i^m \in \mathscr{C}(A_i, V, W)$ with $H_i^m \subset B_i$. Consequently, we have $B_i \backslash H_i^m \subseteqq B_i \backslash A_i$ so that $B_i \backslash H_i^m \in N(\mu_\sigma, g'_i)$. Choose indices $N_i(m) (1 \leqq i \leqq m-1)$ such that

$$B_i \backslash B_{i,k} \in N(\mu_\sigma, g'_i)$$

and

$$H_i^m \backslash H_{i,k}^m \in N(\mu_\sigma, g_i')$$

for indices $k \geqq N_i(m)$.

Then for indices $n \in N$ with $n \geqq N_i(m)$ and $1 \leqq i \leqq m-1$, we have

$$B_{i,n} \div H_{i,n}^m \subset (B_{i,n} \div B_i) \cup (B_i \backslash A_i) \cup (H_i^m \div H_{i,n}^m)$$

or

$$B_{i,n} \div H_{i,n}^m \in N(\mu_\sigma, \text{cls}\, 3g_i') \subset N(\mu_\sigma, g_i) .$$

Consequently for indices $n \geqq \max(N_i(m), i=1, \cdots, m-1)$ we have

$$G_{i,n}^m = B_{i,n} \div H_{i,n}^m \in N(\mu_\sigma, g_i) .$$

Also, for any pair of indices $m, n \in N$, the equality

$$K_{m,n} \div B_{m,n} = \bigcup_{i=1}^{m-1} G_{i,n}^m$$

and Lemma 3.2(1) yields

$$K_{m,n} \backslash B_{m,n} \in N(\mu_\sigma, \text{cls}(g_1 + \cdots + g_{m-1}))$$

for indices $n \geqq N(m) = \max(N_i(m): i=1, \cdots, m-1)$.

For each index $m \in N$, we set $K_m = \bigcup_n K_{m,n}$ and without loss of generality, we assume (via Proposition 3.1(2)) that for indices $n \geqq N(m)$ we have

1. $K_m \backslash K_{m,n} \in N(\mu_\sigma, g_m)$.

The relations $A_m \subset B_m \subset K_m$ and $K_m \backslash B_{m,n} \subset (K_m \backslash K_{m,n}) \cup (K_{m,n} \backslash B_{m,n})$ for indices $m, n \in N$ yield

2. $K_m \backslash B_{m,n} \in N(\mu_\sigma, \text{cls}\, 2g)$

for indices $n \geqq N(m)$.

Also, for each $m, n \in N$, $B_{m,n} \backslash A_m \subset B_m \backslash A_m$ so that $B_{m,n} \backslash A_m \in N(\mu_\sigma, g)$ and the sequence $A_m \backslash B_{m,n} \in V_\delta$, $n \in N$ decreases to the empty set. Consequently, from Proposition 3.2(2), we may assume that for indices $n \geqq N(m)$, $A_m \backslash B_{m,n} \in N(\mu_\sigma, g)$.

Finally, since the sequence $K_m \in V_\sigma$, $m \in N$ increases to the set B_0, there exists an index $m(g)$ such that $s, t \geqq m(g)$ yields

3. $\mu_\sigma(K_s) - \mu_\sigma(K_t) \in g$.

We then have for indices $s, t, r \in N$

$$\begin{aligned}\mu_\delta(A_s)-\mu_\delta(A_t)=(\mu_\delta(A_s)-\mu_\sigma(B_s))+(\mu_\sigma(B_s)-\mu_\sigma(B_{s,r}))\\ +(\mu_\sigma(B_{s,r})-\mu_\sigma(K_s))+(\mu_\sigma(K_s)-\mu_\sigma(K_t))\\ +(\mu_\sigma(K_t)-\mu_\sigma(B_{t,r}))+(\mu_\sigma(B_{t,r})-\mu_\sigma(B_t))\\ +(\mu_\sigma(B_t)-\mu_\delta(A_t)) .\end{aligned}$$

Consequently, for indices $s, t \geqq m(g)$, one has

$$\mu_\delta(A_s)-\mu_\delta(A_t) \in 6g+2 \operatorname{cls} 2g$$

by choosing $r \geqq \max(N(s), N(t))$. This completes the proof of the first part.

The proof of the second part is similar to the proof above; therefore, only the essential points will be mentioned. Let $A_n, A'_n \in V_\delta$, $n \in N$ be two increasing, W-dominated sequences with $A=\bigcup_n A_n=\bigcup_n A'_n$. Let the neighborhoods $g'_n, g_n \in \mathscr{G}$, $n \in N$ be chosen as above. Choose a sequence, $B_n \in \mathscr{C}(A_n, V, W)$, $n \in N$ with the property that for all $n \in N$, $B_n \backslash A_n \in N(\mu_\sigma, g'_n)$ and $\mu_\sigma(B)-\mu_\delta(A_n) \in g'_n$ for all $B \in \mathscr{C}(A_n, V, W)$ with $B \subset B_n$. Choose an analogous sequence $B'_n \in \mathscr{C}(A'_n, V, W)$, $n \in N$ for the sequence $A'_n \in V_\delta$, $n \in N$. Since the unions $\bigcup_n B_n$ and $\bigcup_n B'_n$ contain the set $A \in P(X, W)$, by restricting each sequence to $(\bigcup_n B_n) \cap (\bigcup_n B'_n)$, we may assume that $\bigcup_n B_n=\bigcup_n B'_n$. All the ralations developed in the proof above apply to the sequences $K_m, K'_m \in V_\sigma$, $m \in N$ defined for $m, n \in N$ by the relations $K_m=\bigcup_n K_{m,n}$, $K'_m=\bigcup_n K'_{m,n}$ and

$$K_{m,n}=\bigcup_{r=1}^{m} B_{r,n}$$

$$K'_{m,n}=\bigcup_{r=1}^{m} B'_{r,n} .$$

Since the sequences $K_m, K'_m \in V_\sigma$, $m \in N$ increase to the same limit, Proposition 3.1(3) insures that

$$\lim_m \mu_\sigma(K_m)=\lim_m \mu_\sigma(K'_m) .$$

This relation gives

$$\lim_n \mu_\delta(A_n)=\lim_n \mu_\delta(A'_n) .$$

Let $\mu \in a(V, E)$ be a charge. A net of sets $\{A_\gamma, \gamma \in \Gamma\} \subset P(X)$ is said to converge to the empty set in the μ-topology, written $\mu\text{-}\lim_{\gamma \in \Gamma} A_\gamma=\varnothing$ if for each neighborhood $g \in \mathscr{G}$, there exists an index $\gamma(g) \in \Gamma$ such that

$\gamma \geqq \gamma(g)$ yields $A_\gamma \in N(\mu, g)$. This notion of convergence is an abstraction of convergence in semivariation for vector measures and has been employed for group valued measures by Sion, [17].

Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W (recall that the class W is assumed to consist of V_σ sets and be directed upward by set inclusion). The domain of a completion of the measure μ is defined as follows

$$V_c(\mu) = \{A \in P(X, W) : \mu_\sigma - \lim (D : D \in \mathscr{E}(A, V, V_\delta, W)) = \varnothing\} .$$

That is, a set $A \in P(X, W)$ belongs to the family $V_c(\mu)$ if for each neighborhood $g \in \mathscr{G}$, there exists sets $B \in \mathscr{C}(A, V, W)$ and $C \in \mathscr{I}(A, V, V_\delta)$ such that $D \in \mathscr{E}(A, V, V_\delta, W)$ and $D \subset B \backslash C$ yields $D \in N(\mu_\sigma, g)$.

Before proceeding to the next theorem, it is necessary to develop a characterization of the W-dominated, μ-null sets. Notice that if $A \in P(X, W)$, then $A \in \theta(\mu)$ if and only if for each neighborhood $g \in \mathscr{G}$, there exists a set $B \in \mathscr{C}(A, V, W)$ with $B \in N(\mu_\sigma, g)$. Equivalently, if $A \in P(X, W)$ then $A \in \theta(\mu)$ if and only if $\mu^*(A) = \theta$.

THEOREM 3.2. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W.*

1. *The family $V_c(\mu)$ is a ring containing the classes*

$$V_\sigma \cap P(X, W),\ V_\delta \text{ and } \theta(\mu) \cap P(X, W) .$$

2. *If $A_n \in V_c(\mu)$, $n \in N$ is increasing and W-dominated, then*

$$A = \bigcup_n A_n \in V_c(\mu) .$$

3. *For each set $A \in V_c(\mu)$, the set $\bar{\bar{A}} \backslash \bar{A}$ is μ-null where*

$$\bar{\bar{A}} = \bigcap (B : B \in \mathscr{C}(A, V, W)) \quad \text{and} \quad \bar{A} = \bigcup (C : C \in \mathscr{I}(A, V, V_\delta)) .$$

PROOF. 1. To see that the family $V_c(\mu)$ is a ring, consider two sets $A_1, A_2 \in V_c(\mu)$ and let $g \in \mathscr{G}$ be arbitrary. Choose sets $D_i \in \mathscr{E}(A_i, V, V_\delta, W)$, $D_i \in N(\mu_\sigma, g)$, $i=1, 2$ and assume $D_i = B_i \backslash C_i$, $B_i \in \mathscr{C}(A_i, V, W)$, $C_i \in \mathscr{I}(A_i, V, V_\delta)$ for indices $i=1, 2$. If we set $D_{1,2} = D_1 \cup D_2$ then $D_{1,2} \in \mathscr{E}(A_1 \cup A_2, V, V_\delta, W)$ and $D_{1,2} = N(\mu_\sigma, \operatorname{cls} 2g)$ so that $A_1 \cup A_2 \in V_c(\mu)$. If we set $D_{1,2} = (B_1 \cap B_2) \backslash (C_1 \cap C_2)$, then $D_{1,2} \in \mathscr{E}(A_1 \cap A_2, V, V_\delta, W)$ and $D_{1,2} \in N(\mu_\sigma, \operatorname{cls} 2g)$ so that $A_1 \cap A_2 \in V_c(\mu)$. If we set $D_{1,2} = (B_2 \backslash C_1) \backslash (C_2 \backslash B_1)$, then $D_{1,2} \in \mathscr{E}(A_2 \backslash A_1, V, V_\delta, W)$ and $D_{1,2} \subseteq D_1 \cup D_2$ so that $D_{1,2} \in N(\mu_\sigma, \operatorname{cls} 2g)$ and consequently $A_2 \backslash A_1 \in V_c(\mu)$.

Proposition 3.1(2) insures that $V_\sigma \cap P(X, W) \subset V_c(\mu)$ and Lemma 3.1(1) insures that $V_\delta \subset V_c(\mu)$. To see that $\theta(\mu) \cap P(X, W) \subset V_c(\mu)$ consider a

W-dominated set $A\in\theta(\mu)$ and a neighborhood $g\in\mathscr{G}$. Then from the definition, there exists a set $B\in\mathscr{C}(A, V, W)$ with $B\in N(\mu_\sigma, g)$. Consequently, $D\in\mathscr{C}(A, V, V_\delta, W)$, $D\subset B\backslash\varnothing\in\mathscr{C}(A, V, V_\delta, W)$ yields $D\in N(\mu_\sigma, g)$ so that μ-lim$(D: D\in\mathscr{C}(A, V, V_\delta, W))=\varnothing$ and hence $A\in V_c(\mu)$.

2. Let $A_n\in V_c(\mu)$, $n\in N$ be W-dominated and increasing and let $g\in\mathscr{G}$ be arbitrary. Choose sequences $g_n, g'_n\in\mathscr{G}$, $n\in N$ with $\text{cls}\,3g'_n\subset g_n$, and $\sum_{k=1}^{n} g_k\subset g$ for all $n=1, 2, 3\cdots$ (see Sion, [17], Lemma 2.4). From the definition of the class $V_c(\mu)$, for each $n\in N$, there exist sets $B_n\in\mathscr{C}(A_n, V, W)$ and $C_n\in\mathscr{F}(A_n, V, V_\delta)$ with $B_n\backslash C_n\in N(\mu_\sigma, g'_n)$. By restricting the sequence $B_n\in V_\sigma$, $n\in N$ to the set in W dominating the sequence A_n, $n\in N$, we may assume that the sequence B_n, $n\in N$ is W-dominated and for each index $m\in N$ we assume the representation $B_m=\bigcup_n B_{m,n}$ with $B_{m,n}\in V$, $B_{m,n}\subset B_{m,n+1}$ for indices $n\in N$. For indices $m, n\in N$, set $K_{m,n}=\bigcup_{r=1}^{m} B_{r,n}$. Proceeding as in the proof of Theorem 3.1, for each $m\in N$, there exists an index $N(m)$ such that

$$K_{m,n}\backslash B_{m,n}\in N(\mu_\sigma, \text{cls}\,(g_1+\cdots+g_{m-1}))$$

for all $n\geqq N(m)$.

For each index $m\in N$, set $K_m=\bigcup_n K_{m,n}$ and without loss of generality, we assume (via Proposition 3.1(2)) that for indices $n\geqq N(m)$ we have

$$(1)\qquad K_m\backslash K_{m,n}\in N(\mu_\sigma, g_m)\,.$$

For indices $m, n\in N$, the relations $A_m\subset B_m\subset K_m$ and $K_m\backslash B_{m,n}\subset(K_m\backslash K_{m,n})\cup(K_{m,n}, B_{m,n})$ yield

$$(2)\qquad K_m\backslash B_{m,n}\in N(\mu_\sigma, \text{cls}(g_m+g_m))\subset N(\mu_\sigma, \text{cls}\,2g)$$

for indices $n\geqq N(m)$.

For each $m, n\in N$, $B_{m,n}\backslash C_m\subset B_m\backslash C_m$ so that $B_{m,n}\backslash C_m\in N(\mu_\sigma, g_m)$ and the sequence $C_m\backslash B_{m,n}\in V_\delta$, $n\in N$, decreases to the empty set. Consequently, from Proposition 3.3 we may assume that for indices $n\geqq N(m)$, $C_m\backslash B_{m,n}\in N(\mu_\delta, g_m)$. Finally, since the sequence $K_m\in V_\sigma$, $m\in N$ increases to the set $\bigcup_m K_m=\bigcup_m B_m$, Proposition 3.1(2) insures that there exists an index $m(g)\in N$ and a set $L_g\in V_\sigma$ such that $L_g\in N(\mu_\sigma, g)$ and $m\geqq m(g)$ yields $K/K_m\subset L_g$ where $K=\bigcup_m B_m$. We then have

$$K\backslash C_m\subset(K\div K_m)\cup(K_m\div B_{m,n})\cup(B_{m,n}\div C_m)$$

for indices $m, n\in N$ (take $n\geqq N(m)$)

$$K\backslash C_m\subset L_g\cup(K_m\div B_{m,n})\cup(B_{m,n}\backslash C_m)\cup(C_m\backslash B_{m,n})$$

and consequently from Lemma 3.2(2)

$$K\backslash C_m \in N(\mu_\sigma, \operatorname{cls} 5g)$$

for indices $m \geqq m(g)$.

3. Let

$$\bar{\bar{A}} = \cap(B: B \in \mathscr{C}(A, V, W)) \quad \text{and} \quad \bar{A} = \cup(C: C \in \mathscr{J}(A, V, V_\delta)) .$$

Then $\bar{\bar{A}}\backslash\bar{A} \subset D$ for all sets $D \in \mathscr{E}(A, V, V_\delta, W)$ and since μ-lim$(D: D \in \mathscr{E}(A, V, V_\delta, W) = \varnothing$, for each neighborhood $g \in \mathscr{G}$, there exists a set $D_g \in \mathscr{E}(A, V, V_\delta, W)$ with $D_g \in N(\mu_\sigma, g)$. Consequently, the set $\bar{\bar{A}}\backslash\bar{A}$ has arbitrarily small covers.

PROPOSITION 3.5. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W.*

1. *For each decreasing sequence $A_n \in V_\sigma \cap P(X, W)$, $n \in N$ with $A = \bigcap_n A_n \in V_\sigma$, we have $\mu_\sigma(A) = \lim_n \mu_\sigma(A_n)$.*

2. *For each increasing sequence $A_n \in V_\delta$, $n \in N$ with $A = \bigcup_n A_n \in V_\delta$, we have $\mu_\delta(A) = \lim_n \mu_\delta(A_n)$.*

PROOF. 1. Let $g \in \mathscr{G}$ be arbitrary and choose $g_n \in \mathscr{G}$, with $\sum_{k=1}^n g_k \subset g$ for all $n = 1, 2, 3, \cdots$. For each index $n \in N$, choose $B_n \in V$, $B_n \subset A_n$ such that $A_n\backslash B_n \in N(\mu_\sigma, g_n)$ and define $C_n = \bigcap_{k=1}^n B_k$. Note that the sequence $C_n \in V$, $n \in N$, decreases to its intersection and for each $n \in N$

$$B_n\backslash C_n = \bigcup_{k=1}^{n-1} B_n\backslash B_k \subset \bigcup_{k=1}^{n-1} A_n\backslash B_k \subset \bigcup_{k=1}^{n-1} A_k\backslash B_k .$$

Applying Lemma 3.2(1), for each index $n \in N$, we have

$$B_n\backslash C_n \in N(\mu_\sigma, \operatorname{cls}(g_1 + \cdots + g_{n-1})) ,$$

consequently, for each index $n \in N$, $B_n\backslash C_n \in N(\mu_\sigma, g)$ so that

$$\mu_\sigma(A_n) - \mu(C_n) = (\mu_\sigma(A_n) - \mu(B_n)) + (\mu(B_n) - \mu(C_n))$$

or

$$\mu_\sigma(A_n) - \mu(C_n) = \mu_\sigma(A_n\backslash B_n) + \mu(B_n\backslash C_n) \in 2g .$$

Since the sequence $C_n \in V$, $n \in N$ converges, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $\mu(C_n) \in \lim_n \mu(C_n) + g$. Therefore, for indices $n \geqq n(g)$ we have

$$\mu_\sigma(A_n) \in \lim_n \mu(C_n) + 3g .$$

In particular, if the sequence $A_n \in V_\sigma$, $n \in N$ decreases to the empty set, then the sequence $C_n \in V$, $n \in N$, decreases to the empty set and consequently $\lim_n \mu_\sigma(A_n) = \theta$.

If the sequence $A_n \in V$, $n \in N$ decreases to a set $A \in V_\sigma$, we set $A = \bigcup_n B_n$ with $B_n \in V$, $B_n \subset B_{n+1}$, $n \in N$, and set $C_n = A_n \backslash B_n$, $n \in N$ and note that the sequence $C_n \in V_\sigma$, $n \in N$, decreases to the empty set. From above, $\lim_n \mu_\sigma(A_n \backslash B_n) = \theta$ or $\lim_n (\mu_\sigma(A_n) - \mu(B_n)) = \theta$. Then for any neighborhood $g \in \mathscr{G}$, there exists an index $n(g)$ such that $n \geqq n(g)$ yields

$$\mu_\sigma(A_n) - \mu(B_n) \in g$$

and

$$\mu_\sigma(A) - \mu(B_n) \in g .$$

Consequently, for indices $n \geqq n(g)$, we have

$$\mu_\sigma(A_n) - \mu_\sigma(A) = (\mu_\sigma(A_n) - \mu(B_n)) + (\mu(B_n) - \mu_\sigma(A)) \in 2g$$

so that

$$\lim_n \mu_\sigma(A_n) = \mu_\sigma(A) .$$

2. Let $A_n \in V_\delta$, $n \in N$, increase to a set $A \in V_\delta$, and let $g \in \mathscr{G}$ be arbitrary. Since we assume $A \in V_\delta$, there exists a set $B \in V$ with $A \subset B$. The sequence $B \backslash A_n \in V_\sigma$, $n \in N$ decreases to the set $B \backslash A \in V_\sigma$. Applying part 1 of this proposition, we have $\mu_\sigma(B \backslash A) = \lim \mu_\sigma(B \backslash A_n)$. Therefore, there exists an index $n(g)$ such that $n \geqq n(g)$ yields $\mu_\sigma(B \backslash A_n) - \mu_\sigma(B \backslash A) \in g$. Applying Lemma 3.1(2) and rearranging terms, we have

$$\mu_\sigma(A) - \mu_\delta(A_n) = (\mu(B) - \mu_\sigma(B \backslash A)) + (\mu_\sigma(B \backslash A_n) - \mu(B))$$

so that

$$\mu_\delta(A) - \mu_\delta(A_n) = \mu_\sigma(B \backslash A) - \mu_\sigma(B \backslash A_n) \in g$$

for all $n \geqq n(g)$.

Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W. Then for each set $A \in V_c(\mu)$, the net $\mu_\delta(C)$: $C \in \mathscr{I}(A, V, V_\delta)$ is Cauchy in the group E. Indeed, if this were not the case, then there would exist a neighborhood $g \in \mathscr{G}$ such that for each set $C \in \mathscr{I}(A, V, V_\delta)$, there exists a set $C' \in \mathscr{I}(A, V, V_\delta)$ with $C \subset C'$ and $\mu_\delta(C') - \mu_\delta(C) \notin g$. Starting with any set $C_0 \in \mathscr{I}(A, V, V_\delta)$ we may apply the hypothesis inductively to choose an increasing sequence $C_n \in \mathscr{I}(A, V, V_\delta)$ with $\mu_\delta(C_{n+1}) - \mu_\delta(C_n) \notin g$. The last condition is a contradiction to Theorem 3.1. For each set $A \in V_c(\mu)$, the completion is defined by the relation $\mu_c(A) = \lim (\mu_\delta(C)$: $C \in$

$\mathscr{F}(A, V, V_\delta)$). The additivity of the function μ_δ on the class V_δ insures that the completion $\mu_c(\cdot)$ is finitely additive on the ring $V_c(\mu)$. Moreover, the completion $\mu_c(\cdot)$ represents an extension of the functions $\mu_\sigma(\cdot)$ and $\mu_\delta(\cdot)$.

THEOREM 3.3. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the family W and assume that the family $\mathscr{G}$ is countable. Then the family $V_c(\mu) \subset P(X, W)$ is the smallest delta ring which is closed under W-dominated, increasing convergence and contains the ring V and the family $\theta(\mu) \cap P(X, W)$ of W-dominated μ-null sets.*

PROOF. Let U be a delta ring containing the ring V and closed under W-dominated increasing convergence and let $A \in V_c(\mu)$ be arbitrary. Since the family $\mathscr{G}$ is countable, there exists an increasing sequence $C_n \in \mathscr{F}(A, V, V_\delta)$, $n \in N$ such that μ-$\lim C_n = A$ and $A \backslash \bigcup_n C_n \in \theta(\mu)$. Using the properties of the delta ring U, $C = \bigcup_n C_n \in U$. Since the family U is a ring containing the family $\theta(\mu) \cap P(X, W)$, $A = C \cup (A \backslash C) \in U$.

THEOREM 3.4. *Let $\mu \in ca(V, E)$ be Rickart on the ring V relative to the class W and $(W, \mathscr{B})$-upper complete for an additive family $\mathscr{B}$ of closed neighborhoods of the origin. Then the completion $\mu_c \in ca(V_c, E)$ is $(W, \mathscr{B})$-upper complete.*

PROOF. Let $A_n \in V_c(\mu)$, $n \in N$, be an increasing sequence for which $A_n \in N(\mu_c, b)$, $n \in N$, and a set $b \in \mathscr{B}$. Let $b_n, b'_n \in \mathscr{B}$, $n \in N$, be chosen to satisfy the relations $3b'_n \subset b_n$ and $\sum_{k=1}^{n} b_k \subset b$ for all $n = 1, 2, 3, \cdots$. Using the definition of the ring $V_c(\mu)$, choose sequences $B_n \in \mathscr{C}(A_n, V, W)$, $C_n \in \mathscr{F}(A, V, V_\delta)$, $n \in N$, with $B_n \backslash C_n \in N(\mu_\sigma, b'_n)$ for all indices $n \in N$. For each index $m \in N$, we assume $B_m = \bigcup_n B_{m,n}$, with $B_{m,n} \in V$, $B_{m,n} \subset B_{m,n+1}$ for indices $n \in N$. For any pair of indices $m, n \in N$, set $K_{m,n} = \bigcup_{r=1}^{m} B_{r,n}$. Proceeding as in the proof of Theorem 3.1, for each $m \in N$ there exists an index $N(m)$ such that

$$K_{m,n} \backslash B_{m,n} \in N(\mu_\sigma, \operatorname{cls}(b_1 + \cdots + b_{m-1}))$$

for all $n \geq N(m)$.

For each index $m \in N$, we set $K_m = \bigcup_n K_{m,n}$. Since the family W is directed upward by set inclusion and $K_{m,n} \subset B_1 \cup \cdots \cup B_m$ for all indices $m, n \in N$, the sequence $K_{m,n}$, $n \in N$ is W-dominated. Now, for any pair of indices $m, n \in N$, we have

$$K_{m,n} = B_{m,n} \cup (K_{m,n} \backslash B_{m,n})$$

and

$$B_{m,n} \subset B_m \in N(\mu_\sigma, \text{cls } 2b) .$$

To see the last relation, recall that we have $B_m = C_m \cup (B_m \backslash C_m)$ for all indices $m \in N$. But $C_m \subset A_m \in N(\mu_c, b)$, $m \in N$ and $\mu_\delta \subset \mu_c$ so that $C_m \in N(\mu_\delta, b)$. From Lemma 3.2(2), for all indices $m \in N$, $B_m \in N(\mu_\sigma, \text{cls } 2b)$.

Finally, for indices $m, n \in N$ with $n \geqq N(m)$, we have $K_{m,n} \backslash B_{m,n} \in N(\mu_\sigma, b)$ so that (by Lemma 3.2(1))

$$K_{m,n} \in N(\mu_\sigma, \text{cls } 4b)$$

for indices $n \geqq N(m)$, $m \in N$. Since the sequence $K_{m,n}$, $n \in N$ is W-dominated, for each index $m \in N$, we have

$$K_m = \lim_n K_{m,n} \in N(\mu_\sigma, \text{cls } 4b) .$$

Since the sequence K_m, $m \in N$ is increasing and the measure $\mu \in ca(V, E)$ is $(W, \mathscr{B})$-upper complete, it follows that the sequence $K_m \in V_\sigma$, $m \in N$ is W-dominated. Consequently, the sequence A_m, $m \in N$ is W-dominated.

§ 4. Extensions of Rickart vector measures.

In this section the results of the previous two sections are applied to Banach space valued measures.

Let V be a ring of subsets of an abstract space X and let $(E, | \ |)$ denote a Banach space. Denote by $C(V)$ the space of all subadditive and increasing functions from the ring V into the non-negative reals R^+ which vanish at the empty set. The space $C(V)$ is called the space of contents on the ring V and elements are referred to as contents. Since the ring V is an abelian group with respect to the symmetric difference operation $\div$ defined by $A \div B = (A \backslash B) \cup (B \backslash A)$ for sets $A, B \in V$, each content $p \in C(V)$ generates a semimetric on the group $(V, \div)$ by the relation

$$\rho(A, B) = p(A \div B)$$

for sets $A, B \in V$.

This semimetric is invariant in the sense

$$\rho(A, B) = \rho(A \div C, B \div C)$$

for sets $A, B, C \in V$.

Consequently, any family of contents $P \subset C(V)$ generates a uniform topology on the group $(V, \div)$. A pair (V, P), where the ring V is endowed with the topology generated by the family $P \subset C(V)$, will be called a topological ring of sets.

Topological rings of sets of the above type were studied in detail by Bogdan and Oberle, [6], Drewnowski, [8], and Labuda, [13] in connection with extensions of the Vitali-Hahn-Saks theorem. As will be seen in this section, topological rings occur quite naturally in the study of vector measures.

Let $Q \subset C(V)$ be any family of contents. The topological ring (V, Q) is said to be a dominated convergence ring of sets if for each V-dominated sequence $A_n \in V$, $n \in N$, with $A_n \to A$, we have $A \in V$ and $\lim_n q(A_n \div A) = 0$ for each $q \in Q$. The topological ring (V, Q) is said to be an absolute convergence ring of sets if for each sequence $A_n \in V$, $n \in N$ for which $\sum_n q(A_n) < \infty$ for each content $q \in Q$, we have $A = \bigcup_n A_n \in V$ and $\lim_n q(A \backslash \bigcup_{k=1}^n A_k) = 0$ for each content $q \in Q$. A topological ring which is both a dominated and absolute convergence ring of sets is called a monotone convergence ring of sets.

For each vector charge $\mu \in a(V, E)$, the semivariation is defined for each set $A \in P(X)$ by the relation

$$p(A, \mu) = \sup(|\mu(B)|: B \in V, B \subset A) .$$

The semivariation $p(\cdot, \mu): P(X) \to [0, \infty]$ is increasing on the σ-algebra $P(X)$ and subadditive on the ring V. The space $ab(V, E)(cab(V, E))$ of locally bounded vector charges (measures) consists of those charges (measures) for which the restriction of the semivariation to the ring V is a content. Additionally, if $\mathscr{B}$ is defined to be the family of all multiples of the unit sphere, then each locally bounded vector measure $\mu \in cab(V, E)$ is $(\Sigma(\mu), \mathscr{B})$-upper complete for the class $\Sigma(\mu)$ of V_σ-sets of finite semivariation. The family $\Sigma(\mu)$ is known as the family of "summable V_σ" sets.

Let $\mu \in ab(V, E)$ be a vector charge. Then the topological ring (V, μ) as introduced in section 2 is nothing more than the topological ring $(V, p(\cdot, \mu))$ generated as above by the content $p(\cdot, \mu) \in C(V)$. For a general (without finite semivariation on the ring V) vector charge $\mu \in a(V, E)$, the topological ring (V, μ) is topologically equivalent to the ring $(V, p_\psi(\cdot, \mu))$ where for a set $A \in P(X)$

$$p_\psi(A, \mu) = \sup(\psi(|\mu(B)|): B \in V, B \subset A)$$

and $\psi(r) = r/(1+r)$ for $r \in [0, \infty)$ and $\psi(\infty) = 1$.

THEOREM 4.1. *Let V be a ring of subsets of an abstract space X and let E be a Banach space, and let $\mathscr{B}$ denote the family of positive* *multiples of the unit sphere in E. For a locally bounded vector measure* $\mu \in cab(V, E)$*, the following are equivalent.*

1. *The measure* μ *is Rickart on the ring V relative to the family* $\Sigma(\mu)$ *of summable* V_σ *sets.*

2. *There exists a delta ring* V_c *and a vector measure* $\mu_c \in cab(V_c, E)$ *such that*
   a. $V \subset V_c$
   b. $\mu \subset \mu_c$
   c. *The measure* μ_c *is upper complete on the ring* V_c.

3. *There exists a monotone convergence ring of sets* (V_c, p_c) *such that*
   a. *The ring V is dense in the topological ring* (V_c, p_c)
   b. *The measure* μ *is* p_c*-continuous on V.*

PROOF. From Theorems 3.2 and 3.4 the vector measure μ admits an extension $\mu_c \in cab(V_c, E)$ such that the topological ring (V_c, μ_c) is a $(\Sigma(\mu), \mathscr{B})$ absolute convergence ring of sets. Since the ring V_c is a delta ring and μ_c is countably additive, the topological ring (V_c, μ_c) is a dominated convergence ring of sets.

A Banach space $(E, |\ |)$ is said to satisfy Gould's property (Gould, [9]) if each sequence $e_n \in E$, $n \in N$, which is bounded away from zero, has the following property: For each number $M > 0$, there exists a finite set $\Delta(M) \subset N$ such that $|\sum_{k \in \Delta(M)} e_k| \geqq M$. The characterizations of unconditional and weak unconditional summability given by Bessaga and Pelczynski, [3], insure that the Banach space with Gould's property are precisely those Banach spaces which do not contain a copy of the space c_0 (the space of null convergent sequences of scalars with the uniform norm). The usefulness of Banach spaces with Gould's property results from the fact that each locally bounded charge taking values in a Banach space with Gould's property is Rickart.

COROLLARY. *Let V be a ring of subsets of an abstract space X and let* $(E, |\ |)$ *be a Banach space which does not contain a copy of the space* c_0. *The following are equivalent for a locally bounded vector measure* $\mu \in cab(V, E)$.

1. *There exists a delta ring* V_c *and a vector measure* $\mu_c \in cab(V_c, E)$ *such that*
   a. $V \subset V_c$
   b. $\mu \subset \mu_c$
   c. *The measure* μ_c *is upper complete on the delta ring* V_c.

2. *There exists a monotone convergence ring of sets* (V_c, p_c) *such that*
   a. *The ring V is* p_c*-dense in the ring* V_c

b. *The measure μ is p_c-continuous on the ring V.*

A pointwise bounded sequence $\mu_n \in ab(V, R)$, $n \in N$ is said to be uniformly Rickart on the ring V if the vector charge $\mu: V \to l_\infty$ given by the relation $\mu(A) = \langle \mu_n(A), n \in N \rangle$ is Rickart on the ring V relative to the family $\Sigma(\mu) = \{A \in V_\sigma : \sup_n p(A, \mu_n) < \infty\}$.

The following theorem was constructively established by Areskin, [1], for the case of scalar valued volumes on an algebra of sets.

THEOREM 4.2. *Let $\mu_n \in ca(V, E)$, $n \in N$, be uniformly Rickart and converge pointwise to zero on the ring V. Then there exists a monotone convergence ring of sets (U, p) such that $V \subset U$ and the sequence $\langle \mu_n, n \in N \rangle$ admits a sequence of extensions $\langle \bar{\mu}_n, n \in N \rangle$ which is p-equicontinuous and pointwise convergent to zero on the delta ring U.*

PROOF. For each set $A \in V$, consider the sequence $\langle \mu_n(A), n \in N \rangle$ as the value $\mu(A)$ of a vector charge $\mu: V \to c_0(E)$. From the Rickart condition and the countable additivity of each coordinate charge, the charge $\mu \in a(V, c_0(E))$ is countably additive on the ring V and Rickart on the ring V relative to the family $\Sigma(\mu) = \{A \in V_\sigma : \sup_n p(A, \mu_n) < \infty\}$. Applying Theorem 4.1, there exists a delta ring V_e and a vector measure $\mu_e \in cab(V_e, c_0(E))$ such that $V \subset V_e$, $\mu \subset \mu_e$ and the topological ring $(V_e, p(\cdot, \mu_e))$ is topologically complete. The definition of the range space insures that the measure μ_e has a representation $\mu_e(A) = \langle \mu_n^e(A), n \in N \rangle \in c_0(E)$ for each set $A \in V_e$. Consequently, $\mu_n \subset \mu_n^e$ and for each set $A \in V_e$, $\lim_n \mu_n^e(A) = 0$.

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