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EXTENSIONS OF MONOTONE OPERATOR FUNCTIONS

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ABSTRACT. It is shown that a monotone operator function f defined on an open subset Δ of the real numbers may be extended to a monotone operator function on the convex hull of Δ.

Let f be a bounded real-valued Borel-measurable function defined on a Borel subset Δ of the real numbers. Let A be a bounded selfadjoint operator on a separable complex Hilbert space H such that the spectrum $\sigma(A)$ of A is contained in Δ. Then $f(A)$ is the selfadjoint operator on H defined by

$$\langle f(A)\phi, \psi\rangle = \int_{\sigma(A)} f(\lambda)\langle E(d\lambda)\phi, \psi\rangle, \qquad \phi, \psi \in H,$$

where E is the resolution of the identity corresponding to A.

The function f is said to be a *monotone operator function* on Δ if $f(A) \leq f(B)$ whenever A and B are bounded selfadjoint operators on a Hilbert space H such that $\sigma(A) \subseteq \Delta$, $\sigma(B) \subseteq \Delta$, and $A \leq B$. If f satisfies this monotonicity condition for the totality of finite-dimensional complex Hilbert spaces, then f is said to be a *monotone matrix function* on Δ.

The following characterization of monotone operator functions is essentially due to Loewner [5]. Loewner proved the theorem for monotone matrix functions; Bendat and Sherman [1] proved that a function f is monotone matrix on an open interval (a,b) if and only if f is monotone operator on (a,b).

THEOREM 1. *A real-valued function f defined on an open interval (a,b) is a monotone operator function on (a,b) if and only if f is analytic on (a,b), can be analytically continued onto the upper half-plane, and represents there a holomorphic function with nonnegative imaginary part.*

It will be shown that there is a similar characterization of monotone operator functions defined on an arbitrary open subset Δ of the real numbers.

Let f be a monotone operator function on an open subset Δ of the real numbers. By Theorem 1, f is continuously differentiable on Δ; see also [4, p. 73]. Associated with f is the kernel K, defined on $\Delta \times \Delta$ by

$$(1) \qquad K(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \quad (\lambda \neq \mu), \qquad K(\lambda, \lambda) = f'(\lambda).$$

Loewner made extensive use of this kernel in his study of monotone matrix

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functions. The following theorem is essentially due to Loewner [5]; see also [1] and [4].

THEOREM 2. *Let f be a monotone operator function on an open subset Δ of the real numbers. Then K is a positive matrix; that is,*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} K(\lambda_j,\lambda_k)c_j\bar{c}_k \geq 0$$

whenever $\lambda_1,\ldots,\lambda_n$ *belong to* Δ *and* $c_1,\ldots,c_n$ *are complex numbers, for* $n = 1, 2, 3, \ldots$.

PROOF. Let A and B be self adjoint operators on an n-dimensional Hilbert space H, and let A and B have spectral representations

$$A = \sum_{j=1}^{n} \lambda_j E_j, \qquad B = \sum_{k=1}^{n} \mu_k F_k,$$

where $\lambda_1,\ldots,\lambda_n$ and $\mu_1,\ldots,\mu_n$ belong to Δ. An easy computation shows that

$$f(B) - f(A) = \sum_{j=1}^{n}\sum_{k=1}^{n} K(\lambda_j,\mu_k)E_j(B-A)F_k. \tag{2}$$

Let $B = A + \epsilon P$ be a one-dimensional perturbation of A, say $P = \langle\cdot,\phi\rangle\phi$ for some ϕ in H, such that $\epsilon > 0$ and $\sigma(A+\epsilon P) \subseteq \Delta$. Then B has a spectral representation

$$B = \sum_{k=1}^{n} \lambda_k(\epsilon)E_k(\epsilon),$$

where $\lambda_k(\epsilon) \to \lambda_k$ as $\epsilon \downarrow 0$ $(k = 1,\ldots,n)$. Equation (2) shows that, for any ψ in H,

$$\begin{aligned} 0 &\leq \langle (1/\epsilon)[f(B) - f(A)]\psi,\psi\rangle \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} K(\lambda_j(\epsilon),\lambda_k)\langle E_j P E_k(\epsilon)\psi,\psi\rangle; \end{aligned}$$

take $\epsilon \downarrow 0$ to show that

$$\sum_{j=1}^{n}\sum_{k=1}^{n} K(\lambda_j,\lambda_k)\langle E_k\psi,\phi\rangle\langle\phi,E_j\psi\rangle \geq 0.$$

Now choose $\phi = \sum_{j=1}^{n} c_j e_j$ and $\psi = \sum_{k=1}^{n} e_k$, where $E_j e_j = e_j$ and $\|e_j\| = 1$ $(j = 1,\ldots,n)$, to show that

$$\sum_{j=1}^{n}\sum_{k=1}^{n} K(\lambda_j,\lambda_k)c_j\bar{c}_k \geq 0.$$

Loewner [5] proved that the converse of Theorem 2 is true if Δ is an open interval (a,b). This result, together with Theorem 1, shows that a real-valued continuously differentiable function f, defined on an open interval (a,b), admits an analytic continuation F onto the upper half-plane Π^+ with

$\operatorname{Im}(F) \geq 0$ on Π^+, if and only if the kernel K associated with f is a positive matrix.

This aspect of Loewner's work was generalized by Rosenblum and Rovnyak [6, Theorem 8]; the following theorem is an immediate consequence of their generalization.

THEOREM 3. *Let f be a real-valued continuously differentiable function on an open subset Δ of the real numbers, and let K be the kernel associated with f, as in* (1). *If K is a positive matrix, then there exists a function F such that*

(i) *F is separately holomorphic on the upper and lower half-planes,*
(ii) *F has nonnegative imaginary part on the upper half-plane, and*
(iii) *F may be analytically continued by reflection across Δ so that*

$$f(x) = F(x + i0) = F(x - i0), \qquad x \in \Delta.$$

The next theorem is due to Šmul′jan [7]; it depends heavily on the work of Davis [2], [3] on convex operator functions.

THEOREM 4. *Let f be a monotone operator function on $\{a\} \cup (b,c)$, where $-\infty < a < b < c < \infty$. Then there exists a unique monotone operator function g defined on (a,c) such that $g = f$ on (b,c) and $g(a+0) \geq f(a)$.*

Šmul′jan proved this theorem for the special case $a = 0$, $b = 1$, and $c = 2$. As noted in [7], the general case follows by considering $f \circ h$, where h is the unique linear fractional transformation such that $h(0) = a$, $h(1) = b$, and $h(2) = c$.

Theorems 3 and 4 are now combined to yield a characterization of monotone operator functions defined on arbitrary open subsets of the real numbers.

THEOREM 5. *Let f be a monotone operator function defined on an open subset Δ of the real numbers. Then there exists a monotone operator function h defined on the convex hull D of Δ such that $f = h$ on Δ.*

PROOF. If f is a monotone operator function on Δ, then f is continuously differentiable on Δ. By Theorem 2, the kernel K associated with f is a positive matrix. By Theorem 3, f admits an analytic continuation F onto the upper half-plane that satisfies (i)—(iii) of that theorem.

Let a and c be arbitrary points in Δ with $a < c$. Since Δ is open, there is a point b in Δ such that $a < b < c$ and $(b,c) \subseteq \Delta$. By Theorem 4, there exists a monotone operator function g on (a,c) such that $f = g$ on (b,c). By Theorem 3, g admits an analytic continuation G onto the upper half-plane that satisfies (i)—(iii) of that theorem, with Δ replaced by (a,c).

Let F and G be analytically continued by reflection across Δ and (a,c), respectively, as in Theorem 3(iii). But $F = f$ on Δ, $G = g$ on (a,c), and $f = g$ on (b,c); since the holomorphic functions F and G coincide on (b,c), they coincide on the intersection of their domains. Thus $F = G$ on Π^+, and so F may be analytically continued across $\Delta \cup (a,c)$ by reflection. Since a and c are arbitrary points in Δ, it follows that F may be analytically continued across the convex hull D of Δ by reflection.

Let H be this extension of F to the complex plane slit along $(-\infty, \infty) - D$, and let h be the restriction of H to D. Since H has nonnegative imaginary part in the upper half-plane and since h is real-valued and analytic on D, h is a monotone operator function on D, by Theorem 1. Since $h = f$ on Δ, h is the desired extension of f to D.

The converse of Theorem 5 follows directly from the definition of "monotone operator function": if g is a monotone operator function on an interval D and if Δ is an open subset of D, then $g|\Delta$ is a monotone operator function on Δ.

Note that Theorem 3 was essential to the proof of Theorem 5; Loewner's original theorem (Theorem 1 above) does not provide the necessary single analytic continuation of f onto the upper half-plane.

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