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EXTENSIONS OF ONE PRIMITIVE INVERSE SEMIGROUP BY ANOTHER

JANET E. AULT

1. Introduction and summary. Every inverse semigroup containing a primitive idempotent is an ideal extension of a primitive inverse semigroup by another inverse semigroup. Consequently, in developing the theory of inverse semigroups, it is natural to study ideal extensions of primitive inverse semigroups (cf. [**3**; **7**]). Since the structure of any primitive inverse semigroup is known, an obvious type of ideal extension to consider is that of one primitive inverse semigroup by another. In this paper, we will construct all such extensions and give an abstract characterization of the resulting semigroup.

The problem of extending one primitive inverse semigroup by another can be essentially reduced to that of extending one Brandt semigroup by another Brandt semigroup. The latter problem has been solved by Lallement and Petrich in [**3**] in case the first Brandt semigroup has only a finite number of idempotents. The method employed there depends heavily on the finiteness condition. Using an entirely different approach, we are able to drop that restriction, and show that the solution in the general case is analogous to that of the finite case in [**3**]. That is, if V is an ideal extension of one Brandt semigroup by another, then multiplication in V is completely determined by a cardinal number, a group homomorphism, and several independent functions between sets (with no algebraic restrictions placed on the functions). Therefore, in so far as semigroups are concerned (i.e., modulo groups), we consider the problem solved. The generalization to primitive inverse semigroups is made easily, requiring just an extra condition on one of the functions.

Those semigroups which can be constructed as an ideal extension of one primitive inverse semigroup by another are exactly those inverse semigroups whose idempotents have height less than three. More generally, for $n \geqq 1$, an inverse semigroup whose idempotents have height less than $n + 1$ is one which can be constructed as n repeated ideal extensions of one primitive inverse semigroup after another. Unfortunately, the notation needed to actually construct all such semigroups becomes very involved and the conditions lengthy. Such a construction will not be given here for n greater than two.

2. Terminology. For a nonempty set I, let $|I|$ be the cardinality of I and $\mathscr{I}_I$ denote the semigroup of all partial one-to-one transformations on I (written as operators on the right). For $\alpha \in \mathscr{I}_I$, $\mathbf{d}\alpha$ and $\mathbf{r}\alpha$ denote the domain

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and range of α, respectively, and rank $\alpha = |\mathbf{r}\alpha|$. For $A \subseteq I$, $\mathfrak{S}(A)$ is the symmetric group over A.

For G a group and P a subsemigroup of $\mathscr{I}_I$, the *wreath product of G with P*, denoted by G wr P, is defined on the set $\{(\theta, \alpha)|\alpha \in P, \theta: \mathbf{d}\alpha \to G\}$, with multiplication as follows:

$$(\theta, \alpha)(\psi, \beta) = (\theta \cdot {}^{\alpha}\psi, \alpha\beta),$$

where $i(\theta \cdot {}^{\alpha}\psi) = (i\theta)(i\alpha\psi)$ for all $i \in \mathbf{d}\alpha\beta$. We will be interested in the situation where P is the symmetric group, and thus G wr P is also a group, and the instance when $P = \mathscr{I}_I$. For $(\theta, \alpha) \in G$ wr P, we define $\mathbf{d}(\theta, \alpha)$ to be $\mathbf{d}\alpha$.

In general, the terminology used here is that of Clifford and Preston [**1**]. For a semigroup S, $\Omega(S)$ is the translational hull of S and $\Pi(S)$ is the inner part of $\Omega(S)$. If S has a zero, that is, $S = S^0$, then S^* denotes $S\backslash 0$. A function $\theta: S^* \to T$, T a semigroup, satisfying $(a\theta)(b\theta) = (ab)\theta$ if $ab \neq 0$, is called a partial homomorphism. For S, E_S is the set of idempotents.

For aesthetic reasons which will become apparent in Theorem 2, we make the following departure from the notation of [**1**]. A Brandt semigroup $\mathscr{M}^0(I, G, I; \Delta)$ over a group G is defined on the set $(I \times G \times I) \cup 0$ with multiplication: $(i, g, j)(j, h, k) = (i, gh, k)$, and all other products equal to 0. The class of completely 0-simple inverse semigroups is (up to isomorphism) the class of Brandt semigroups.

A primitive inverse semigroup $S = S^0$ is one in which every nonzero idempotent is primitive. Every such semigroup S can be written uniquely as the orthogonal sum of Brandt semigroups $B_\alpha, \alpha \in A$, that is, $S = \cup B_\alpha$, $B_\alpha \cap B_\beta = B_\alpha B_\beta = 0$ if $\alpha \neq \beta$. (See [**2**, Corollary 5.17] or [**1**, Theorem 6.39].)

A semigroup V is an ideal extension of a semigroup S by a semigroup $T = T^0$, if S is an ideal of V and T is isomorphic to the Rees quotient V/S. From now on, we shall merely call V an extension of S by T.

Let C be a class of semigroups with zero, and let $S_1, S_2, \ldots, S_n$ be members of C. Let $V_1 = S_1$, and for $k > 1$, let V_k be an extension of S_k by V_{k-1}. Then $V = V_n$ is called an *n-chain of C-semigroups*, $S_1, S_2, \ldots, S_n$.

3. Preliminary results. Let V be an ideal extension of a semigroup S by a semigroup T. If S is weakly reductive, the extension is completely determined by a partial homomorphism θ of T^* into $\Omega(S)$ satisfying $(a\theta)(b\theta) \in \Pi(S)$ if $ab = 0$ in T (see [**3**] and [**1**, Theorem 4.21]). Such a function is called an *extension function for S by T*. More generally, if R is another semigroup and $\sigma: R \to \Omega(S)$ is an isomorphism, with $\psi: T^* \to R$ a function such that $\psi\sigma$ is an extension function for S by T, then we will say that ψ is an extension function for S by T.

Consequently, knowledge of the structure of the translational hull of S is crucial to the problem of finding all extensions of S. Petrich has shown that a Brandt semigroup $\mathscr{M}^0(I, G, I; \Delta)$ has its translational hull isomorphic to G wr $\mathscr{I}_I$, and under this isomorphism $\Pi(S) \simeq \{(\theta, \alpha)|\text{ rank } \alpha \leqq 1\}$ (see

[**8**, Theorem 8; **9**, Theorem 1]). With this in mind, we restate [**1**, Theorem 4.21] for our particular situation.

THEOREM 1. *Let* $S = \mathscr{M}^0(I, G, I; \Delta)$ *be a Brandt semigroup and* T *be a semigroup with zero. Let* $\chi: T^* \to G \text{ wr } \mathscr{I}_I$ *be a partial homomorphism, denoted by* $\chi: t \to (\psi_t, \theta_t)$, *satisfying the property that* rank $\theta_a\theta_b \leqq 1$ *if* $ab = 0$ *in* T. *On* $V = S \cup T^*$, *define a multiplication* $\cdot$ *as follows: for* $s, t \in T^*$, $(i, g, j) \in S$,

$$(i, g, j) \cdot t = (i, g(j\psi_t), j\theta_t) \qquad \text{if } j \in \mathbf{d}\theta_t;$$
$$t \cdot (i, g, j) = (i\theta_t^{-1}, (i\theta_t^{-1}\psi_t)g, j) \quad \text{if } i \in \mathbf{r}\theta_t;$$

if $st = 0$ *in* T,

$$s \cdot t = (k\theta_s^{-1}, (k\theta_s^{-1}\psi_s)(k\psi_t), k\theta_t) \quad \text{if } k = \mathbf{r}\theta_s \cap \mathbf{d}\theta_t;$$
$$a \cdot b = ab \quad \text{if } a, b \in S, \text{ or } a, b \in T^* \text{ and } ab \neq 0;$$

and all other products equal to 0. *Then under this multiplication,* V *is an extension of* S *by* T. *Conversely, every extension of* S *by* T *can be obtained in this way for some such function* χ.

It is clear that, because of the isomorphism between $\Omega(S)$ and G wr $\mathscr{I}_I$, χ is an extension function for S by T. In particular, χ is a partial homomorphism, $\chi: t \to (\psi_t, \theta_t)$, if and only if the function $\theta: t \to \theta_t$ is a partial homomorphism of T^* into $\mathscr{I}_I$ and for $st \neq 0$,

$$(1) \qquad (i\psi_s)(i\theta_s\psi_t) = i\psi_{st} \quad \text{if } i \in \mathbf{d}\theta_{st}.$$

In the general situation, if S is a primitive inverse semigroup, then S is the orthogonal sum of Brandt semigroups, $B_\alpha, \alpha \in A$, and the translational hull of S is isomorphic to the direct product of the translational hulls of the B_α. Furthermore, under the same isomorphism, $\Pi(S)$ corresponds to the set

$$\{(x_\alpha)_{\alpha \in A} | \quad \text{for some } \beta, x_\beta \in \Pi(B_\beta)\}.$$

Using this, we see that for a semigroup $T = T^0$,

$$\theta: T^* \to \bigtimes_{\alpha \in A} \Omega(B_\alpha)$$

is an extension function for S by T if and only if $t\theta = (t\theta_\alpha)_{\alpha \in A}$ where $\theta_\alpha: T^* \to \Omega(B_\alpha)$ is an extension function for B_α by T, for all $\alpha \in A$, and if $st = 0$ in T,

$$(2) \qquad (s\theta_\alpha)(t\theta_\alpha) \neq 0 \quad \text{for at most one } \alpha \in A.$$

Consequently, the problem of finding extensions of a primitive inverse semigroup S is essentially reduced to that of finding extension functions of Brandt semigroups (with the restriction imposed by (2)).

4. The construction. We first construct all extensions of one Brandt semigroup by another Brandt semigroup, and then generalize the result to that of primitive inverse semigroups. All functions will be written as operators on the right.

THEOREM 2. *Let* $S = \mathcal{M}^0(I, G, I; \Delta)$ *and* $T = \mathcal{M}^0(X, H, X; \Delta)$ *be two disjoint Brandt semigroups. Let* ν *be a cardinal number, with* $\nu \leqq |I|$, *and let* $\mathcal{A}_\nu = \{A \subseteq I \mid |A| = \nu\}$, *with* $A_0 \in \mathcal{A}_\nu$ *fixed. Define the following:*

(i) *let* $\pi: X \to \mathcal{A}_\nu$ *be any function such that* $|x\pi \cap y\pi| \leqq 1$ *if* $x \neq y$;

(ii) *let* Φ *be a homomorphism of* H *into* G wr $\mathfrak{S}(A_0)$, *with* $(h)\Phi$ *denoted by* (σ_h, f_h);

(iii) *for* $x \in X$, *let* $\tau_x: A_0 \to x\pi$ *be a one-to-one correspondence, and* $\rho_x: A_0 \to G$ *be any function.*

Then $\chi: T^* \to G$ wr $\mathcal{I}_I$, *defined by* $t\chi = (\psi_t, \theta_t)$ *with*

$$\theta_{(x,h,y)} = \tau_x^{-1} f_h \tau_y \quad \textit{for all } (x, h, y) \in T^*, \tag{3}$$

$$i\psi_{(x,h,y)} = (i\tau_x^{-1}\rho_x)^{-1}(i\tau_x^{-1}\sigma_h)(i\tau_x^{-1}f_h\rho_y) \quad \textit{for all } i \in x\pi, \tag{4}$$

is an extension function for S *by* T.

Conversely, every extension function for S *by* T *can be so obtained for some* ν, π, Φ, τ_x, ρ_x.

Proof. A function $\Phi: H \to G$ wr $\mathfrak{S}(A_0)$ is a homomorphism with $\Phi: h \to (\sigma_h, f_h)$, if and only if the function $f: H \to \mathfrak{S}(A_0)$ defined by $f: h \to f_h$ is a homomorphism and for $g, h \in H$,

$$(i\sigma_g)(if_g\sigma_h) = i\sigma_{gh} \quad \text{for all } i \in A_0. \tag{5}$$

Let θ_t, ψ_t be as defined in (3) and (4). We shall show that χ satisfies the conditions of Theorem 1. Define θ by $\theta: t \to \theta_t$. It is clear that θ maps T^* into $\mathcal{I}_I$. Also, since $\tau_y\tau_y^{-1}$ is the identity map on A_0 and f is a group homomorphism, the function θ is a partial homomorphism. The condition on π insures that θ satisfies the property: rank $\theta_a\theta_b \leqq 1$ if $ab = 0$ in T. For, if $(x, g, y), (w, h, z) \in T^*$ and $y \neq w$, then $|y\pi \cap w\pi| \leqq 1$. Since $\mathbf{r}\tau_y = y\pi$, $\mathbf{r}\tau_w = w\pi$, we have

$$\begin{aligned} \text{rank } \theta_{(x,g,y)}\theta_{(w,h,z)} &= |\mathbf{r}(\tau_x^{-1}f_g\tau_y)(\tau_w^{-1}f_h\tau_z)| \\ &\leqq |\mathbf{r}\tau_y\tau_w^{-1}| = |\mathbf{r}\tau_y \cap \mathbf{d}\tau_w^{-1}| \\ &= |\mathbf{r}\tau_y \cap \mathbf{r}\tau_w| = |y\pi \cap w\pi| \leqq 1. \end{aligned}$$

Now, it must be shown that θ_t and ψ_t satisfy (1). If $(x, g, y) \in T^*$, then $\psi_{(x,g,y)}$ maps $\mathbf{d}\theta_{(x,g,y)}$ into G since $\mathbf{d}\theta_{(x,g,y)} = x\pi$. Further, using (3) and (4), we see that for $i \in x\pi$,

$$\begin{aligned} i\theta_{(x,g,y)}\psi_{(y,h,z)} &= \{(i\tau_x^{-1}f_g\tau_y)\tau_y^{-1}\rho_y\}^{-1}\{(i\tau_x^{-1}f_g\tau_y)\tau_y^{-1}\sigma_h\}\{(i\tau_x^{-1}f_g\tau_y)\tau_y^{-1}f_h\rho_z\} \\ &= (i\tau_x^{-1}f_g\rho_y)^{-1}(i\tau_x^{-1}f_g\sigma_h)(i\tau_x^{-1}f_{gh}\rho_z). \end{aligned}$$

Using this, (4), and (5), we get

$$\begin{aligned} &(i\psi_{(x,g,y)})(i\theta_{(x,g,y)}\psi_{(y,h,z)}) \\ &\quad = (i\tau_x^{-1}\rho_x)^{-1}(i\tau_x^{-1}\sigma_g)(i\tau_x^{-1}f_g\rho_y)(i\tau_x^{-1}f_g\rho_y)^{-1}(i\tau_x^{-1}f_g\sigma_h)(i\tau_x^{-1}f_{gh}\rho_z) \\ &\quad = (i\tau_x^{-1}\rho_x)^{-1}(i\tau_x^{-1}\sigma_g)(i\tau_x^{-1}f_g\sigma_h)(i\tau_x^{-1}f_{gh}\rho_z) \\ &\quad = (i\tau_x^{-1}\rho_x)^{-1}(i\tau_x^{-1})\sigma_{gh}(i\tau_x^{-1})f_{gh}\rho_z \\ &\quad = i\psi_{(x,gh,z)} = i\psi_{(x,g,y)(y,h,z)}. \end{aligned}$$

Hence $\psi_{(x,g,y)}$ satisfies (1), and consequently, χ satisfies the conditions of Theorem 1, and is an extension function for S by T.

Conversely, let χ: $T^* \to G$ wr $\mathscr{I}_I$ be an extension function for S by T. Then $t\chi = (\psi_t, \theta_t)$, and θ: $T^* \to \mathscr{I}_I$, defined by θ: $t \to \theta_t$, is a partial homomorphism. Since T is 0-simple, and partial homomorphisms preserve $\mathscr{J}$-classes, it follows that θ maps T^* into a $\mathscr{J}$-class of $\mathscr{I}_I$. A $\mathscr{J}$-class in $\mathscr{I}_I$ consists of all elements with the same rank (cf. [**3**]). Thus, there exists a cardinal number ν, $\nu \leqq |I|$, so that rank $\theta_t = \nu$, for all $t \in T^*$. Since θ_t is one-to-one, $|\mathbf{d}\theta_t| = |\mathbf{r}\theta_t| = \nu$.

Let $\mathscr{A}_\nu$ be as before, and define π: $X \to \mathscr{A}_\nu$ by

$$x\pi = \mathbf{d}\theta_{(x,1,x)} \quad \text{for all } x \in X.$$

Since χ satisfies the conditions of Theorem 1, for $x \neq y$, rank $\theta_{(x,1,x)}\theta_{(y,1,y)} \leqq 1$. In addition, $\theta_{(x,1,x)}$ is an idempotent, so $\mathbf{d}\theta_{(x,1,x)} = \mathbf{r}\theta_{(x,1,x)} = x\pi$, and thus $|x\pi \cap y\pi| \leqq 1$. Hence π satisfies (i).

Let $x_0 \in X$ be fixed and $A_0 = x_0\pi$. Define a function f on H by

$$f_h = \theta_{(x_0,h,x_0)} \quad \text{for all } h \in H.$$

Then f maps H into $\mathfrak{S}(A_0)$, since

$$\mathbf{d}\theta_{(x_0,h,x_0)} = \mathbf{r}\theta_{(x_0,h,x_0)} = \mathbf{d}\theta_{(x_0,1,x_0)} = \mathbf{r}\theta_{(x_0,1,x_0)} = x_0\pi = A_0,$$

and further, f is a homomorphism.

For $h \in H$, define σ_h: $A_0 \to G$ by the following:

$$\sigma_h = \psi_{(x_0,h,x_0)}.$$

Then, using (1), for $i \in A_0$,

$$\begin{aligned}(i\sigma_g)(if_g\sigma_h) &= (i\psi_{(x_0,g,x_0)})(i\theta_{(x_0,g,x_0)}\psi_{(x_0,h,x_0)})\\ &= i\psi_{(x_0,g,x_0)(x_0,h,x_0)} = i\psi_{(x_0,gh,x_0)} = i\sigma_{gh}.\end{aligned}$$

Thus, by (5), Φ: $H \to G$ wr $\mathfrak{S}(A_0)$, defined by $(h)\Phi = (\sigma_h, f_h)$ is a homomorphism.

Further, for $x \in X$, let

$$\tau_x = \theta_{(x_0,1,x)}, \quad \text{and } \rho_x = \psi_{(x_0,1,x)}.$$

Then, for $(x, h, y) \in T^*$,

$$\begin{aligned}\theta_{(x,h,y)} &= \theta_{(x,1,x_0)(x_0,h,x_0)(x_0,1,y)}\\ &= \theta_{(x,1,x_0)}\theta_{(x_0,h,x_0)}\theta_{(x_0,1,y)} = \tau_x^{-1}f_h\tau_y,\end{aligned}$$

as in (3).

Now to see that ψ can be defined as in (4), let $x \in X$ and $i \in x\pi$. Then

$$\begin{aligned}i\psi_{(x,1,x)} &= (i\psi_{(x,1,x)})(i\theta_{(x,1,x)})\psi_{(x,1,x)}\\ &= (i\psi_{(x,1,x)})(i\psi_{(x,1,x)}).\end{aligned}$$

Since $i\psi_{(x,1,x)}$ is in G, and it is idempotent, it follows that $i\psi_{(x,1,x)} = e$, the identity of G, for all $i \in x\pi$. Moreover,

$$\begin{aligned}(i\psi_{(x,1,x_0)})(i\tau_x^{-1}\psi_{(x_0,1,x)}) &= (i\psi_{(x,1,x_0)})(i\theta_{(x,1,x_0)}\psi_{(x_0,1,x)}) \\ &= i\psi_{(x,1,x_0)(x_0,1,x)} \\ &= i\psi_{(x,1,x)} = e.\end{aligned}$$

Hence $i\psi_{(x,1,x_0)} = (i\tau_x^{-1}\psi_{(x_0,1,x)})^{-1} = (i\tau_x^{-1}\rho_x)^{-1}$. Finally, for $(x, h, y) \in T^*$, $i \in x\pi$,

$$\begin{aligned}i\psi_{(x,h,y)} &= i\psi_{(x,h,x_0)}(i\theta_{(x,h,x_0)}\psi_{(x_0,1,y)}) \\ &= (i\psi_{(x,1,x_0)})(i\theta_{(x,1,x_0)}\psi_{(x_0,h,x_0)})(i\theta_{(x,h,x_0)}\psi_{(x_0,1,y)}) \\ &= (i\tau_x^{-1}\rho_x)^{-1}(i\tau_x^{-1}\sigma_h)(i\tau_x^{-1}f_h\rho_y),\end{aligned}$$

as in (4). Hence the extension function χ can be constructed by means of the functions π, Φ, τ_x, ρ_x, $x \in X$.

We shall say that the function χ (and thus the extension V) is determined by $\{\nu, \pi, \Phi; \tau_x, \rho_x\}$. We note that the parameters used are either functions between sets with no algebraic restrictions placed on the functions, or else a homomorphism of one group into another. These functions are actually independent of one another since we can define $\tau_A: A_0 \to A$ for all $A \in \mathscr{A}_\nu$, and denote $\tau_{x\pi}$ by τ_x. Consequently, in so far as semigroups are concerned, every extension of one Brandt semigroup by another can be constructed via the independent parameters π, Φ, τ_x, ρ_x, $x \in X$.

By changing the conditions on the function π, we can change the nature of the resultant function χ. If there is no restriction on π, then χ ranges over all partial homomorphisms. If the restriction that $x\pi \cap y\pi = \emptyset$ if $x \neq y$ is added, then every homomorphism is exactly of the form χ. In particular, with this added restriction on π, the nontrivial homomorphisms of T into $\mathscr{I}_I$ are just those of the form θ, where

$$(x, h, y)\theta = \tau_x^{-1}f_h\tau_y, \quad 0\theta = 0.$$

Having found all extensions of one Brandt semigroup by another, we now use (2) for the general case where S and T are arbitrary primitive inverse semigroups.

THEOREM 3. *Let S be the orthogonal sum of Brandt semigroups S_j,*

$$S_j = \mathscr{M}^0(I_j, G_j, I_j; \Delta), \quad j \in J,$$

and T be the orthogonal sum of Brandt semigroups T_α,

$$T_\alpha = \mathscr{M}^0(X_\alpha, H_\alpha, X_\alpha; \Delta), \quad \alpha \in A,$$

with the index sets I_j, X_α, all disjoint. For every $\alpha \in A, j \in J$, let $\chi^{\alpha j}: T^* \to G_j \text{ wr } \mathscr{I}_{I_j}$ be an extension function for S_j by T_α, determined by the set $\{\nu^{\alpha j}, \pi^{\alpha j}, \Phi^{\alpha j}, \tau_x^{\alpha j}, \rho_x^{\alpha j}\}$, with the functions $\pi^{\alpha j}$ satisfying the following conditions: for $\alpha, \beta \in A$ and $x \in X_\alpha, y \in X_\beta$, with $x \neq y$,

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(i) $|x\pi^{\alpha j} \cap y\pi^{\beta j}| \leqq 1$ *for all* $j \in J$;
(ii) *there exists at most one* $j \in J$ *such that* $x\pi^{\alpha j} \cap y\pi^{\beta j} \neq \emptyset$.
Then

$$\chi: T^* \to \bigtimes_{j \in J} G_j \operatorname{wr} \mathscr{I}_{I_j}$$

defined by $\chi: t \to (t\chi^{\alpha j})_{j \in J}$ *if* $t \in T_\alpha^*$, *is an extension function for* S *by* T.
Conversely, every extension of S *by* T *gives rise to a function* χ *so defined.*

Proof. The first condition on $\pi^{\alpha j}$, $\alpha \in A$, $j \in J$, is necessary and sufficient that $\chi^j: T^* \to G_j \operatorname{wr} \mathscr{I}_{I_j}$ defined by

$$(x, g, y)\chi^j = (x, g, y)\chi^{\alpha j} \quad \text{if } (x, g, y) \in T_\alpha^*,$$

be an extension function for S_j by T.
Now using (2) and the fact that

$$\Omega(S) \simeq \bigtimes_{j \in J} G_j \operatorname{wr} \mathscr{I}_{I_j},$$

we see that χ is an extension function for S by T if and only if χ^j is an extension function for S_j by T for each $j \in J$, and if $st = 0$ in T,

$$(6) \qquad (s\chi^j)(t\chi^j) \neq 0 \quad \text{for at most one } j \in J.$$

But, by definition of χ^j, (6) can be stated as follows: if $s \in T_\alpha^*$, $t \in T_\beta^*$, $st = 0$, then

$$(7) \qquad (s\chi^{\alpha j})(t\chi^{\beta j}) \neq 0 \quad \text{for at most one } j \in J.$$

Since $\chi^{\alpha j}$ is determined by $\{\nu^{\alpha j}, \pi^{\alpha j}, \Phi^{\alpha j}; \tau_x^{\alpha j}, \rho_x^{\alpha j}\}$, the definition of $\chi^{\alpha j}$ in Theorem 2 shows that for $x \in X_\alpha$, $\mathbf{d}(x, 1, x)\chi^{\alpha j} = \pi^{\alpha j}$. Hence (7) is equivalent to the statement: if $x \in X_\alpha$, $y \in X_\beta$, $x \neq y$, then $(x\pi^{\alpha j}) \cap (y\pi^{\beta j}) \neq \emptyset$ for at most one $j \in J$. Thus χ is an extension function for S by T if and only if $\chi^{\alpha j}$ is an extension function for S_j by T_α for every $j \in J$, $\alpha \in A$, with (i) and (ii) satisfied.

We have constructed all extensions V of an arbitrary primitive inverse semigroup S by any other primitive inverse semigroup T. Now, given another primitive inverse semigroup R, we might attempt to construct all extensions of R by V. This would involve extension functions $\chi: S^* \to \Omega(R)$, $\chi': T^* \to \Omega(R)$, satisfying certain relationships. The functions χ, χ' can be constructed as in Theorem 3, with the restrictions being placed on the various functions defined. However, even with just three primitive inverse semigroups S, T, R, the notation becomes very cumbersome and the restrictions lengthy. Consequently, we merely note here that it is possible to repeat the process in Theorems 2 and 3 a finite number of times, with many conditions at each step, resulting in a (not very satisfactory) construction of all semigroups which can be considered as n-chains of primitive inverse semigroups. If interest is restricted to certain types of extensions, for example, strict, then our method might be used repeatedly with more success.

5. Characterization of the extension. It is unfortunate that we do not have a nice construction of all semigroups which are n-chains of primitive inverse semigroups, for an arbitrary natural number n, since the resultant semigroup can be described abstractly in a very natural way. For the case $n = 2$, we have an abstract characterization of those semigroups constructed in Theorem 3.

We need the following definition which can be found in [**6**]. Let P be a partially ordered set with zero, 0. For $e \in P$, height e is defined by

$$\text{height } e = \max\{n | \, e = e_n > e_{n-1} > \ldots > e_0 = 0\}$$

if the maximum exists; otherwise, height $e = \infty$. For $A \subseteq P$,

$$\text{height } A = \max\{\text{height } e | \, e \in A\}.$$

THEOREM 4. *A semigroup $V = V^0$ is a semigroup which is an n-chain of primitive inverse semigroups if and only if V is an inverse semigroup with height $E_V \leqq n$.*

Proof. If V is an n-chain of primitive inverse semigroups $S_1, \ldots, S_n$, then V is an inverse semigroup since each S_i is an inverse semigroup. Now if $e \in V$ and $e = e_m > \ldots > e_0 = 0$, then each $e_i \neq 0$ is in a distinct $S_{i'}$, and thus $m \leqq n$.

Conversely, let V be an inverse semigroup with height $E_V \leqq n$. The result will be proved by induction on n. If $n = 1$, then V is a primitive inverse semigroup. Let $n > 1$. Let $\mathscr{J}$ be the usual Green's relation on V, and J_i, $i \in I$, be the distinct nonzero $\mathscr{J}$-classes. Now J_i^0 is a 0-simple inverse semigroup and has a primitive idempotent, since every idempotent is of finite height in V and thus in J_i^0. Hence J_i^0 is a primitive inverse semigroup, in particular, a Brandt semigroup.

Now let $e \in J_i$ with height $e = 1$. Then if f is another idempotent in J_i, we claim that height $f = 1$. For, if height $f > 1$, then there exists $f_0 \in E_V$ such that $0 < f_0 < f$. Since J_i^0 is a Brandt semigroup, $f_0 \notin J_i$. In addition, J_i^0 is 0-simple, so there exists $x \in J_i$ such that $xx^{-1} = e$ and $x^{-1}x = f$. Now $f_0 < f$, so

$$(xf_0)^{-1}(xf_0) = f_0x^{-1}xf_0 = f_0ff_0 = f_0,$$

and $xf_0 \neq 0$. On the other hand,

$$0 \neq (xf_0)(xf_0)^{-1} = xf_0x^{-1} \leqq xx^{-1} = e.$$

Since height $e = 1$, it must be that $e = (xf_0)(xf_0)^{-1}$. But then e and f_0 are in the same $\mathscr{J}$-class. However, this is contrary to the assumption, so it must be that height $f = 1$ and all idempotents in J_i have height equal to one.

Let $P = (\bigcup_{i \in I'} J_i) \cup 0$, where height $E_{J_i} = 1$ if and only if $i \in I'$. It is clear that under the multiplication in V, P is a primitive inverse semigroup. To see that P is an ideal of V, let $x \in P$ and $s \in V$. Then $(xs)(xs)^{-1} \leqq xx^{-1}$, and height $xx^{-1} \leqq 1$. Hence $(xs)(xs)^{-1} \in J_i$ for some $i \in I'$ and $xs \in J_i \subseteq P$. It is similarly shown that P is a left ideal and thus an ideal of V.

Now V is an extension of P by V/P and V/P is an inverse semigroup with height $E_{(V/P)} \leqq n - 1$. By induction, V/P is an $(n - 1)$-chain of primitive inverse semigroups, $S_1, \ldots, S_{n-1}$. Consequently, V is an n-chain of primitive inverse semigroups, $S_1, \ldots, S_{n-1}, P$.

COROLLARY. *A semigroup V is an extension of one primitive inverse semigroup by another if and only if V is an inverse semigroup with* height $E_V \leqq 2$.

The results in § 4 are contained in the author's doctoral dissertation written under the direction of Professor Mario Petrich at The Pennsylvania State University.

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University of Florida,
Gainesville, Florida