EXTENSIONS OF PURE STATES.[*]

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1. **Introduction and preliminaries.** The main concern of this paper is the problem of uniqueness of extensions of pure states from maximal abelian self-adjoint algebras of operators on a Hilbert space to the algebra of all bounded operators on that space. The answer, as many of us have suspected for several years, is in the negative. To the best of our knowledge, the problem is not recorded in the literature. We heard of it first from I. E. Segal and I. Kaplansky, though it is difficult to credit a problem which stems naturally from the physical interpretation and the inherent structure of a subject. This problem has arisen, in one form or another, in our work on several different occasions; and we have been gathering bits of information related to it, over the years. (The present solution is prompted by just such a reappearance.)

To state the problem precisely, let $\mathcal{H}$ be a (complex) Hilbert space and $\mathfrak{A}$ an algebra of bounded operators invariant under the adjoint operation $(A \rightarrow A^*)$, containing the identity operator, I, and closed in the uniform (operator bound) topology. The algebra, $\mathfrak{A}$, is a C^*-algebra, and a linear functional, f, on $\mathfrak{A}$ which is 1 at I and real, non-negative on positive operators (those operators, A, such that $(Ax, x) \geqq 0$ for each x in $\mathcal{H}$), is a state of $\mathfrak{A}$. The set of states of $\mathfrak{A}$ is a convex subset of the dual of $\mathfrak{A}$ and is compact in the w^*-topology on the dual (the weak topology induced by $\mathfrak{A}$). The Krein-Milman theorem tells us that the set of states is the closed convex hull of its extreme points—these are the pure states of $\mathfrak{A}$. An argument of the Hahn-Banach type enables us to extend states from a C^*-subalgebra of $\mathfrak{A}$ to $\mathfrak{A}$ [4]. The set of extensions of such a state forms a compact convex subset of the dual whose extreme points can easily be shown to be pure states of $\mathfrak{A}$ [4], provided that the state of the subalgebra is pure. Thus, if a pure state has a unique pure state extension from a C^*-subalgebra of a C^*-algebra to the algebra, then the closed convex hull of this extension, viz.

itself, is the set of all state extensions of the given pure state. Thus a pure state of a C^*-subalgebra has a unique state extension to the algebra if and only if it has a unique pure state extension.

If the C^*-algebra $\mathfrak{A}$ is abelian, the set of pure states is compact, and the natural mapping of $\mathfrak{A}$ into the second dual, followed by the restriction mapping to the set of pure states, is an algebraic, isometric, order isomorphism of $\mathfrak{A}$ onto the algebra of continuous, complex-valued functions on this compact space (the pure state space). (This isomorphism carries the adjoint operation in $\mathfrak{A}$ into complex conjugation in the function space.) An easy Zorn's lemma argument shows that $\mathfrak{A}$ is contained in a maximal abelian, self-adjoint subalgebra, $\mathcal{C}$, of the algebra, $\mathcal{B}$, of all bounded operators on $\mathcal{H}$. (Making use of the decomposition of operators as a sum of a self-adjoint and a skew-adjoint operator, it is not difficult to show that $\mathcal{C}$ is maximal with respect to the property of being abelian.) The fact that bounded families of operators in $\mathcal{C}$ have least upper bounds causes the pure state space of $\mathcal{B}$ to be extremely disconnected (i. e., the closure of each open set is open [8]). Examples of maximal abelian, self-adjoint algebras arise from the multiplication algebras on $L_2(X, \mu)$, with μ a measure on the space X, i. e. the algebra of operators T_g, with g an essentially bounded, μ-measurable function on X, where $T_g(h) = gh$, for each h in $L_2(X, \mu)$. In particular, with X the unit interval and μ Lebesgue measure on X, the algebra, $\mathcal{C}_c$, which arises will be referred to as 'the continuous maximal abelian algebra'; and with X, the integers, and $\mu(n) = 1$, for all n, the algebra, $\mathcal{C}_d$, which arises will be referred to as 'the discrete algebra.' (The maximal abelian algebras on finite-dimensional spaces are constructed as $\mathcal{C}_d$ is, with the integers replaced by a finite set. The algebra, $\mathcal{C}_d$, can also be viewed as the set of bounded diagonal matrices relative to a complete orthonormal basis.) Each maximal abelian algebra on a separable Hilbert space is unitarily equivalent to $\mathcal{C}_c$, $\mathcal{C}_d$, a finite-dimensional maximal abelian algebra, or the direct sum of $\mathcal{C}_c$ with one of the last two types. The problem of extensions of pure states from maximal abelian algebras to $\mathcal{B}$ (we consider mainly the separable case throughout this paper) reduces then, to a study of extensions from $\mathcal{C}_c$ and $\mathcal{C}_d$. Although we refer to $\mathcal{C}_d$ as 'the discrete maximal abelian algebra,' it should be noted that there is a great deal of 'non-discreteness' about it. In fact, its pure state space is easily identified with the β-compactification of the integers [9].

Each unit vector, x, gives rise to a state of $\mathcal{B}$ by means of the mapping, $A \rightarrow (Ax, x)$; and it easily seen that this state, ω_x, is a pure state of $\mathcal{B}$. We refer to ω_x as a 'vector state' (also 'discrete state'). From our previous

remarks, we see that a pure state of an abelian C^*-algebra is multiplicative (the converse is true for all C^*-algebras), and from this, that a vector state is pure on an abelian C^*-algebra if and only if the vector which induces it is an eigenvector for each operator in the algebra. (Note that two vector states of $\mathcal{B}$ are equal if and only if the two vectors differ by a scalar multiple of modulus 1.) Since some operators in $\mathcal{A}_c$ have no eigenvectors, none of the pure states of $\mathcal{A}_c$ is a vector state—yet, each such pure state has a pure state extension to $\mathcal{B}$. Thus, there are pure states of $\mathcal{B}$ which are not vector states—we call these 'singular pure states.' Indeed, the points of the β-compactification of the integers which do not correspond to integers give rise to pure states of $\mathcal{A}_d$ which are not vector states and, therefore, have singular pure state extensions to $\mathcal{B}$. A well-known fact about singular pure states (which can be read out of the results of [3], for example) tells us that the singular pure states are precisely those which annihilate all completely continuous operators.

The uniqueness problem for extensions of pure states is the following: is there a unique state extension of a pure state of a maximal abelian self-adjoint algebra of the algebra, $\mathcal{B}$, of all bounded operators to $\mathcal{B}$? There are, of course, the two subdivisions of this problem—the question for $\mathcal{A}_d$ and the question for $\mathcal{A}_c$. It is quite easy to see that uniqueness of extension cannot be expected for abelian C^*-algebras other than the maximal abelian ones. In fact, if $\mathfrak{A}$ is an abelian C^*-algebra and $\mathcal{A}$ is a maximal abelian one containing it properly, then, making use of the function representation (of $\mathcal{A}$), the Stone-Weierstrass theorem assures us that $\mathfrak{A}$ does not separate pure states of $\mathcal{A}$, i.e., that there are two distinct pure states of $\mathcal{A}$ which agree on $\mathfrak{A}$ (and this restriction to $\mathfrak{A}$, being multiplicative, is pure). Naturally, one wonders why uniqueness should be expected in the maximal abelian case. Classically, our maximal abelian algebra, $\mathcal{A}$, would be that associated with an orthonormal basis, viz. $\mathcal{A}_d$, and the pure state, one due to a basis vector, x. Since the one-dimensional projections on the basis elements lie in $\mathcal{A}_d$, another pure state, ρ, which agrees with ω_x on $\mathcal{A}_d$ will annihilate all these projections with the exception of the one whose range contains x; so that if ρ is a vector state, that vector differs from x by a scalar multiple of modulus 1. Thus $\rho = \omega_x$, when we note that if ρ were not a vector state, it would annihilate all completely continuous operators and, in particular, all one-dimensional projections. More generally, if $\mathcal{A}$ is a maximal abelian algebra, ω_x is a pure state of $\mathcal{A}$, E is the one-dimensional projection whose range contains x, and ω is a pure state extension of ω_x from $\mathcal{A}$ to all bounded operators, then $\omega = \omega_x$.

In fact, x is an eigenvector for each operator in $\mathcal{Q}$, so that $\mathcal{Q}$ leaves the range of E invariant and, hence, commutes with E (since $\mathcal{Q}$ is a self-adjoint algebra). Thus E lies in $\mathcal{Q}$, and ω is a vector state ω_y, since $\omega(E) = \omega_x(E) = 1 \neq 0$. From $\omega_y(E) = \omega_x(E) = 1$, we conclude that $\| E \| = 1$ and that y lies in the range of E. Being a unit vector, y is a scalar multiple of modulus 1 of x, and $\omega_y = \omega_x$, on all bounded operators.

Having these results for vector pure states, it is not unreasonable to expect them to hold for arbitrary pure states in the same way that one passes from certain properties of the point spectrum to those of the general spectrum. Indeed, a casual handling of limit processes (just allowing oneself the minor luxury of a sequential limit in place of a directed limit) leads to a "proof" of the uniqueness of pure state extensions—false but provocative. Add to this evidence the elusiveness of a counter-example and one has the case for the conjecture.

In [5], von Neumann introduces a process for taking the "diagonal part" of certain operators in a von Neumann algebra (strongly closed C^*-algebra) relative to a maximal abelian self-adjoint subalgebra. Among other things, this process is linear, order preserving, and idempotent, and, so, provides a continuous way of simultaneously extending all the pure states of a maximal abelian algebra (provides a cross-section in the sheaf-like structure of state extensions over the pure state space of the maximal abelian algebra, so to speak). Two distinct diagonal processes will, of course, settle the pure state extension problem negatively for a particular maximal abelian algebra. In Section 2, we give a discussion of diagonal processes suitable for our applications, and in 3, we prove the uniqueness of diagonal processes for $\mathcal{Q}_d$ and the non-uniqueness of diagonal processes for $\mathcal{Q}_c$. The non-uniqueness proof is a mixture of abstract and classical techniques which produces a specific operator with distinct "diagonal parts" relative to $\mathcal{Q}_c$ (and, so, to which some pure state of $\mathcal{Q}_c$ can be extended in more than one way). In the last section, we discuss related questions concerning pure states.

2. Diagonal processes. The lemma which follows provides the means for constructing diagonal processes relative to maximal abelian algebras.

LEMMA 1. *If $\mathcal{Q}$ is an abelian von Neumann algebra generated by the projections $\{E\}_{n \in \mathcal{N}}$, $\mathcal{N}$ the set of positive integers, and p is a point of $\beta(\mathcal{N}) - \mathcal{N}$, where $\beta(\mathcal{N})$ is the β-compactification of $\mathcal{N}$, then there is a linear operator, $\mathcal{D}_p$, whose domain is the set of bounded operators and which is such that:*

(a) $\mathcal{D}_p(B)$ commutes with each E_n (so, lies in $\mathcal{a}'$, the commutant of $\mathcal{a}$), and $\mathcal{D}_p(B)$ is a weak closure point of operators $\{B^{|E_1|E_2|\cdots|E_n}\}$, where $T^{|E}$ is $ETE + (I-E)T(I-E)$.

(b) $\mathcal{D}_p(AB) = A\mathcal{D}_p(B)$, for each A in $\mathcal{a}$ (and $\mathcal{D}_p(BA) = \mathcal{D}_p(B)A$).

(c) $\mathcal{D}_p(I) = I$, and $\mathcal{D}_p(B) \geqq 0$ if $B \geqq 0$.

Proof. Note that

$$\| B^{|E} \| = \| EBE + (I-E)B(I-E) \|$$
$$= \max\{\| EBE \|, \|(I-E)B(I-E)\|\} \leqq \| B \|,$$

so that the function, f, defined on $\mathcal{d}$ by $f(n) = B^{|E_1|\cdots|E_n}$ maps $\mathcal{d}$ into the (weakly compact) ball of radius $\| B \|$ about 0 in the set of bounded operators. Note also that $B^{|E|F} = B^{|F|E}$ when $EF = FE$ and that $E(B^{|E}) = (B^{|E})E$; so that $f(n)$ commutes with $E_1, \cdots, E_n$. From the properties of $\beta(\mathcal{d})$, we have that f has a unique extension, f_1, from $\mathcal{d}$ to $\beta(\mathcal{d})$ which is continuous and whose range is contained in the ball of radius $\| B \|$ about 0. We define $\mathcal{D}_p(B)$ to be $f_1(p)$. The observation that $(\alpha B + C)^{|E} = \alpha(B^{|E}) + C^{|E}$, together with the fact that $\mathcal{d}$ is dense in the Hausdorff space $\beta(\mathcal{d})$, yields the linearity of $\mathcal{D}_p$ and the fact that $\mathcal{D}_p(B)$ is a weak closure point of $\{B^{|E_1|\cdots|E_n}\}$. If B is I, then $B^{|E_1|\cdots|E_n} = I$, for all n, so that $\mathcal{D}_p(B) = I$; and if $B \geqq 0$, then $B^{|E_1|\cdots|E_n} \geqq 0$, for all n, whence each weak closure point of $\{B^{|E_1|\cdots|E_n}\}$ is positive and, in particular, $\mathcal{D}_p(B) \geqq 0$. Moreover, $(AB)^{|E_k} = A(B^{|E_k})$ (and $(BA)^{|E_k} = (B^{|E_k})A$), with A in $\mathcal{a}$, so that $\mathcal{D}_p(AB) = A\mathcal{D}_p(B)$ (and $\mathcal{D}_p(BA) = \mathcal{D}_p(B)A$) (recall that left and right multiplication by A is weakly continuous). For a given n_0, $\mathcal{D}_p(B)$ is a weak closure point of $B^{|E_1|\cdots|E_m}$, with $m \geqq n_0$, each of which commutes with E_{n_0}. Thus $\mathcal{D}_p(B)$ commutes with E_{n_0}, for each n_0; so that $\mathcal{D}_p(B)$ lies in $\mathcal{a}'$.

DEFINITION 1. *A linear order preserving mapping from all bounded operators into the commutant of an abelian von Neumann algebra, $\mathcal{a}$, which is the identity on $\mathcal{a}$ is a "diagonal process relative to $\mathcal{a}$." If the image of each operator, B, is a weak closure point of operators $B^{|E_1|\cdots|E_n}$, with $E_1, \cdots, E_n$ in $\mathcal{a}$, the diagonal process is "proper"; otherwise, it is "improper."*

Remark 1. If $\mathcal{D}$ is proper and $B \in \mathcal{a}'$, $\mathcal{D}(B)$ is a weak closure point of $B^{|E_1|\cdots|E_n} = B$, so that $\mathcal{D}(B) = B$.

LEMMA 2. *If $\mathcal{D}$ is a diagonal process relative to $\mathcal{a}$ then $\mathcal{D}(AB)$*

$= A \mathcal{D}(B)$ $(and$ $\mathcal{D}(BA) = \mathcal{D}(B)A)$ for $each$ A in $\mathcal{A}$ and $each$ $bounded$
$operator,$ $B.$ If $\mathcal{D}$ is $weakly$ $continuous$ on the $unit$ $ball,$ $then$ it is the
$unique$ $proper$ $diagonal$ $process$ $relative$ to $\mathcal{A},$ and $\mathcal{D}(B)$ is the $weak$ $limit$
of $\{B_n\},$ $where$ $B_n = B^{|E_1| \cdots |E_n}$ and $\{E_n\}$ is a $generating$ $family$ of $projec$-
$tions$ for $\mathcal{A}.$

Proof. We remark first, that if T and S are distinct operators in $\mathcal{A}'$, there is an extension to $\mathcal{A}'$ of some pure state of $\mathcal{A}$ (in fact, a pure state extension) which differs on T and S. In fact, for each cardinal, n, there is a projection P_n in $\mathcal{A}$ such that $\mathcal{A}P_n$ is an n-fold copy of some maximal abelian algebra, $\mathcal{A}_n$, acting on a Hilbert space, $\mathcal{H}_n$ (i. e. $\mathcal{A}'P_n$ acting on $P_n(\mathcal{H})$ is unitarily equivalent to the algebra of $n \times n$ matrices with entries in $\mathcal{A}_n$ acting on the direct sum of $\mathcal{H}_n$ with itself n times, in the usual way), and $\sum_n P_n = I$. With T and S distinct, $TP_n \neq SP_n$, for some n. If we can establish our result for $\mathcal{A}P_n$ and its commutant $\mathcal{A}'P_n$ (on $P_n(\mathcal{H})$), there is a pure state ρ_n of $\mathcal{A}P_n$ and a state extension, ρ_n', of it to $\mathcal{A}'P_n$ such that $\rho_n'(TP_n) \neq \rho_n'(SP_n)$. Defining ρ and ρ' by $\rho(A) = \rho_n(AP_n)$ and $\rho'(A') = \rho_n'(A'P_n)$, respectively, we note that ρ, being multiplicative, is a pure state of $\mathcal{A}$, ρ' is a state extension of it and $\rho'(T) \neq \rho'(S)$. We may assume therefore that $\mathcal{A}$ is an n-fold copy of the maximal abelian algebra $\mathcal{A}_0$ acting on $\mathcal{H}_0$; from which $\mathcal{A}'$ is the algebra of all $n \times n$ matrices, with entries in $\mathcal{A}_0$, which give bounded operators acting upon the direct sum of $\mathcal{H}_0$ with itself n times. Let X be the pure state space of $\mathcal{A}_0$, so that $\mathcal{A}_0$ is algebraically isomorphic to the algebra, $C(X)$, of complex-valued continuous functions on X. Some entry in the matrix representations of T and S are distinct and, so, differ at a pure state ρ_0 of $\mathcal{A}_0$. Let $\bar{\rho}_0(A')$ be the matrix obtained by replacing each entry of A' by its value at ρ_0, for A' in $\mathcal{A}'$. The operator corresponding to this matrix is bounded and positive if A' is positive. Indeed, with n finite, the boundedness is automatic and the positivity then follows from the fact that $\bar{\rho}_0$ is adjoint-preserving and multiplicative, since ρ_0 is. (Boundedness and positivity can also be established when ρ_0 is not assumed pure by making use of [1].) Thus $\| \bar{\rho}_0(A') \| \leq \| A' \|$, when n is finite. Applying this to the infinite case, we see that each finite minor has norm not exceeding $\| A' \|$, and again $\| \bar{\rho}_0(A') \| \leq \| A' \|$. As in the finite case, it now follows that $\bar{\rho}_0(A')$ is positive if A' is. Since $\bar{\rho}_0(T) \neq \bar{\rho}_0(S)$, there is a unit vector, x_0, in the Hilbert space direct sum of the complex numbers with itself n times such that $(\bar{\rho}_0(T)x_0, x_0) \neq (\bar{\rho}_0(S)x_0, x_0)$. Now $\bar{\rho}_0(A)$ is a scalar multiple of I, for each A in $\mathcal{A}$ so that $A \rightarrow (\bar{\rho}_0(A)x_0, x_0)$ is a pure state, ρ, of $\mathcal{A}$ and $A' \rightarrow (\bar{\rho}_0(A')x_0, x_0)$ is an extension, ρ', of it to $\mathcal{A}'$. By construction,

$\rho'(T) \neq \rho'(S)$. (The set of all state extensions of ρ to $\mathcal{a}'$ is a compact convex subset of the set of states of $\mathcal{a}'$ whose extreme points are pure states of $\mathcal{a}'$. If T and S coincide on each of these pure state extensions of ρ, they coincide on their finite convex combinations, so, on their (w*-) closed convex hull, i.e., on all state extensions of ρ—in particular, on ρ'. Thus T and S differ on some pure state extension to $\mathcal{a}'$ of a pure state of $\mathcal{a}$.)

If ρ is a pure state of $\mathcal{a}$ and ρ' a state extension of ρ to all bounded operators, then $\rho'(AB) = \rho'(A)\rho'(B)$ (and $\rho'(BA) = \rho'(B)\rho'(A)$), for each A in $\mathcal{a}$. In fact, if E is a projection in $\mathcal{a}$, $\rho'(E) = 0$ or 1, since ρ' is multiplicative on $\mathcal{a}$. Thus $\rho'(EB)$ or $\rho'[(I-E)B]$ is 0 (as $\rho'(E)$ is 0 or 1, respectively), by an application of Schwarz's inequality to the inner product $K, H \to \rho'(H^*K)$ on the algebra of bounded operators. In either case, $\rho'(EB) = \rho'(E)\rho'(B)$. Thus $\rho'(AB) = \rho'(A)\rho'(B)$ for operators A in $\mathcal{a}$ which are linear combinations of projections in $\mathcal{a}$, and, by continuity of ρ' in the uniform topology, for uniform limits of such operators. From the spectral theorem, each self-adjoint operator in $\mathcal{a}$ is such a limit, so that $\rho'(AB) = \rho'(A)\rho'(B)$, for each A in $\mathcal{a}$ and each bounded operator, B.

In particular, if ρ'' is a state extension of ρ to $\mathcal{a}'$ and $\rho' = \rho'' \circ \mathcal{D}$, then ρ' is a state extension of ρ to all bounded operators. (Recall that $\mathcal{D}$ is the identity transform on $\mathcal{a}$.) Thus,

$$\rho''(\mathcal{D}(AB)) = \rho'(AB) = \rho'(A)\rho'(B) = \rho'(A)\rho''(\mathcal{D}(B))$$
$$= \rho''(A)\rho''(\mathcal{D}(B)) = \rho''(A\mathcal{D}(B)),$$

with A in $\mathcal{a}$ and B a bounded operator. (Note that the last equality follows from the considerations of the preceding paragraph applied to an extension of ρ'' from $\mathcal{a}'$—and hence of ρ from $\mathcal{a}$—to all bounded operators.) Since $\mathcal{D}(AB)$ and $A\mathcal{D}(B)$ are in $\mathcal{a}'$ and ρ'' is an arbitrary state extension to $\mathcal{a}'$ of an arbitrary pure state of $\mathcal{a}$, $\mathcal{D}(AB) = A\mathcal{D}(B)$, from the results of the first paragraph of this proof.

Suppose, now, that $\mathcal{D}$ is weakly continuous on the unit ball (and, so, on each bounded ball), and that $\mathcal{D}'$ is a proper diagonal process relative to $\mathcal{a}$. In this case, $\mathcal{D}'(B)$ is a weak closure point of $\{B^{|E_1|\cdots|E_n}\}$. Each such weak closure point, A', is such that $\mathcal{D}(A')$ is a weak closure point of $\{\mathcal{D}(B^{|E_1|\cdots|E_n})\} = \{\mathcal{D}(B)\}$, whence $\mathcal{D}(A') = \mathcal{D}(B)$. With A' in $\mathcal{a}'$, $\mathcal{D}(A') = A'$, since $\mathcal{D}$ is proper (cf. Remark 1). Thus $\mathcal{D}'(B) = \mathcal{D}(B)$ and $\mathcal{D}' = \mathcal{D}$. Moreover, each weak limiting point, A', of the sequence $(B^{|E_1|\cdots|E_n})$, lies in $\mathcal{a}'$, since it commutes with each E_n, and is a weak closure point of $\{B^{|E_1|\cdots|E_n}\}$, so that $A' = \mathcal{D}(B)$. Since $\{B^{|E_1|\cdots|E_n}\}$ is

contained in the weakly compact ball of radius $\parallel B \parallel$ about 0, $(B^{|E_1|\cdots|E_n})$ has a weak limiting point which must be $\mathscr{D}(B)$. Thus $\mathscr{D}(B)$ is the weak limit of $(B^{|E_1|\cdots|E_n})$.

The next lemma notes the possibility of extending a positive linear mapping from a linear space of bounded operators containing I into an abelian von Neumann algebra to such mappings of all bounded operators into the abelian von Neumann algebra. The proof is a direct copy of the proof of Krein's extension theorem for states [4] making use of the boundedly complete lattice properties of abelian von Neumann algebras.

LEMMA 3. *If $\mathcal{C}$ is an abelian von Neumann algebra, $\mathfrak{L}_0$ a self-adjoint linear space of bounded operators containing I, and ϕ_0 an adjoint-preserving, positive, linear mapping of $\mathfrak{L}_0$ into $\mathcal{C}$, then ϕ_0 has an adjoint-preserving, positive linear extension with range in $\mathcal{C}$ to the algebra, $\mathscr{B}$, of all bounded operators.*

Proof. Partially order the set of adjoint-preserving, positive, linear mappings with range in $\mathcal{C}$, domain a self-adjoint linear subspace of $\mathscr{B}$ containing $\mathfrak{L}_0$, and which extend ϕ_0, by "function extension". Zorn's lemma applies, and there exists a maximal element, ϕ, with domain $\mathfrak{L}$. If $\mathfrak{L} \neq \mathscr{B}$, there is a self-adjoint operator B not in $\mathfrak{L}$. Choose a positive integer n, such that $nI \geqq B \geqq -nI$. Then $-n\phi(I)$ is a lower bound for the subset $\{\phi(A): A \in \mathfrak{L} \text{ and } A \geqq B\}$ of $\mathcal{C}$ and $n\phi(I)$ is an upper bound for $\{\phi(C): C \in \mathfrak{L} \text{ and } C \leqq B\}$. These subsets have a greatest lower bound, A_1, and least upper bound, A_0, respectively, in $\mathcal{C}$, since $\mathcal{C}$ is a boundedly complete lattice. Since $\phi(A) \geqq \phi(C)$, when $A \geqq B \geqq C$, with A and C in $\mathfrak{L}$, $A_1 \geqq A_0$. Choose A in $\mathcal{C}$ such that $A_1 \geqq A \geqq A_0$, and define ϕ' on the linear space generated by B and $\mathfrak{L}$ as follows: $\phi'(\alpha B + C) = \alpha A + \phi(C)$, with C in $\mathfrak{L}$. Then ϕ' is an adjoint-preserving linear mapping with range in $\mathcal{C}$ and is an extension of ϕ. If $\alpha B + C \geqq 0$, making use of the choice of A in each of the cases, $\alpha = 0$, $\alpha > 0$, $\alpha < 0$, we conclude that ϕ' is a positive mapping. Since $B \notin \mathfrak{L}$, the existence of ϕ' contradicts the maximality of ϕ. Thus $\mathfrak{L} = \mathscr{B}$ and ϕ is an adjoint-preserving, positive, linear extension from $\mathfrak{L}$ to $\mathscr{B}$ of ϕ_0.

Remark 2. With the notation of the preceding lemma, ϕ_0 has a unique positive extension from $\mathfrak{L}_0$ to $\mathscr{B}$ if and only if the greatest lower bound of $\{\phi_0(A): A \text{ in } \mathfrak{L}_0 \text{ and } A \geqq B\}$ is equal to the least upper bound of $\{\phi_0(C): C \text{ in } \mathfrak{L}_0 \text{ and } B \geqq C\}$, for each self-adjoint operator, B, in $\mathscr{B}$. In fact, if they are equal, the positivity condition forces each positive extension of ϕ_0 to take this value at B; and if they are not equal for some self-adjoint B, we may

extend ϕ_0 to the space generated by B and $\mathfrak{L}_0$ by assigning either of these values to B, and then extend the resulting mappings to two distinct positive mappings of $\mathcal{B}$ into $\mathcal{C}$, each of which extends ϕ_0.

Remark 3.   If we take $\mathcal{C}$ as $\mathfrak{L}_0$ and ϕ_0 as the identity mapping on $\mathcal{C}$, the extension lemma guarantees the existence of a diagonal process with range in $\mathcal{C}$, and this diagonal process is improper if $\mathcal{C}$ is not maximal abelian (for a proper process is the identity on $\mathcal{C}'$, the commutant of $\mathcal{C}$). In case $\mathcal{C}$ is maximal abelian, and we proceed as just noted, the extension lemma and the the preceding remark provide another criterion for uniqueness of the diagonal process.

LEMMA 4.   *If $\mathcal{D}$ is a diagonal process relative to the maximal abelian algebra $\mathcal{C}$, there is a *-representation, ϕ, of the algebra, $\mathcal{B}$, of all bounded operators which is an isomorphism on $\mathcal{C}$, and a projection E on the representation space such that*

$$\mathcal{D}(B) = \phi^{-1}[E(\phi(B))E]$$

for each B in $\mathcal{B}$.

Proof.   Let $\{F_\alpha\}$ be a maximal orthogonal family of countably decomposable projections in $\mathcal{C}$, so that $\sum_\alpha F_\alpha = I$. Note that $\mathcal{D}_\alpha$, defined by $\mathcal{D}_\alpha(B) = \mathcal{D}(B)F_\alpha$, for B in the algebra, $\mathcal{B}_\alpha$, of bounded operators on $F_\alpha(\mathcal{H})$ ($\mathcal{H}$ the underlying Hilbert space of $\mathcal{B}$), is a diagonal process relative to the maximal abelian algebra $\mathcal{C}F_\alpha$ (on $F_\alpha(\mathcal{H})$). If we can find a representation ϕ_α of $\mathcal{B}_\alpha$ and a projection E_α with the properties described in the lemma (relative to $\mathcal{C}_\alpha$), then the direct sum, ϕ, of the representations and E, the sum of E_α, establish the result for $\mathcal{D}$.

We may assume that $\mathcal{C}$ is countably decomposable, so that there exists a (unit) separating vector, x, for $\mathcal{C}$. We define a state, ρ, of $\mathcal{B}$ by: $\rho(B) = \omega_x(\mathcal{D}(B))$. From Gelfand-Neumark [1] and Segal [6], ρ gives rise to a *-representation ϕ of $\mathcal{B}$ constructed as follows. The set of operators B in $\mathcal{B}$ such that $\rho(B^*B) = 0$ is a left ideal, $\mathfrak{I}$. The quotient vector space $\mathcal{B}/\mathfrak{I}$ has a positive definite inner product on it defined by

$$[B + \mathfrak{I}, C + \mathfrak{I}] = \rho(C^*B),$$

so that the completion, $\mathcal{H}_0$, of $\mathcal{B}/\mathfrak{I}$ relative to this inner product is a Hilbert space. The mapping, $B + \mathfrak{I} \rightarrow AB + \mathfrak{I}$, on $\mathcal{B}/\mathfrak{I}$ to $\mathcal{B}/\mathfrak{I}$ extends to a bounded operator $\phi(A)$, for each A in $\mathcal{B}$ and ϕ is the *-representation in question. That ϕ is an isomorphism on $\mathcal{C}$ (with range $\mathfrak{A}_0$, let us say) is a

consequence of the definition of ρ. In fact, if $\phi(A) = 0$, then

$$0 = [A + \mathfrak{A}, A + \mathfrak{A}] = \rho(A^*A) = \omega_x(\mathscr{D}(A^*A)) = \| Ax \|^2,$$

for A in $\mathcal{A}$, whence $A = 0$. (Recall that x was chosen as a separating vector for $\mathcal{A}$.)

Let E be the projection on the closure of $\{A + \mathfrak{A} : A \in \mathcal{A}\}$. Our final assertion is that $\phi[\mathscr{D}(B)] = E\phi(B)E$, both operators restricted to $E(\mathscr{H}_0)$. We have

$$[\phi[\mathscr{D}(B)](A + \mathfrak{A}), C + \mathfrak{A}] = \rho(C^*\mathscr{D}(B)A)$$
$$= \omega_x[C^*\mathscr{D}(B)A] = \omega_x[\mathscr{D}(C^*\mathscr{D}(B)A)]$$

and

$$[E\phi(B)E(A + \mathfrak{A}), C + \mathfrak{A}] = [\phi(B)(A + \mathfrak{A}), C + \mathfrak{A}]$$
$$= \rho(C^*BA) = \omega_x[\mathscr{D}(C^*BA)] = \omega_x[C^*\mathscr{D}(B)A],$$

with C and A in $\mathcal{A}$. Thus, as operators on $E(\mathscr{H}_0)$, $\phi[\mathscr{D}(B)] = E\phi(B)E$.

Remark 4. Relative to the scalar algebra, $\{\lambda I\}$ the identity mapping on $\mathscr{B}$ is the unique proper diagonal process, and each state, ρ, of $\mathscr{B}$ yields an improper diagonal process by means of the mapping $B \to \rho(B)I$.

Remark 5. If $\mathscr{D}$ is a diagonal process relative to $\mathcal{A}_c$, then $\mathscr{D}(C) = 0$ for each completely continuous operator, C. In fact, if ρ is a pure state of $\mathcal{A}_c$, $\rho \circ \mathscr{D}$ is a state extension of ρ from $\mathcal{A}_c$ to $\mathscr{B}$ and so, the finite convex combinations of pure state extensions, ρ', of ρ to $\mathscr{B}$ have $\rho \circ \mathscr{D}$ as a w^*-limit point. Now $\rho'(C) = 0$ or else ρ' is a vector state, ω_x. But then ω_x is pure on $\mathcal{A}_c$, so that x is a simultaneous eigenvector for $\mathcal{A}_c$—a contradiction. Thus $\rho'(C) = 0$, so that finite convex combinations of such ρ' annihilate C and $\rho \circ \mathscr{D}(C) = \rho[\mathscr{D}(C)] = 0$. Hence $\mathscr{D}(C) = 0$.

3. Uniqueness and non-uniqueness of diagonal processes. We consider $\mathcal{A}_d$ first and show that there is a unique diagonal process relative to it. Let $\{x_k\}$ be an orthonormal basis for $\mathscr{H}$, the Hilbert space upon which $\mathcal{A}_d$ acts, relative to which each operator in $\mathcal{A}_d$ is diagonal. Let us define $\mathscr{D}(B)$ for a bounded operator, B, to be the operator whose matrix representation relative to $\{x_k\}$ is the diagonal matrix with diagonal that of the matrix representation for B relative to $\{x_k\}$. Clearly, then, $\mathscr{D}$ is a diagonal process relative to $\mathcal{A}_d$. With $x = \sum_k \alpha_k x_k$ and $\| B \| \leq 1$,

$$|(\mathscr{D}(B)x, x)| \leq \sum |\alpha_k|^2 |(\mathscr{D}(B)x_k, x_k)| = \sum |\alpha_k|^2 |(Bx_k, x_k)|,$$

and for suitably large N, $\sum_{k \geq N} |\alpha_k|^2 |(Bx_k, x_k)| < \epsilon/2$. Thus, with

$$|(Bx_k, x_k)| < \epsilon/2 \|x\|, \qquad\qquad k = 1, \cdots, N,$$

we have $|(\mathcal{D}(B)x, x)| \leq \sum |\alpha_k|^2 |(Bx_k, x_k)| < \epsilon$, so that $\mathcal{D}$ is a continuous mapping at 0 on the unit ball of the algebra, $\mathcal{B}$, of all bounded operators in the weak operator topology into $\mathcal{B}$ in this topology. Since $\mathcal{B}$ is a topological linear space in the weak operator topoloygy and $\mathcal{D}$ is linear, $\mathcal{D}$ is continuous on the unit ball of $\mathcal{B}$ in this topology. From Lemma 2, it follows that $\mathcal{D}$ is the unique proper diagonal process relative to $\mathcal{C}_d$. By other considerations, we show that $\mathcal{D}$ is the *unique diagonal process* relative to $\mathcal{C}_d$.

THEOREM 1. *The unique diagonal process relative to $\mathcal{C}_d$ is $\mathcal{D}$.*

Proof. If $\mathcal{D}'$ is a diagonal process distinct from $\mathcal{D}$, then $\mathcal{D}'(B) \neq \mathcal{D}(B)$ for some B in $\mathcal{B}$; so that $(\mathcal{D}'(B)x_k, x_k) \neq (\mathcal{D}(B)x_k, x_k)$, for some k—whence $\omega_{x_k} \circ \mathcal{D}' \neq \omega_{x_k} \circ \mathcal{D}$. But ω_{x_k} is a vector pure state of $\mathcal{C}_d$ and has a unique state extension to $\mathcal{B}$. Thus $\mathcal{D}$ is the unique diagonal process relative to $\mathcal{C}_d$.

Of course, this does not establish that the pure states of $\mathcal{C}_d$ which are not vector states (the points of the β-compactification of the integers other than integer points) have unique state extensions to $\mathcal{B}$.

THEOREM 2. *There is more than one proper diagonal process relative to $\mathcal{C}_c$; pure state extension is not unique relative to $\mathcal{C}_c$.*

Proof. If we represent our Hilbert space, $\mathcal{H}$ as $L_2(0,1)$ under Lebesgue measure and $\mathcal{C}_c$ as the multiplication algebra of this measure space, then the set of projections $\{E_{km} : m = 1, 2, \cdots, k = 1, \cdots, m\}$ corresponding to multiplication by the characteristic function of the closed intervals $[(k-1)/m,$ $k/m]$ generate $\mathcal{C}_c$. Now $I = \sum_{k=1}^{m} E_{km}$, so that $B^{|E_{1m}| \cdots |E_{mm}} = \sum_{k=1}^{m} E_{km} B E_{km}$, and $B^{|E_{1m}| \cdots |E_{mm}|E_{1mn}| \cdots |E_{mnmn}} = \sum_{k=1}^{mn} E_{kmn} B E_{kmn}$. From Lemma 2, if there is a unique diagonal process $\mathcal{D}$ of the form $\mathcal{D}_p$, p in the β-compactification of the integers but not an integer, in particular, if there is a unique proper diagonal process, then $\mathcal{D}(B)$ is the weak limit (with respect to j) of $\sum_{k=1}^{m} E_{km} B E_{km}$, where $m = 2^j$. In fact, if this is not the case for some B, then $\mathcal{D}_p(B) \neq \mathcal{D}_{p'}(B)$ for some points p and p' in $\beta(\mathcal{H}) - \mathcal{H}$. We shall exhibit such a B.

The functions, f_n, defined by $f_n(x) = e^{2\pi inx}$, for $n = 0, \pm 1, \cdots$, form an orthonormal basis for $\mathcal{H}$. As B, we shall take the projection, G, on the subspace spanned by certain of these elements $\{f_{n_j} : j = 1, 2, \cdots\}$ (to be

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specified later). We have

$$(E_{km}GE_{km}(1),1) = \sum_{j=1}^{\infty} |(f_{n_j}, E_{km}(1))|^2 = \sum_{j=1}^{\infty} |\int_{(k-1)/m}^{k/m} e^{2\pi i n_j x}\, dx|^2$$

$$= \sum_{j=1}^{\infty} |(1/2\pi i n_j)[e^{2\pi i n_j k/m} - e^{2\pi i n_j (k-1)/m}]|^2$$

$$= \sum_{j=1}^{\infty} (1/4\pi^2 n_j^2)|e^{2\pi i n_j/m} - 1|^2,$$

whence

$$((\sum_{k=1}^{m} E_{km}GE_{km})(1),1) = \sum_{j=1}^{\infty} (m/4\pi^2 n_j^2)|e^{2\pi i n_j/m} - 1|^2$$

$$= \sum_{j=1}^{\infty} (m/\pi^2 n_j^2)\sin^2(\pi n_j/m).$$

We show that, for a suitable choice of $n_1, n_2, \cdots$,

$$(1) \qquad\qquad (m/\pi^2) \sum_{j=1}^{\infty} (1/n_j^2)\sin^2(\pi n_j/m)$$

does not tend to a limit as m ($= 2^r$) tends to ∞. For our set, $\{n_j\}$, choose all integers in the closed intervals $[2^{2k-2}, 2^{2k-1}]$, $k = 1, 2, \cdots$ (so that $n_1 = 1$, $n_2 = 2, n_3 = 4, n_4 = 5, \cdots$).

Note that (1) may be rewritten as

$$(1/\pi) \sum_{j=1}^{\infty} (\pi n_j/m)^{-2}[\sin^2(\pi n_j/m)](\pi/m) (= a_m)$$

which is the integral over $[0,\infty]$ of the step function, s_m, defined as $(\pi n_j/m)^{-2}[\sin^2(\pi n_j/m)] \cdot (1/\pi)$ on the interval $[\pi(n_j-1)/m, \pi n_j/m]$, $j = 1, 2, \cdots$, and 0 elsewhere, and that, with $m \equiv 0$ (4),

$$(1/\pi) \sum_{m/4 < n_j \leq m/2} (\pi n_j/m)^{-2}[\sin^2(\pi n_j/m)] \cdot (\pi/m)$$

$$= \int_{\pi/4}^{\pi/2} s_m(x)\, dx \ (= b_m).$$

Now, with $m_k = 2^{2k}$, s_{m_k} is a Riemann approximating step function to $\pi^{-1}x^{-2}\sin^2 x$, on the interval $[\pi/4, \pi/2]$. Thus, if a_{2^m} tends to a limit as m tends to ∞, so does a_{m_k} as k tends to ∞, and

$$\lim_m a_{2^m} = \lim_k a_{m_k} \geq \lim_k b_{m_k} = \pi^{-1}\int_{\pi/4}^{\pi/2} x^{-2}\sin^2 x\, dx > \pi^{-2}.$$

(For the last inequality, note that the derivative, $2x^{-3}\sin x\ (x\cos x - \sin x)$, of $x^{-2}\sin^2 x$ is negative on $[\pi/4, \pi/2]$, so that $x^{-2}\sin^2 x > 4\pi^{-2}$ on $[\pi/4, \pi/2)$.)

On the other hand, there are no terms n_j in $(2^{2k-1}, 2^{2k+1-2})$. Thus, with $r_k = 2^{2k+1-2}$, s_{r_k} is 0 on $[\pi 2^{2k-1}(2^{2k+1-2})^{-1}, \pi k/(k+1)]$, whose left end point tends to 0 as k tends to ∞. Since each s_m is bounded (e.g. by 1) on $[0, \pi]$, we have $\lim_{k \to \infty} \int_0^\pi s_{r_k}(x)\, dx = 0$. Thus, if $\lim_m a_{2^m}$ exists, then

$$\pi^{-2} < \lim_k a_{m_k} = \lim_k a_{r_k} = \lim_k \int_0^\infty s_{r_k}(x)\, dx$$

$$= \lim_k \int_\pi^\infty s_{r_k}(x)\, dx \leqq \lim_k (1/\pi) \int_\pi^\infty x^{-2}\, dx = \pi^{-2},$$

a contradiction. (Note that $s_m(x) \leqq \pi^{-1} x^{-2}$, for x in $[0, \infty)$.) Thus $\lim_k a_{2^k}$ does not exist, and G does not have a unique diagonal part relative to $\mathcal{C}_c$ (G is the projection on the space spanned by $e^{2\pi i n_j x}$, where $\{n_j\}$ is as described). From our earlier discussion, there are pure states of $\mathcal{C}_c$ which do not have unique state (and pure state) extensions to all bounded operators (in fact, which have distinct values on G).

4. The pure states. We have noted that non-uniqueness of diagonal processes implies non-uniqueness of pure state extension and that uniqueness of the diagonal process does not lead to uniqueness of pure state extension. The problem of uniqueness of pure state extension (and even that of diagonal processes) may be raised in more refined form. Given a maximal abelian self-adjoint algebra $\mathcal{C}$; for which operators, B, is it the case that all extensions of the same pure state of $\mathcal{C}$ coincide on B?

LEMMA 5. *If $\mathcal{C}$ is a maximal abelian algebra then there exists a sequence of projections $\{E_n\}$ in $\mathcal{C}$ such that $B^{|E_1| \cdots |E_n}$ converges to an operator of $\mathcal{C}$ in the uniform topology if and only if $\rho_1(B) = \rho_2(B)$ for each pair of states, ρ_1, ρ_2, of all bounded operators such that $\rho_1 \mid \mathcal{C} = \rho_2 \mid \mathcal{C}$ is a pure state of $\mathcal{C}$.*

Proof. Suppose that a sequence such as $\{E_n\}$ exists, for the operator B. Then, with ρ_1 and ρ_2 states of all bounded operators whose restrictions to $\mathcal{C}$ are pure and equal, $\rho_1(B^{|E}) = \rho_1(B)$ and $\rho_2(B^{|E}) = \rho_2(B)$ for each projection, E, in $\mathcal{C}$. In fact, $\rho_1(E)$ is 0 or 1, since E is a projection in $\mathcal{C}$ and ρ_1 is pure on $\mathcal{C}$, while $\rho_1(B^{|E}) = \rho_1(E)\rho_1(B)\rho_1(E) + \rho_1(I - E)\rho_1(B)\rho_1(I - E) = \rho_1(B)$ (cf. Lemma 2, second paragraph of the proof). Thus,

$$\rho_1(B^{|E_1| \cdots |E_n}) = \rho_1(B) \text{ and } \lim \rho_1(B^{|E_1| \cdots |E_n}) = \rho_1(A) = \rho_1(B),$$

where $B^{|E_1| \cdots |E_n}$ tends uniformly to the operator A in $\mathcal{C}$ (recall that states of C^*-algebras are continuous in the uniform topology). Similary, $\rho_2(B) = \rho_2(A)$ $(= \rho_1(A) = \rho_1(B))$, so that $\rho_1(B) = \rho_2(B)$.

Suppose now that all extensions of each given pure state of $\mathcal{C}$ coincide on B. Clearly then each diagonal process relative to $\mathcal{C}$ has the same value, A, at B. We shall find $\{E_n\}$ such that $B^{|E_1|\cdots|E_n}$ tends uniformly to A, or equivalently, that $(B-A)^{|E_1|\cdots|E_n} = B^{|E_1|\cdots|E_n} - A$ tends uniformly to 0. Of course, extensions of a given pure state of $\mathcal{C}$ coincide on $B-A$, and have value 0 (since $B-A$ has diagonal 0 under each diagonal process relative to $\mathcal{C}$). We may assume, therefore, that each extension of a pure state of $\mathcal{C}$ has value 0 on B, and that B is self adjoint.

Now $\mathcal{C}$ is *-isomorphic with $C(X)$, where X is extremely disconnected—each point, x_0, of X corresponds to a pure state, ρ_{x_0}, of $\mathcal{C}$ (and conversely). Since each state extension of ρ_{x_0} has the value 0 on B, we have

$$0 = \inf\{\rho_{x_0}(A): A \text{ in } \mathcal{C}, A \geq B\} = \sup\{\rho_{x_0}(A): A \text{ in } \mathcal{C}, B \geq A\}.$$

Thus, we can choose operators A_{x_0} and A^{x_0} in $\mathcal{C}$ such that $A^{x_0} \geq B \geq A_{x_0}$ and $1/n > \bar{A}^{x_0}(x_0) \geq 0 \geq \bar{A}_{x_0}(x_0) > -1/n$, where $\bar{A}$ is the function in $C(X)$ corresponding to an operator, A, in $\mathcal{C}$. It follows that there is a closed-open set, containing x_0, whose characteristic function corresponds to a projection E_{n,x_0} in $\mathcal{C}$, on which $\bar{A}^{x_0}$ is less than $1/n$ and $\bar{A}_{x_0}$ is greater than $-1/n$. Then

$$(1/n) E_{n,x_0} \geq E_{n,x_0} A^{x_0}$$
$$= E_{n,x_0} A^{x_0} E_{n,x_0} \geq E_{n,x_0} B E_{n,x_0} \geq E_{n,x_0} A_{x_0} \geq (-1/n) E_{n,x_0},$$

so that $\| E_{n,x_0} B E_{n,x_0} \| \leq 1/n$. Since X is compact, the closed-open sets corresponding to E_{n,x_0}, for each x_0, cover X and have a finite subcovering. Denote the corresponding projections by $E_{n1}, \cdots, E_{nk_n}$. We may replace this set of projections (using intersections and relative complements) by an orthogonal set of projections in $\mathcal{C}$ each of which is contained in some E_{nj} and such that each E_{nj} is the sum of projections in the new finite set. If E is one of the new projections, contained, say, in E_{n1}, then

$$\| EBE \| = \| EE_{n1} B E_{n1} E \| \leq \| E \|^2 \| E_{n1} B E_n \| \leq 1/n.$$

We may assume that $E_{n1}, \cdots, E_{nk_n}$ are orthogonal, so that their sum is I (their corresponding closed-open sets cover X). Thus

$$\| B^{|E_{n1}|\cdots|E_{nk_n}} \| = \| \sum_{j=1}^{k_n} E_{nj} B E_{nj} \| \leq 1/n.$$

The sequence $E_{11}, E_{12}, \cdots, E_{1k_1}, E_{21}, \cdots, E_{2k_2}, \cdots$ will serve as the desired sequence, $\{E_n\}$.

Remark 6. If the state extensions to $\mathcal{B}$ of a state of a maximal abelian

algebra, $\mathcal{Q}$, coincide on each operator of a certain set, $\mathscr{S}$, (i.e. the restriction of these extensions to $\mathscr{S}$ defines a single-valued function on $\mathscr{S}$) then the same is true for the uniform closure of the self-adjoint linear space generated by $\mathscr{S}$ and for the set of those operators, T, for which there is a family, $\{E_n\}$ of projections in $\mathcal{Q}$ such that $T^{|E_1|\cdots|E_n}$ has a uniform limit in $\mathscr{S}$.

THEOREM 3. *All state extensions to $\mathcal{B}$ of a pure state of $\mathcal{Q}_d$ coincide on each permutation matrix* (i.e. *each linear operator which permutes the eigenvectors of $\mathcal{Q}_d$*).

Proof. Let $\{x_n\}_{n=1,2,\cdots}$ be a basis of eigenvectors for $\mathcal{Q}_d$, and let $Tx_n = x_{\alpha(n)}$, where α is a permutation of $\mathfrak{A}$. Then, the matrix of T relative to $\{x_n\}$ has 1 at each entry $\alpha(n)$, n and zeros at the other entries (it is a permutation matrix). Let $\mathfrak{A}_1$ be the fixed points of α. We shall define three other sets of integers, $\mathfrak{A}_2$, $\mathfrak{A}_3$, and $\mathfrak{A}_4$. Assign to $\mathfrak{A}_2$ the first element of $\mathfrak{A}$ not in $\mathfrak{A}_1$ and suppose that each element of $\mathfrak{A}$ less than n has been assigned to one of $\mathfrak{A}_1, \cdots, \mathfrak{A}_4$ in such a way that j and $\alpha(j)$ are not in the same set if they are distinct. Assign n to the first one of $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$ which contains neither $\alpha(n)$ nor $\alpha^{-1}(n)$, unless $\alpha(n) = n$, in which case, assign n to $\mathfrak{A}_1$. In this way, we construct four pairwise disjoint sets $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$ with union $\mathfrak{A}$ such that $\alpha(n)$ and n lie in no one of them, unless they are equal (in which case it lies in $\mathfrak{A}_1$). Let E_j, $j = 1, \cdots, 4$ be the projection (in $\mathcal{Q}_d$) on the subspace spanned by $\{x_k: k \text{ in } \mathfrak{A}_j\}$. From the construction, $E_1 T E_1 = E_1$ and $E_j T E_j = 0$, $j = 2, 3, 4$, while $E_1, \cdots, E_4$ are mutually orthogonal and have sum I. Thus $T^{|E_1|E_2|E_3|E_4} = E_1$, which lies in $\mathcal{Q}_d$. An application of Lemma 5 completes the proof.

Combining Theorem 3 with Remark 6, we see that pure state extension from $\mathcal{Q}_d$ is unique to the algebra of linear combinations of permutation matrices (and to its uniform closure). We note that the proof of Theorem 3 applies to more general 0, 1 matrices (e.g. to those operators which annihilate some basis vectors and are one-one mappings of the others into the set of basis vectors).

5. Related questions. The results that we have obtained leave the question of uniqueness of extension of the singular pure states of $\mathcal{Q}_d$ open. We incline to the view that such extension is non-unique (although the diagonal process is unique—Theorem 1, and there is a large class of operators to which extension is unique—Theorem 3). Our considerations also raise the question of whether or not each pure state of $\mathcal{B}$ is the extension of some pure state of some maximal abelian algebra. (This is true for the vector

states of $\mathcal{B}$.)   With regard to this last question, one can partially order the
states of $\mathcal{B}$ by comparing the sets of elements on which they give "definite
information."   (We say that the state, ω of $\mathcal{B}$ is definite on a self-adjoint
operator, A, when ω is pure on the C^*-algebra generated by A-equivalently,
when $\omega(A^2) = \omega(A)^2$.   The set of operators on which ω is definite is "the
definite set" of ω.)

THEOREM 4.   *A state of $\mathcal{B}$ is pure if and only if its definite set is
maximal (with respect to inclusion).*

Proof.   If $\mathcal{K}$ is the left kernel of ω, then the definite set of ω is
$\{\mathcal{K}_* + \lambda I\}$, where $\mathcal{K}_*$ is the set of self-adjoint operators in $\mathcal{K}$ and λ is a
real number.   (Note that the set of self-adjoint operators in a uniformly
closed left ideal determines that ideal—in fact, the positive operators in the
ideal determine it [7].)   Indeed, if A is in $\mathcal{K}_*$, then $0 = \omega(A) = \omega(A)^2$
$= \omega(A^2)$ (by definition of $\mathcal{K}$), so that ω is definite on $\mathcal{K}_*$ and hence on
$\{\mathcal{K}_* + \lambda I\}$.   On the other hand, if ω is definite on B, then it is definite
on $B - \omega(B)I$, so that $\omega([B - \omega(B)I]^2) = \omega(B - \omega(B)I)^2 = 0$ and $B - \omega(B)I$
is in $\mathcal{K}_*$, B is in $\{\mathcal{K}_* + \lambda I\}$.   If $\mathcal{K}$ is not a maximal left ideal and $\mathcal{J}$ is a
left ideal in $\mathcal{B}$ containing $\mathcal{K}$ properly, choose A in $\mathcal{J}_*$ not in $\mathcal{K}$.   If
$A = B + \lambda I$, with B in $\mathcal{K}_*$, then $A - B = \lambda I$ is in $\mathcal{J}$, so that $\lambda = 0$, $A = B$
is in $\mathcal{K}$—a contradiction.   Thus $\{\mathcal{J}_* + \lambda I\}$ contains $\{\mathcal{K}_* + \lambda I\}$ properly.
It follows that the definite set of ω is maximal only if its left kernel is a
maximal left ideal—which implies that ω is pure [2].

Suppose, now, that $\mathcal{K}$ is a maximal left ideal and that $\mathcal{J}$ is a left ideal
such that $\{\mathcal{J}_* + \lambda I\}$ contains $\{\mathcal{K}_* + \lambda I\}$, the definite set of some pure state.
We show that $\{\mathcal{J}_* + \lambda I\}$, the definite set of an arbitrary state, coincides
with $\{\mathcal{K}_* + \lambda I\}$, in this case.   Passing to a maximal left ideal containing $\mathcal{J}$,
we may assume that $\mathcal{J}$ itself is maximal; so that $\mathcal{J}$ is the left kernel of a
pure state, ρ, of $\mathcal{B}$.   If $\mathcal{J}$ annihilates a vector, y, then so must $\mathcal{K}$; for other-
wise, $\mathcal{K}_*$ contains all self-adjoint completely continuous operators, and in par-
ticular, one which maps y onto a non-zero vector orthogonal to y-contradicting
the fact that $\{\mathcal{J}_* + \lambda I\}$ has y as an eigenvector and contains $\{\mathcal{K}_* + \lambda I\}$.
Thus $\mathcal{K}_*$ consists of all self-adjoint operators annihilating some vector, z,
and has y as an eigenvector; so that z is a scalar multiple of y, $\mathcal{K}_*$ anni-
hilates y, $\mathcal{K} = \mathcal{J}$, and $\{\mathcal{K}_* + \lambda I\} = \{\mathcal{J}_* + \lambda I\}$.

We may assume that $\mathcal{J}$ does not annihilate a vector and, so, contains $\mathcal{C}$,
the ideal of completely continuous operators in $\mathcal{B}$.   Thus, ϕ, the irreducible
representation of $\mathcal{B}$ associated with ρ has $\mathcal{C}$ as kernel.   If A is in $\mathcal{K}_*$ but
not in $\mathcal{J}_*$, then $\rho(A^2) = \rho(A)^2 \neq 0$; so that $\rho(E) \neq 0$ for some spectral

projection, E, of A corresponding to an interval whose closure does not contain 0. In fact, from the Spectral Theorem, A is a uniform limit of finite linear combinations of such spectral projections, and if ρ annihilates each of them, then since ρ is uniformly continuous, $\rho(A) = 0$. Since E corresponds to an interval whose closure does not contain 0, $I - E + AE$ has an inverse, B; so that $BEA = E[B(I - E + AE)] = E$ is in $\mathcal{K}_*$ and hence in $\{\mathcal{J}_* + \lambda I\}$. Moreover, ECE is in $\mathcal{K}_*$, hence in $\{\mathcal{J}_* + \lambda I\}$, for each self-adjoint C in $\mathcal{B}$. Now ϕ maps $\{\mathcal{J}_* + \lambda I\}$ into the set of self-adjoint operators which have x as an eigenvector, where x is a vector such that $\omega_x \phi = \rho$; so that the projection, $\phi(E)$, has x in its range (since $\rho(E) \neq 0$), and $\phi(E)\phi(C)\phi(E)x = \phi(E)\phi(C)x = \alpha x$. Since ϕ is an irreducible representation of $\mathcal{B}$, $\phi(E)$ must be the one-dimensional projection whose range contains x. But ρ annihilates $\mathcal{C}$, and $\rho(E) \neq 0$. Thus E is infinite dimensional, and $E = F + I - F$, where F and $I - F$ are infinite dimensional; so that $\phi(E) = \phi(F) + \phi(I - F)$, with $\phi(F)$ and $\phi(I - F)$ non-zero orthogonal projections. (Recall that the kernel of ϕ is $\mathcal{C}$). Hence $\phi(E)$ cannot be one-dimensional, each A in $\mathcal{K}_*$ lies in $\mathcal{J}_*$, $\mathcal{K}$ is contained in $\mathcal{J}$ and $\mathcal{K} = \mathcal{J}$, by maximality, and $\{\mathcal{K}_* + \lambda I\}$ is a maximal definite set.

Presumably, the definite set of each pure state contains the set of self-adjoint elements of some (perhaps many) maximal abelian algebras. A general question of obvious interest is that of the classification of the irreducible representations of $\mathcal{B}$. We know from [3] that the separable ones are all unitarily equivalent (to the algebra of bounded operators on separable Hilbert space) and are associated with vector states. The vector states of $\mathcal{B}$ are unitarily equivalent. Is this the case for the singular pure states of $\mathcal{B}$? A clever counting argument of Kaplansky's shows that this is not so. In fact, each pure state of $\mathcal{A}_d$ has a pure state extension to $\mathcal{B}$, so that there are at least 2^C pure states of $\mathcal{B}$ (the pure state space of $\mathcal{A}_d$ is $\beta(\mathfrak{A})$ which has cardinality 2^C), while there are only C operators (as can easily be seen from the matrix representation relative to a countable orthonormal basis.) Each unitary equivalence class contains at most C states, so that there are 2^C inequivalent singular pure states.

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